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A NOTE ON E

ALEKSANDAR KRON

Since there is no characteristic matrix for E so far, there is no possibility of investigating whether E has the finite model property in the sense of [1]. The aim of this note is to prove that for any wff D of E there is a finite set of wffs having properties similar to some properties of a finite model.

I shall suppose that E is formulated as in [2] or [3], but I shall write $\neg$ for negation instead of $^{-}$. Let $X_1, X_2, \ldots$ be the sequence of all finite non-empty sets of wffs of E. If $X_i = \{A_1, \ldots, A_n\}$, $i = 1, 2, \ldots$, then $\overline{X_i}$ shall denote the wff $A_1 \,\&\, \ldots \,\&\, A_n$. Let us write X instead of X_i. X will be called *consistent* iff $\dashv_E \neg\overline{X}$; X is *inconsistent* iff $\vdash_E \neg\overline{X}$. Clearly, if X is consistent, then for no wff B $\vdash_E \overline{X} \to B \,\&\, \neg B$.

Lemma 1. *For any X, B and C, if X is consistent and $\vdash_E \overline{X} \to B \vee C$, then either $X \cup \{B\}$ or $X \cup \{C\}$ is consistent.*

Proof. Suppose that the contrary is the case. Then we have both $\vdash_E \neg(\overline{X} \,\&\, B)$ and $\vdash_E \neg(\overline{X} \,\&\, C)$. By adjunction we obtain $\vdash_E \neg(\overline{X} \,\&\, B) \,\&\, \neg(\overline{X} \,\&\, C)$ and thus $\vdash_E \neg(\overline{X} \,\&\, B \vee \overline{X} \,\&\, C)$. But then we easily derive $\vdash_E \neg(\overline{X} \,\&\, (B \vee C))$ and $\vdash_E \neg\overline{X} \vee \neg(B \vee C)$. Since $\vdash_E \overline{X} \to B \vee C$, we have $\vdash_E \neg(B \vee C) \to \neg\overline{X}$. Therefore, $\vdash_E \neg\overline{X}$, contrary to the assumption of the lemma.

Lemma 2. *For all X, B, C and D, if $\vdash_E \overline{X} \to B \vee C$ and $\dashv_E \overline{X} \to D$, then either $\dashv_E \overline{X} \,\&\, B \to D$ or $\dashv_E \overline{X} \,\&\, C \to D$.*

Proof. Suppose that both $\vdash_E \overline{X} \,\&\, B \to D$ and $\vdash_E \overline{X} \,\&\, C \to D$. We first easily obtain $\vdash_E (\overline{X} \,\&\, B) \vee (\overline{X} \,\&\, C) \to D$ and then $\vdash_E \overline{X} \,\&\, (B \vee C) \to D$. Since $\vdash_E \overline{X} \to B \vee C$, we have $\vdash_E \overline{X} \to \overline{X} \,\&\, (B \vee C)$ and thus $\vdash_E \overline{X} \to D$, contrary to the hypothesis of the lemma.

Let D be an arbitrary wff of E, let $\mathrm{P}^+(D)$ be the set of all subformulae of D, let $\mathrm{P}^-(D)$ be the set of all negations of the wffs of $\mathrm{P}^+(D)$ and let $\mathrm{P}(D) = \mathrm{P}^+(D) \cup \mathrm{P}^-(D)$. Furthermore, let $\mathrm{X}(D) = \{C_j \vee \neg C_j\colon\ C_j \in \mathrm{P}^+(D)\}$, for all $1 \leq j \leq r$, where r is the number of subformulae of D. In the sequel I shall consider only the members $Y_1, \ldots, Y_{2r}$ of the sequence $X_1, X_2, \ldots$ satisfying the following two conditions:

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(1) $\mathsf{X}(D) \subseteq Y_k$
(2) $Y_k \subseteq \mathsf{P}(D)$,

$1 \leq k \leq 2^{2^r}$. If $Y_m \subseteq Y_n$, then Y_n is called an *extension* of Y_m. Thus, every Y_k is an extension of $\mathsf{X}(D)$. Let us write Y instead of Y_k, $1 \leq k \leq 2^{2^r}$ and C instead of C_j, $1 \leq j \leq 2r$, and let us introduce Y', Y'', Z, etc., for the same purpose.

A set Y will be called *D-normal* iff it is consistent and for every $C \in \mathsf{P}^+(D)$ either $C \in Y$ or $\neg C \in Y$.

Lemma 3. *For any consistent Y there is a D-normal extension Z.*

Proof. Since $\mathsf{X}(D) \subseteq Y$, we have $\vdash_E \overline{Y} \rightarrow C \vee \neg C$, for all $C \in \mathsf{P}^+(D)$. By Lemma 1, either $Y' = Y \cup \{C\}$ or $Y'' = Y \cup \{\neg C\}$ is consistent. Since $\mathsf{X}(D)$ is finite, repeating the same argument we could show that there is a D-normal extension Z of Y.

I shall note that the preceding lemma states only the existence of a D-normal extension Z of Y; it does not provide a construction of Z given Y.

Let M_D be the set of all normal extensions of $\mathsf{X}(D)$. Obviously, M_D is not empty. Let us say that $C \in \mathsf{P}(D)$ is *valid* in M_D iff $C \in Y$ for all $Y \in M_D$; it is *refutable* in M_D iff there is an Y such that $C \notin Y$.

Lemma 4. *For all $C \in \mathsf{P}(D)$, if $\dashv_E C$, then C is refutable in M_D.*

Proof. If $\dashv_E C$, then $\dashv_E \overline{\mathsf{X}}(D) \rightarrow C$. But $\vdash_E \overline{\mathsf{X}}(D) \rightarrow C \vee \neg C$. Therefore, by Lemma 2, $\dashv_E \overline{\mathsf{X}}(D) \;\&\; \neg C \rightarrow C$. I have to show that $\mathsf{X}(D) \cup \{\neg C\}$ is consistent. Suppose that the contrary is the case. Then $\vdash_E \neg \overline{\mathsf{X}}(D) \vee \neg\neg C$ and by the rule γ (see [4]), since $\vdash_E \overline{\mathsf{X}}(D)$, we have $\vdash_E \neg\neg C$ and thus $\vdash_E C$, contrary to the hypothesis that $\dashv_E C$. By Lemma 3 there is a D-normal extension of $\mathsf{X}(D) \cup \{\neg C\}$. Therefore, there is an $Y \in M_D$ such that $C \notin Y$, and C is thus refutable in M_D

Corollary. *If $C \in \mathsf{P}(D)$ is valid in M_D, then $\vdash_E C$.*

Lemma 5. *For all $C \in \mathsf{P}(D)$, if $\vdash_E C$, then C is valid in M_D.*

Proof. Suppose that C is not valid in M_D. Then there is an $Y \in M_D$ such that $\neg C \in Y$. Obviously, $\vdash_E \overline{Y} \rightarrow \neg C$. But $\vdash_E Y \rightarrow \neg C \rightarrow .C \rightarrow \neg \overline{Y}$ and thus $\vdash_E C \rightarrow \neg \overline{Y}$. Now if $\vdash_E C$, we have $\vdash_E \neg \overline{Y}$ and Y is inconsistent, which is impossible, since $Y \in M_D$. Therefore, $\dashv_E C$, and this proves the lemma.

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University of Beograd
Beograd, Yugoslavia