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DUALS OF SMULLYAN TREES

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1. As readers of Jeffrey or Smullyan know, the consistency of a finite set S of wffs from the sentential calculus (SC) can be tested by means of a tree, called here a *Smullyan tree*.[1] The branches of the tree, which are gotten by breaking up each member of S into shorter wffs, breaking up these shorter wffs into still shorter ones, and so on, represent the various ways in which the members of S could be true. Those branches (if any) on which both an atomic wff (one of the letters 'P', 'Q', 'R', etc.) and its negation occur are said to be *closed*, the rest to be *open*. And the method guarantees that:

(1) If every branch of the tree is closed, S (the set tested) is inconsistent, whereas

(2) If at least one branch stays open, S is consistent, and a truth-value assignment on which all the members of S are true can be read off any open branch of the tree.

When '$\sim$', '&', 'v', and '$\supset$' serve as primitive connectives, the rules for breaking up truth-functional compounds are seven in number:[2]

1. Concerning Smullyan trees, see [4], [5], and [6]. We of course take a set S of the sort described to be (semantically) consistent if there is a truth-value assignment to the atomic components of the members of S on which all these members are true (i.e., get a **T**).

2. When '$\equiv$' also serves as a primitive connective, two extra rules serve to break up compounds of the sort $A \equiv B$ or the sort $\sim(A \equiv B)$:

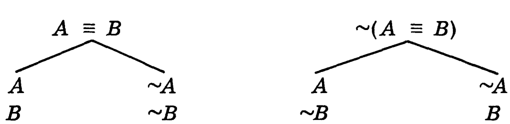

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TABLE I

R1	R2	R3
$A \,\&\, B$	$A \vee B$	$A \supset B$
A / B	A | B (branch)	$\sim A$ | B (branch)

R4	R5	R6	R7
$\sim\sim A$	$\sim(A \,\&\, B)$	$\sim(A \vee B)$	$\sim(A \supset B)$
A	$\sim A$ | $\sim B$ (branch)	$\sim A$ / $\sim B$	A / $\sim B$

We illustrate things by means of two trees. All the branches of the first close, and hence the set tested is inconsistent. Three branches of the second stay open, and hence the set tested is consistent. (We write 'x' at the tip of every closed branch, and—*to avoid pointless entries—do so the* minute an atomic wff and its negation have both occurred on the branch.)

EXAMPLE 1: $\{P \supset (Q \,\&\, R), (\sim P \vee Q) \supset \sim R, P\}$.

1	$P \supset (Q \,\&\, R)$			
2	$(\sim P \vee Q) \supset \sim R$			
3	P			
4	$\sim P$	$Q \,\&\, R$		(R3, 1)
5	x	Q		(R1, 4)
6		R		(R1, 4)
7		$\sim(\sim P \vee Q)$	$\sim R$	(R3, 2)
8		$\sim\sim P$	x	(R6, 7)
9		$\sim Q$		(R6, 7)
		x		

EXAMPLE 2: $\{\sim(P \,\&\, \sim Q), \sim Q \vee R, \sim(R \supset P)\}$.

1	$\sim(P \,\&\, \sim Q)$				
2	$\sim Q \vee R$				
3	$\sim(R \supset P)$				
4	R				(R7, 3)
5	$\sim P$				(R7, 3)
6	$\sim Q$		R		(R2, 2)
7	$\sim P$	$\sim\sim Q$	$\sim P$	$\sim\sim Q$	(R5, 1)
8		Q		Q	(R4, 7)
		x			

As we shall soon see, lines 1-3 here are sure to be true on the result of assigning the truth-value **F** to 'P', the truth-value **F** to 'Q', and the truth-value **T** to 'R', an assignment picked off the left-most branch; also on the result of assigning **F** to 'P', **T** to 'R', and either one of **T** and **F** to 'Q', an assignment picked off the third branch.

Proof of (1) is easily had. Note indeed that if the uppermost wff in rule R4 is true on some truth-value assignment α, the one wff into which it decomposes is sure to be true on α; if the uppermost wff in rules R2, R3, and R5 is true on α, at least one of the two wffs into which it decomposes is sure to be true on α; and if the uppermost wff in rules R1, R6, and R7 is true on α, both of the two wffs into which it decomposes are sure to be true on α. Hence, if the members of the set tested are true on any truth-value assignment, then *either* all the wffs on the left-most branch of a tree must be true on that assignment, *or* all those on the next branch, *or* all those on the next, and so on. But the wffs on a closed branch cannot *all* be true on any truth-value assignment: one of them is the negation of another. So, if all the branches of a tree are closed, the set tested is sure to be inconsistent.

Proof of (2) uses the notion of a "model set," a model set S being such that:

(a) If the negation $\sim A$ of an atomic wff A belongs to S, then A does not belong to S;

(b) If a conjunction $A \& B$ belongs to S, then both A and B belong to S;

(c) If a disjunction $A \vee B$ belongs to S, then at least one of A and B belongs to S;

(d) If a conditional $A \supset B$ belongs to S, then at least one of $\sim A$ and B belongs to S;

(e) If the negation $\sim\sim A$ of a negation $\sim A$ belongs to S, then A belongs to S;

(f) If the negation $\sim(A \& B)$ of a conjunction $A \& B$ belongs to S, then at least one of $\sim A$ and $\sim B$ belongs to S;

(g) If the negation $\sim(A \vee B)$ of a disjunction $A \vee B$ belongs to S, then both $\sim A$ and $\sim B$ belong to S; and

(h) If the negation $\sim(A \supset B)$ of a conditional $A \supset B$ belongs to S, then both A and $\sim B$ belong to S.

Now let S^* consist of the atomic components of the various members of a model set S of wffs. It is easily shown by mathematical induction on the length of an arbitrary member of S that each and every member of S is sure to be true on the result of assigning **T** to every member of S^* which belongs to S, **F** to every one whose negation belongs to S, and either **T** or **F** to every other member of S^*. So any model set is consistent.[3] But the set made up of the wffs on any open branch of a tree is readily seen to be a model set. Hence that set (plus of course each one of its subsets) is sure to be consistent. So, if any branch of a tree is open, then the set tested is

3. The notion of a model set is due to Hintikka, see [3], pp. 22-29; see also [6], p. 57, where model sets are called Hintikka sets.

sure to be consistent, and its members are sure to be true on the following truth-value assignment: **T** to every atomic wff which occurs unnegated on that branch, **F** to every one which occurs negated, and either **T** or **F** to every other atomic component of the members of the set. (The two truth-value assignments under Example 2 were gotten that way: '*R*' occurred unnegated, and both '*P*' and '*Q*' occurred negated, on the left-most branch of the tree; '*R*' occurred unnegated, and '*P*' occurred negated, on the third branch.)

2. Whether or not a wff *A* from SC is valid (i.e., is a tautology) can likewise be tested by means of a Smullyan tree. Indeed, *A* is valid if and only if $\{\sim A\}$ is inconsistent. So, if $\{\sim A\}$ has a closed tree, *A* is sure to be valid; whereas, if $\{\sim A\}$ has an open tree, *A* is sure not to be valid, and a truth-value assignment on which *A* is false (i.e. $\sim A$ is true) can be read off any open branch of the tree. But the validity of *A* can be tested in another and more direct way.

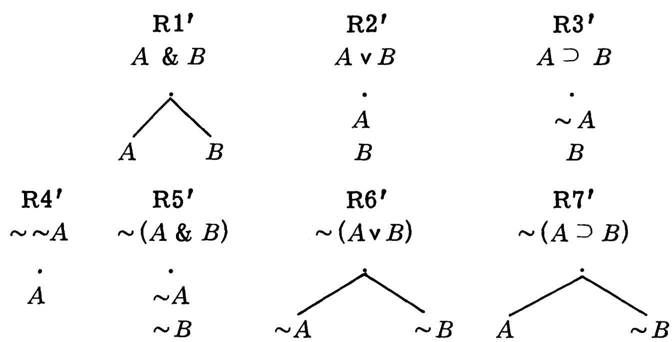
Smullyan trees have duals, to be known here as *dual trees*. The branches of a dual tree, gotten by breaking up a wff *A* into shorter wffs, breaking up these shorter wffs into still shorter ones, and so on, represent the various ways in which *A* could be false. And proof will be given below that:

(3) *If every branch of a dual tree is closed, A (the wff tested) is valid, whereas*

(4) *If at least one branch stays open, A is not valid, and a truth-value assignment on which A is false can be read off any open branch of the tree.*

With '$\sim$', '&', 'v', and '$\supset$' serving again as primitive connectives, the rules for breaking truth-functional compounds now run:[4]

TABLE II



4. When '$\equiv$' also serves as a primitive connective, two extra rules serve to break up compounds of the sort $A \equiv B$ or the sort $\sim(A \equiv B)$:

Illustrations are as follows. Since every branch of the first tree is closed, the wff tested is valid; since four branches of the second stay open, the wff tested is not valid.

EXAMPLE 3: $(P \supset (\sim Q \,\&\, R)) \supset ((Q \supset \sim P) \,\&\, (P \supset R))$.

1	$(P \supset (\sim Q \,\&\, R)) \supset ((Q \supset \sim P) \,\&\, (P \supset R))$				
2	$\sim(P \supset (\sim Q \,\&\, R))$				(R3′, 1)
3	$(Q \supset \sim P) \,\&\, (P \supset R)$				(R3′, 1)
4	P		$\sim(\sim Q \,\&\, R)$		(R7′, 2)
5			$\sim\sim Q$		(R5′, 4)
6			$\sim R$		(R5′, 4)
7			Q		(R4′, 5)
8	$Q \supset \sim P$	$P \supset R$	$Q \supset \sim P$	$P \supset R$	(R′1, 3)
9	$\sim Q$	$\sim P$	$\sim Q$	$\sim P$	(R3′, 8)
10	$\sim P$	R	$\sim P$	R	(R3′, 8)
	x	x	x	x	

EXAMPLE 4: $((P \supset Q) \vee (P \supset R)) \supset (\sim(\sim P \supset Q) \vee R)$.

1	$((P \supset Q) \vee (P \supset R)) \supset (\sim(\sim P \supset Q) \vee R)$								
2	$\sim((P \supset Q) \vee (P \supset R))$								(R3′, 1)
3	$\sim(\sim P \supset Q) \vee R$								(R3′, 1)
4	$\sim(\sim P \supset Q)$								(R2′, 3)
5	R								(R2′, 3)
6	$\sim P$				$\sim Q$				(R7′, 4)
7	$\sim(P \supset Q)$		$\sim(P \supset R)$		$\sim(P \supset Q)$		$\sim(P \supset R)$		(R6′, 2)
8	P	$\sim Q$	P	$\sim R$	P	$\sim Q$	P	$\sim R$	(R7′, 7)
	x		x	x				x	

Line 1 here is sure to be true on the result, for example, of assigning **T** to both 'P' and 'Q', and **F** to 'R', an assignment picked off the second branch.

Proof of (3) is as follows. If the uppermost wff in rule R4′ is false on some truth-value assignment α, the one wff into which it decomposes is sure to be false on α; if the uppermost wff in rules R1′, R6′, and R7′ is false on α, at least one of the two wffs into which it decomposes is sure to be false on α; and if the uppermost wff in rules R2′, R3′, and R5′ is false on α, both of the two wffs into which it decomposes are sure to be false on α. Hence, if the wff tested is false on any truth-value assignment, then *either* all the wffs on the left-most branch of a dual tree must be false on that assignment, *or* all those on the next branch, *or* all those on the next, and so on. But the wffs on a closed branch of a dual tree cannot *all* be false on any truth-value assignment. So, if all the branches of a dual tree are closed, the wff tested cannot be false either on any truth-value assignment, and hence has to be valid.

Proof of (4) calls for the notion of a "dual model set," a dual model set S being such that:

(a′) If the negation $\sim A$ of an atomic wff A belongs to S, then A does not belong to S;

(b′) If a conjunction $A \,\&\, B$ belongs to S, then at least one of A and B belongs to S;

(c′) If a disjunction $A \vee B$ belongs to S, then both A and B belong to S;

(d′) If a conditional $A \supset B$ belongs to S, then both $\sim A$ and B belong to S;

(e′) If the negation $\sim\sim A$ of a negation $\sim A$ belongs to S, then A belongs to S;

(f′) If the negation $\sim(A \,\&\, B)$ of a conjunction $A \,\&\, B$ belongs to S, then both $\sim A$ and $\sim B$ belong to S;

(g′) If the negation $\sim(A \vee B)$ of a disjunction $A \vee B$ belongs to S, then at least one of $\sim A$ and $\sim B$ belongs to S; and

(h′) If the negation $\sim(A \supset B)$ of a conditional $A \supset B$ belongs to S, then at least one of A and $\sim B$ belongs to S.

Now let S^* consist this time of the atomic components of the various members of a dual model set S of wffs. An obvious induction will show that each and every member of S is sure to be false on the result of assigning **F** to every member of S^* which belongs to S, **T** to every one whose negation belongs to S, and either **T** or **F** to every other member of S^*. But the set made up of the wffs on any open branch of a dual tree is a dual model set. So, if any branch of a dual tree is open, then the wff tested is sure to be false on the following truth-value assignment: **F** to every atomic wff which occurs unnegated on that branch, **T** to every one which occurs negated, and either **T** or **F** to every other atomic component of the members of the set. (The truth-value assignment under Example 4 was gotten that way: 'P' and 'Q' occurred negated, and 'R' occurred unnegated, on the second branch of the tree.)[5]

3. Dual trees can also be used to test the consistency of a finite set of wffs from SC. Indeed, $\{A_1, A_2, \ldots, A_n\}$ is inconsistent if and only if $\sim(A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n)$ is valid. So, if this negated conjunction has a closed dual tree, the original set $\{A_1, A_2, \ldots, A_n\}$ is sure to be inconsistent; whereas, if $\sim(A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n)$ has an open dual tree, the set is sure to be consistent, and a truth-value assignment on which $A_1, A_2, \ldots, A_n$ are sure to be true can be read off any open branch of the tree. But Smullyan trees

5. The reduction technique of [2] resembles our dual trees. In [1] it is extended to the quantificational calculus and abbreviated so as to require less rewriting of formulas. In [7] it is extended to quantified modal logic.

are a more direct means of testing consistency, exactly as their duals are a more direct means of testing validity.

The techniques presented above extend to the quantificational calculus. Finite sets of wffs from that calculus can again be tested for consistency, and individual wffs tested for validity, by means of trees. The rules for breaking up quantifications and their negations are the same whether the tree constructed be a Smullyan tree or the dual of one:

TABLE III

R8	R9	R10	R11
$(\forall X)A$	$\sim(\forall X)A$	$(\exists X)A$	$\sim(\exists X)A$
.	.	.	.
$A(C/X)$	$\sim A(C/X)$	$A(C/X)$	$\sim A(C/X)$

Note: In all four rules $A(C/X)$ is the result of putting the individual constant (= parameter) C for every free occurrence of X in A. In rule R8 and R11, C may be any individual constant, and the compound may be decomposed as often as desired. In rules R9 and R10, C must be foreign to any branch on which $\sim A(C/X)$ and $A(C/X)$ are entered, and the compound may be decomposed only once.

A full account of Smullyan trees at the quantificational level will be found in [4], [5], and [6].

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