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ε-CALCULUS BASED AXIOM SYSTEMS FOR SOME PROPOSITIONAL MODAL LOGICS

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1 *Introduction*: Instead of considering propositional S4 (say) as a theory built on top of classical *propositional* logic, we can consider it as the classical *first order* theory of its Kripke models. This first order theory can be formulated in any of the ways first order theories usually are: tableau, Gentzen system, natural deduction, conventional axiom system, ε-calculus. If the first order theory of Kripke S4 models can be given in a formulation which is technically easy to use, the result is a convenient S4 proof system. Thus, in [2] we gave a tableau formulation of the S4 model theory which dealt automatically with the peculiarities of Kripke models, and produced a proof system for S4 (and similarly for other modal logics) which is simple to apply. In this paper we give what is essentially an ε-calculus formulation of the Kripke model theory of S4 (and S5, T, B, and also first order versions) which also automatically treats the peculiarities of the Kripke models. It is less convenient in use than the tableau system of [2] but still has a curious intrinsic interest.

2 *Propositional systems*: We begin with S4, which we formulate as a classical theory in which atomic formulas, as defined below, are strings of symbols which include modal formulas as substrings. We take $\sim$, $\wedge$, and $\Diamond$ as primitive. Modal formulas are defined as usual. We use $P_1, P_2, \ldots,$ $Q_1, Q_2, \ldots$ to stand for modal formulas beginning with an occurrence of $\Diamond$, and X and Y to stand for arbitrary modal formulas. Let * be some new symbol. By an *atomic formula* (of the classical theory we are constructing) we mean a string of symbols of the form $*P_1, \ldots, P_n*X$ (we adopt the convention that n may be 0, i.e. $**X$ is an atomic formula). *Formulas* are built up in the usual way from atomic formulas using $\sim$ and $\wedge$ as primitive.

We have the single rule of modus ponens, and as axioms we take all classical tautologies, and all formulas of the form (where again n may be 0):

(1) $*P_1, \ldots, P_n*X \wedge Y \equiv *P_1, \ldots, P_n*X \wedge *P_1, \ldots, P_n*Y$
(2) $*P_1, \ldots, P_n*\sim X \equiv \sim *P_1, \ldots, P_n*X$

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(3) $*P_1, \ldots, P_n*\Diamond X \supset *P_1, \ldots, P_n, \Diamond X*X$
(4) $*P_1, \ldots, P_i, \ldots, P_n*X \supset *P_1, \ldots, P_i*\Diamond X$ (i may be 0 or n)

We claim the modal formula X is valid in all S4 models if and only if $**X$ is provable in the above system. Thus, the following is a proof of $**\Diamond\Diamond X \supset \Diamond X$, and hence of the S4 validity of $\Diamond\Diamond X \supset \Diamond X$. We make use of the deduction theorem.

(1)	$**\Diamond\Diamond X$	hypothesis
(2)	$**\Diamond\Diamond X \supset *\Diamond\Diamond X*\Diamond X$	axiom 3
(3)	$*\Diamond\Diamond X*\Diamond X$	by 1, 2
(4)	$*\Diamond\Diamond X*\Diamond X \supset *\Diamond\Diamond X, \Diamond X*X$	axiom 3
(5)	$*\Diamond\Diamond X, \Diamond X*X$	by 3, 4
(6)	$*\Diamond\Diamond X, \Diamond X*X \supset **\Diamond X$	axiom 4
(7)	$**\Diamond X$	by 5, 6
(8)	$**\Diamond\Diamond X \supset **\Diamond X$	using deduction theorem on 1-7
(9)	$**\Diamond\Diamond X \supset \Diamond X$	from 8 using axioms 1 and 2.

Intuitively, the atomic formula $*P_1, \ldots, P_n*X$ may be thought of as saying in some Kripke S4 model, $*P_1, \ldots, P_n*$ names a possible world in which the modal formula X is true. (The idea of putting names for possible worlds into the language is due to Fitch [1]). $*P_1, \ldots, P_n, \Diamond X*$ is intended to be the name of a world related to the world named by $*P_1, \ldots, P_n*$ and in which X is true, if there is one. Thus axiom 3 is typical of the ε-calculus, despite its appearance.

For other modal logics we proceed as follows: For S5, change axiom 4 to

(4′) $*P_1, \ldots, P_n*X \supset *Q_1, \ldots, Q_k*\Diamond X$ (n and k may be 0)

For T, change axiom 4 to

(4″a) $*P_1, \ldots, P_n, P_{n+1}*X \supset *P_1, \ldots, P_n*\Diamond X$
(4″b) $*P_1, \ldots, P_n*X \supset *P_1, \ldots, P_n*\Diamond X$

Finally, for B, add to the system for T

(4″c) $*P_1, \ldots, P_n*X \supset *P_1, \ldots, P_n, P_{n+1}*\Diamond X$.

3 *Correctness and completeness*: We talk in terms of S4, but the other logics are treated similarly. Given a particular Kripke S4 model (see [3], [4], or [5]) we may define an *interpretation* and *truth under the interpretation* in the following way. Let $I(**)$ be some arbitrary world in the model. Having defined $I(*P_1, \ldots, P_n*)$ to be some possible world, if there is a world in the model, related to $I(*P_1, \ldots, P_n*)$ and in which X is true, choose one such world and let it be $I(*P_1, \ldots, P_n, \Diamond X*)$; if there is no such world, let $I(*P_1, \ldots, P_n, \Diamond X*)$ be $I(*P_1, \ldots, P_n*)$. Thus $I(*P_1, \ldots, P_n*)$ is defined for all strings $*P_1, \ldots, P_n*$. For atomic formulas, call $*P_1, \ldots, P_n*X$ true under the interpretation I if X is true in the world $I(*P_1, \ldots, P_n*)$. Truth under I is extended to all formulas in the usual way. It is easy to see that all the axioms are true under this interpretation, and that truth is preserved under modus ponens, and thus that all theorems are true. Thus if $**X$ is provable, X is true in $I(**)$. Since $I(**)$ was arbitrary, X must be true in all worlds of the model, and since the model itself was arbitrary, X must be S4 valid.

Completeness is easily shown using maximal consistent sets. Thus, if $**X$ is not provable, we may extend $\{\sim **X\}$ to a maximal consistent set S of formulas. Using S we define a countermodel. Let the collection of possible worlds be the set of all strings $*P_1, \ldots, P_n*$. Define $*P_1, \ldots, P_n* \; R \; *Q_1, \ldots, Q_k*$ to hold if $P_1, \ldots, P_n$ is an initial segment of $Q_1, \ldots, Q_k$. Call Y true in the world $*P_1, \ldots, P_n*$ if $*P_1, \ldots, P_n*Y$ is in S. It is straightforward that this defines a Kripke S4 model. Then since $\sim **X$ is in S, X is not valid in this model.

4 *First order systems*: We give first order versions of the logics of section **2**. We do not prove correctness or completeness, but the methods of section **3** easily extend.

For a first order version of S4 see [5] where both a model theory and a proof system are given. We assert that by adding the following two axioms to the S4 system above we get that system (we take $\exists$ as primitive).

(5) $*P_1, \ldots, P_n* \, (\exists x)X(x) \supset *P_1, \ldots, P_n*X(*P_1, \ldots, P_n*X(x))$

(6) $*P_1, \ldots, P_n*Y\,(*P_1, \ldots, P_i*X(x)) \supset *P_1, \ldots, P_n* \,(\exists x)\,Y(x) \;\; (i \leq n)$.

This is clearly seen to be an ε-calculus formulation. In axiom 5, $*P_1, \ldots, P_n*X(x)$ is intended to name a constant of the world named by $*P_1, \ldots, P_n*$ which makes $X(x)$ true in that world, if any constant does.

The following is a proof in this system of $(\exists x)\Diamond X(x) \supset \Diamond\,(\exists x)X(x)$.

(1)	$**(\exists x)\Diamond X(x)$	hypothesis
(2)	$**\Diamond X(**X(x))$	from 1 and axiom 5
(3)	$*\Diamond X(**X(x))*X(**X(x))$	from 2 and axiom 3
(4)	$*\Diamond X(**X(x))*(\exists x)X(x)$	from 3 and axiom 6
(5)	$**\Diamond\,(\exists x)X(x)$	from 4 and axiom 4

In [6] it is shown that adding the Barcan formula, $\Diamond(\exists x)X(x) \supset (\exists x)\Diamond X(x)$ to the first order S4 system of [5] produces a 'constant domain' S4, one in which all the possible worlds of a model have the same associated domain of constants. To treat this logic in a system like that above, change axiom 6 to:

(6′) $*P_1, \ldots, P_n*Y(*Q_1, \ldots, Q_k*X(x)) \supset *P_1, \ldots, P_n*(\exists x)Y(x)$.

Similarly, for 'constant domain' first order versions of S5, T, and B, we may add axioms 5 and 6′ to the respective propositional systems.

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