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EFFECTIVE INNER PRODUCT SPACES

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1 *Introduction* Dekker ([1]) introduced and studied a recursive enumerable vector space U_F over a recursive field F which is universal for all countable dimensional vector spaces over F. Many further results were gotten by Guhl [3], Metakides and Nerode ([7]), and others. The purpose of this paper* is to introduce a natural inner product on U_F and to show that the analogues of classical finite dimensional inner product space theory fail even for the recursive spaces.

2 *Preliminaries* We assume that the reader is familiar with the notations, conventions, and results of [1]. We let ε denote the set of nonnegative integers, and we note that 0 plays the role of both the Gödel number of the zero element of F and the zero vector of U_F. If β is a repère (a linearly independent set) in U_F and x is a member of $\mathbf{L}(\beta)$, we write $\mathrm{supp}_\beta(x)$ for the set of all elements of β which have nonzero coefficients when x is expressed as a linear combination of elements in β. We let $\eta = \rho e$ be the canonical basis for U_F and write $\mathrm{supp}(x)$ for $\mathrm{supp}_\eta(x)$. Following [8], Chapter 11, we call the field F *formally real* if -1_F is not expressible in F as a sum of squares. Note that F is formally real if and only if a sum of squares of elements of F vanishes only when each element is zero. All formally real fields have characteristic 0; $\boldsymbol{Q}$, $\boldsymbol{Q}(\sqrt{2})$, $\boldsymbol{Q}(\pi)$ are formally real, while $\boldsymbol{Q}(\mathrm{i})$ is not.

Definition D1: Let F be any countable formally real field for which there exists a one-to-one mapping ϕ from F onto ε under which the field operations correspond to (partial) recursive functions. We consider the recursively presented vector space U_F over F constructed in [1]. We define a function $\langle , \rangle$ from $\varepsilon \times \varepsilon \rightarrow \varepsilon$ by

$$\langle x, y\rangle = \phi\left(\sum_{i=1}^{k} x_i y_i\right) ,$$

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where $x = (x_0, x_1, \ldots)^{\#}$, $y = (y_0, y_1, \ldots)^{\#}$, and both x_n and y_n are 0_F for $n > k$. We call $\langle , \rangle$ the standard inner product on U_F and we note that it is recursive.

From now on, all our fields F will be formally real fields for which the function ϕ exists.

Proposition P1 *The standard inner product on U_F satisfies the following:*

(1) *for all $u, v \in U_F$, $\langle u, v\rangle = \langle v, u\rangle$,*
(2) *if $u, v, w \in U_F$, then $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$, where the addition on the right is that induced on ε by ϕ,*
(3) *if $\alpha \in F$ and $u, v \in U_F$, then $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$, where the multiplication on the right (really $\phi(\alpha)\cdot\langle u, v\rangle$) is that induced on ε by ϕ,*
(4) *if $\langle v, w\rangle = 0$ for all $w \in U_F$, then $w = 0$,*
(5) *if $\langle u, v\rangle = 0$, then $v = 0$, and conversely,*
(6) *$\langle e_i, x\rangle \neq 0$ if and only if $e_i \in \mathrm{supp}(x)$.*

Proof: Linear algebra. Q.E.D.

3 *Eight propositions* The elements $u, v \in U_F$ are said to be *orthogonal*, denoted by $u \perp v$, if $\langle u, v\rangle = 0$. If $S \subseteq U_F$, we denote by $S^{\perp}$ the set of all elements $w \in U_F$ for which $\langle w, s\rangle = 0$ for each $s \in S$. The proofs of the first seven propositions follow exactly as in the classical cases (with the added observation that the Gram-Schmidt orthogonalization process is effective on r.e. repères) and are omitted.

Proposition P2 (a) *Let $S \subseteq U_F$. Then $S^{\perp}$ is a subspace of U_F and $S^{\perp} \cap \mathbf{L}(S) = \{0\}$.*

(b) *If $S \subseteq T \subseteq U_F$, then $T^{\perp} \leqslant S^{\perp}$.*

Proposition P3 *If $S \leqslant U_F$ is finite dimensional, then $S \oplus S^{\perp} = U_F$.*

Proposition P4 *If W_1 and W_2 are subspaces of $\overline{U}_F$, then*

(i) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$,

and

(ii) $(W_1^{\perp} + W_2^{\perp}) \leqslant (W_1 \cap W_2)^{\perp}$.

Proposition P5 *Let W_1 and W_2 be subspaces of U_F. If for each $S \leqslant U_F$, $(S^{\perp})^{\perp} = S$, then equality holds in* P4 (ii).

Proposition P6 *If $a_0, a_1, \ldots, a_n, \ldots$ is a (finite or infinite) sequence of pairwise mutually orthogonal non-zero elements in U_F, then $\gamma = \rho a$ is a repère.*

Proposition P7 (a) *if $\langle x, a_i\rangle = 0$ for $0 \leqslant i \leqslant n$ and $y \in \mathbf{L}(a_0, \ldots, a_n)$, then $\langle x, y\rangle = 0$.*

(b) *If $\langle x, a_i\rangle = 0$ for $0 \leqslant i \leqslant n$ and $x \in \mathbf{L}(a_0, \ldots, a_n)$, then $x = 0$.*

Proposition P8 *Suppose* b *is a* 1-1 *recursive function ranging over an* *infinite* r.e. *repère* β. *Then there is a* 1-1 *recursive function* $\overline{\mathsf{b}}$ *whose range is an infinite* r.e. *repère* $\overline{\beta}$ *such that*

(i) $(\forall n)[\mathbf{L}(b_0, \ldots, b_n) = \mathbf{L}(\overline{b}_0, \ldots, \overline{b}_n)]$,
(ii) *The elements of* $\overline{\beta}$ *are pairwise mutually orthogonal.*

We note that in the proof of P8, $\overline{\beta}$ can be gotten uniformly from β by refining β according to the order of presentation by b. We call the process *refining* β *into an orthogonal repère according to* b.

Proposition P9 *Every* r.e. *space over* F *has a recursive orthogonal basis.*

Proof: It suffices to note that since F has characteristic 0, it is infinite. Then by suitable scalar multiplication, if necessary, the function $\overline{\mathsf{b}}$ of P8 can be made strictly increasing.

4 *The main construction* The r.e. space $W \leqslant U_F$ is called a *recursive space* if there is some r.e. space V such that $W \oplus V = U_F$. In the past, recursive spaces have proved to be the easiest to work with effectively. We show below that under these conditions the worst pathologies exist.

Lemma L1 *Define the recursive function* a *by*

$$\mathsf{a}(0) = \mathsf{a}(1) = 1$$
$$\mathsf{a}(n) = \mathsf{a}(n-1) + [\mathsf{a}(n-1)]^2, \textit{ for } n \geqslant 2.$$

Then for all $n \geqslant 1$, $1 + [\mathsf{a}(1)]^2 + \ldots + [\mathsf{a}(n)]^2 - \mathsf{a}(n+1) = 0$.

Proof: By induction on n. We note here that $\mathsf{a}(2) = 2$, $\mathsf{a}(3) = 6$, $\mathsf{a}(4) = 42$, etc. Q.E.D.

Proposition P10 (a) *There exists a recursive space* S *such that* S *is an infinite dimensional proper subspace of* U_F *and* $S^{\perp} = \{0\}$.

(b) *There exists a recursive space* S *such that* $(S^{\perp})^{\perp} \neq S$.
(c) *There exists a recursive space* S *such that* $S \oplus S^{\perp} \not\leqslant U_F$.

Proof: Clearly (a) implies (b) and (c). We focus on (a) and define the recursive function d by

$$\mathsf{d}(0) = e_0 + e_1$$
$$\mathsf{d}(1) = e_0 - e_1 + e_2$$
$$\mathsf{d}(2) = e_0 - e_1 - 2e_2 + e_3$$
$$\mathsf{d}(3) = e_0 - e_1 - 2e_2 - 6e_3 + e_4$$
$$\mathsf{d}(4) = e_0 - e_1 - 2e_2 - 6e_3 - 42e_4 + e_5$$
$$\mathsf{d}(n) = e_0 - \left(\sum_{i=1}^{n} \mathsf{a}(i)e_i\right) + e_{n+1}, \text{ for } n \geqslant 1.$$

We note that $\eta = \rho e$ is a orthonormal basis for U_F under $\langle , \rangle$. Let $\delta = \rho\mathsf{d}$, $S = \mathbf{L}(\delta)$. We claim:

(1) δ is a recursive repère,
(2) $\delta \cup \{e_0\}$ is a recursive basis for U_F,
(3) S is a recursive space,

(4) for all $0 \leqslant k \neq n$, $\langle \mathsf{d}(k), \mathsf{d}(n)\rangle = 0$,
(5) $S^{\perp} = \{0\}$.

The first three are straightforward.

Re (*4*). The case when $k = 0$ is immediate. Suppose $k \geqslant 1$. Then

$$\mathsf{d}(k) = e_0 - \left(\sum_{i=1}^{k} \mathsf{a}(i)e_i\right) + e_{k+1}$$
$$\mathsf{d}(n) = e_0 - \left(\sum_{i=1}^{k} \mathsf{a}(i)e_i\right) - \mathsf{a}(k+1)e_{k+1} - \left(\sum_{i=k+2}^{n} \mathsf{a}(i)e_i\right) + e_{n+1}.$$

We have $\langle \mathsf{d}(k), \mathsf{d}(n)\rangle = 1 + \sum_{i=1}^{k} [\mathsf{a}(i)]^2 - \mathsf{a}(k+1) = 0$ by L1.

Re (*5*). Let $x \in S^{\perp}$. Then $x \in U_F$ implies that there exists a p such that $x \in \mathbf{L}(e_0, e_1, \ldots, e_p)$. Note that $d_0, d_1, \ldots, d_p - e_{p+1}$ are $p+1$ mutually orthogonal elements in $\mathbf{L}(e_0, \ldots, e_p)$, and hence by P6 form a basis for $\mathbf{L}(e_0, \ldots, e_p)$. Furthermore, $\langle x, \mathsf{d}(i)\rangle = 0$ for all $i \geqslant 0$ and $e_{p+1} \notin \mathsf{supp}(x)$ imply $\langle x, d_p - e_{p+1}\rangle = 0$. Thus $x = 0$ by P7 (b). Q.E.D.

We will modify the proof above several times in what follows.

Proposition P11 *There exist recursive spaces* S_1 *and* S_2 *such that* $(S_1 \cap S_2)^{\perp} \not\leq S_1^{\perp} + S_2^{\perp}$, *and hence the inequality in* P4 (ii) *cannot be strengthened to equality even for recursive spaces.*

Proof: Define recursive functions b and d similar to the definition of d in the proof of P10 as follows:

$$\mathsf{b}(0) = e_0 + e_1, \qquad \mathsf{d}(0) = e_0 + e_1,$$
$$\mathsf{b}(1) = e_0 - e_1 + e_2, \qquad \mathsf{d}(1) = e_0 - e_1 + e_3,$$

and for $n \geqslant 2$:

$$\mathsf{b}(n) = e_0 - e_1 - \left(\sum_{i=1}^{n-1} \mathsf{a}(i+1)e_{2i}\right) + e_{2n},$$
$$\mathsf{d}(n) = e_0 - e_1 - \left(\sum_{i=1}^{n-1} \mathsf{a}(i+1)e_{2i+1}\right) + e_{2n+1}.$$

Let $\beta = \rho\mathsf{b}$, $\delta = \rho\mathsf{d}$, $S_1 = \mathbf{L}(\delta)$, $S_2 = \mathbf{L}(\beta)$. The proofs of the following claims are left to the reader.

(1) $\mathbf{L}(e_0, e_2, e_4, \ldots)$ and $\mathbf{L}(e_0, e_3, e_5, \ldots)$ are r.e. complementary spaces for S_1 and S_2 respectively,
(2) β and δ are infinite r.e. repères, hence S_1 and S_2 are recursive spaces,
(3) $S_1 \cap S_2 = \mathbf{L}(e_0 + e_1)$,
(4) $S_1^{\perp} = \mathbf{L}(e_2, e_4, e_6, \ldots)$, $S_2^{\perp} = \mathbf{L}(e_3, e_5, e_7, \ldots)$,
(5) $(S_1 \cap S_2)^{\perp} = \mathbf{L}(e_0 - e_1, e_2, e_3, e_4, \ldots)$,
(6) $(S_1^{\perp} + S_2^{\perp}) = \mathbf{L}(e_2, e_3, e_4, e_5, \ldots)$.

Clearly (5) and (6) give us the desired conclusion. Q.E.D.

5 *Orthogonal complements* In light of P10 (c), we denote by **O.C.** the family of all subspaces W of U_F such that $W \oplus W^{\perp} = U_F$.

Proposition P12 Card(**O.C.**) $\geqslant \mathfrak{c}$.

Proof: Let $\sigma \subseteq \varepsilon$. Then $\mathbf{L}(e(\sigma)) \in$ **O.C.** Q.E.D.

We show below that even in **O.C.** the theory is not smooth by showing that there exist recursive spaces in **O.C.** whose orthogonal complements are not r.e.

Definition D2: For $S \subseteq U_F$ and $x \in U_F$, we say that *x is orthogonal to S*, denoted $\langle x, S\rangle = 0$, if $x \in S^{\perp}$.

Proposition P13 *Let $W \leqslant U_F$ and let $W = \mathbf{L}(\beta)$, then $\langle x, W\rangle = 0$ if and only if $\langle x, \beta\rangle = 0$.*

Proof: Linear algebra. Q.E.D.

Proposition P14 *Let $W \in$ **O.C.** be* r.e. *Then $W^{\perp}$ is* r.e. *if and only if for each $x \in U_F$ we can effectively test $\langle x, W\rangle = 0$.*

Proof: If $W^{\perp}$ is r.e., then W is recursive. Given $x \in U_F$, we can effectively express x as $w + \overline{w}$ where $w \in W$ and $\overline{w} \in W^{\perp}$. Then $\langle x, W\rangle = 0 \Leftrightarrow w = 0$. Conversely, if we can effectively test for each $x \in U_F$ whether or not $\langle x, W\rangle = 0$, then clearly $W^{\perp}$ is r.e. Q.E.D.

Definition D3: For $x \in U_F - \{0\}$, we define

(i) $\mathbf{z}(x)$ as the element of least index (w.r.t. the function e) in $\mathrm{supp}(x)$,
(ii) $\mathbf{t}(x)$ as the index of $\mathbf{z}(x)$,
(iii) $\mathbf{m}(x)$ as the element of largest index in $\mathrm{supp}(x)$,
(iv) $\mathbf{u}(x)$ as the index of $\mathbf{m}(x)$.

Clearly, $\mathbf{z}(x), \mathbf{t}(x), \mathbf{m}(x)$, and $\mathbf{u}(x)$ are partial recursive functions of x with domains $\varepsilon - \{0\}$. If $S \subseteq U_F$, we let $\mathbf{m}(S)$ denote the set $\{\mathbf{m}(x) \mid x \in S - \{0\}\}$ and similarly for $\mathbf{z}(S)$. We note the following properties of the functions **m** and **z**:

(a) Let W be a r.e. space. Then $\mathbf{m}(W)$ is an r.e. set and W is recursive if and only if $\mathbf{m}(W)$ is a recursive space [3], P1. 14.
(b) Let W be any space and β any basis for W. If **m** is 1-1 on β, then $\mathbf{m}(\beta) = \mathbf{m}(W)$ [3], P1. 15.
(c) Every space has a basis on which the function **m** is 1-1.
(d) Every r.e. space has a r.e. basis on which the function **m** is 1-1 [3], P1. 17.
(e) Let W be any space and β any basis of W. If **z** is 1-1 on β, then $\mathbf{z}(\beta) = \mathbf{z}(W)$ [3], P1. 26.

Proposition P15 *There exists a recursive space $W \in$ **O.C.** such that $W^{\perp}$ is not* r.e.

Proof: Let f be a 1-1 recursive function ranging over a non-recursive subset of $\{2, 4, 6, 8, \dots\}$. Let $g(n) = 1 + \sum_{i=0}^{n} f(i)$. Note that g is a 1-1 strictly increasing function which is recursive and $\rho g \subseteq \{1, 3, 5, 7, \dots\}$.

Furthermore, for all n, $f(n) < g(n)$, and $e \circ f$ and $e \circ g$ are 1-1 recursive functions, the latter strictly increasing. Hence $\rho(e \circ f)$ and $\rho(e \circ g)$ are r.e. and recursive respectively. Define $\mathrm{c}(n) = e_{f(n)} + e_{g(n)}$, $\mathrm{d}(n) = e_{f(n)} - e_{g(n)}$, $\gamma = \rho\mathrm{c}(n)$, $W = \mathbf{L}(\gamma)$, $\delta = \rho\mathrm{d}(n)$, $V = \mathbf{L}(\delta)$.

Note that $W \oplus V \oplus \mathbf{L}(\eta - (\rho(e \circ f) \cup \rho(e \circ g))) = U_F$ and $W^\perp = V \oplus \mathbf{L}(\eta - (\rho(e \circ f) \cup \rho(e \circ g)))$. Since $\mathbf{m}(\mathrm{c}(n)) = \mathbf{m}(\mathrm{d}(n)) = e(g(n))$ is a 1-1 strictly increasing recursive function of n, γ, and δ are bases for the recursive spaces W and V respectively. If $W^\perp$ were r.e., we could effectively test $\langle e_{2n}, W\rangle$ for each n and thus $\rho(e \circ f)$ would be recursive. Q.E.D.

6 *Decidable spaces* The r.e. space W is said to be *decidable* if the set $U_F - W$ is r.e. Guhl [4] has shown that if F is infinite, then there are decidable spaces which are not recursive. In light of our previous examples we ask the following two questions:

(i) If W is r.e. and $W^\perp$ is r.e., is $W \oplus W^\perp$ decidable?
(ii) If W is r.e. and for each $x \in U_F$ we can effectively test $\langle x, W\rangle = 0$, is $W \oplus W^\perp$ decidable?

It is clear that a positive answer to (i) implies a positive answer to (ii). Proposition P18 below gives a negative answer to (ii).

Proposition P16 *Suppose $W \oplus W^\perp \neq U_F$ and $x \in U_F - (W \oplus W^\perp)$. Let β be an orthogonal basis for W where $\beta = \rho\mathrm{b}$, a 1-1 function. Suppose*

$$x = \sum_{j=0}^{k} \alpha_j e_{i_j}, \text{ where wolog we assume that } \langle e_{i_j}, W\rangle \neq 0 \text{ for } 0 \leq j \leq k.$$

Then:

(a) $\langle x, b_n\rangle \neq 0$ *for infinitely many n,*
(b) *for at least one $j (0 \leq j \leq k)$, $\langle e_{i_j}, b_n\rangle \neq 0$ for infinitely many n.*

Proof: Clearly (a) $\Rightarrow$ (b). Now suppose (a) is false, say $\langle x, b_p\rangle = 0$ *for all but* $p = j_1, \ldots, j_s$. Let $u = x - z$ where

$$z = \frac{\langle x, b_{j_i}\rangle}{\langle x, b_{j_i}\rangle} b_{j_i} + \ldots + \frac{\langle x, b_{j_s}\rangle}{\langle x, b_{j_s}\rangle} b_{j_s}.$$

Then $x = u + z$, $z \in W$, $u \in W^\perp$ and thus $x \in W \oplus W^\perp$, contrary to the choice of x. Q.E.D.

Proposition P17 *Let $W \neq U_F$ and β be an orthogonal basis for W where $\beta = \rho\mathrm{b}$, a 1-1 function. Let $e_n \in \eta$. Then $\langle e_n, b_k\rangle \neq 0$ for infinitely many k if and only if $e_n \in U_F - (W \oplus W^\perp)$.*

Proof: The "if" part follows directly from P16. The converse will follow from: if $x \in W \oplus W^\perp$, then $\langle x, b_k\rangle \neq 0$ for at most finitely many k. Suppose $x = u + v$ where $u \in W$, $v \in W^\perp$. Then $\langle x, b_k\rangle = \langle u, b_k\rangle$. If $u = \alpha_1 b_{i_1} + \ldots + \alpha_n b_{i_n}$, then $\langle u, b_k\rangle \neq 0$ if and only if $k \in \{i_1, \ldots, i_n\}$. Q.E.D.

Proposition P18 *There exists a* r.e. *space W such that*

(i) *for all* $x \in U_F$, *we can effectively test* $\langle x, W\rangle = 0$,
(ii) $U_F - (W \oplus W^{\perp})$ *is not* r.e.

Proof: Let a be the function defined in L1. Let p be the function which enumerates the primes in order, i.e., $p(0) = 2$, $p(n) = n$th odd prime. It is well known that p is 1-1, strictly increasing and recursive; let $\tau = \rho p$. For each n, define

$$P_n = \{p^k(n) \mid k \geqslant 1\} \text{ and } P_n^* = \{e(x) \mid x \in P_n\}$$

Let $\Gamma = \varepsilon - \left(\bigcup_{n \in \varepsilon} P_n\right)$, $\Gamma^* = \{e(x) \mid x \in \Gamma\} = \eta - \left(\bigcup_{n \in \varepsilon} P_n^*\right)$. Let t be the principal function of Γ; note that t is 1-1, strictly increasing and recursive. Define the 1-1 recursive function $\mathbf{d}(m,n)$ of two variables as follows:

$$\mathbf{d}(0,0) = e_0 + e_1 = e_{t(0)} + e_{t(1)}$$
$$\mathbf{d}(0,n) = e_0 - \left(\sum_{i=1}^{n} a(i)e_{t(i)}\right) + e_{t(n+1)}, \text{ for } n \geqslant 1,$$

for $m \geqslant 1$, we proceed as follows:

$$\mathbf{d}(m,0) = e_{p(m-1)} + e_{(p(m-1))^2},$$
$$\mathbf{d}(m,n) = e_{p(m-1)} - \left(\sum_{i=1}^{n} a(i)e_{(p(m-1))^{i+1}}\right) + e_{(p(m-1))^{n+2}}, \text{ for } n \geqslant 1.$$

For a fixed m, let $Q_m = \rho\mathbf{d}(m,n)$. We note the following four facts:

(i) for all m, n, if $m \neq n$, then $\operatorname{supp}(Q_m) \cap \operatorname{supp}(Q_n) = \emptyset$,
(ii) $\operatorname{supp}(Q_0) \subseteq \Gamma^*$, and if $m \geqslant 1$, then $\operatorname{supp}(Q_m) \subseteq P_{m-1}^*$,
(iii) $\eta \subseteq \bigcup_{m \in \varepsilon} \operatorname{supp}(Q_m)$,
(iv) $\rho\mathbf{d}$ is an orthogonal repère.

Now let f be a 1-1 recursive function ranging over a non-recursive subset α of τ. Let $\alpha' = \tau - \alpha$; thus α' is not r.e. The goal of the following construction is to modify the definition of $\mathbf{d}(m,n)$ above in such a way that the resulting orthogonal repère spans W and $e(\tau) \cap (U_F - (W \oplus W^{\perp}))$ is $e(\alpha')$. We define two 1-1 functions $\bar{\mathbf{d}}$ and $\mathbf{c}$ such that $W = \mathbf{L}(\rho\bar{\mathbf{d}})$ and $W^{\perp} = \mathbf{L}(\rho\mathbf{c})$. $\bar{\mathbf{d}}$ will be very similar to $\mathbf{d}$; the only change is if $f(k) = p(m-1)$, then we define

$$\bar{\mathbf{d}}(m,n) = e_{(p(m-1))^{n+2}}, \text{ for all } n \geqslant k.$$

Otherwise, $\bar{\mathbf{d}}(m,n) = \mathbf{d}(m,n)$. Note that $\bar{\mathbf{d}}$ is recursive: to compute $\bar{\mathbf{d}}(m,n)$, first compute $f(0), \ldots, f(n)$. If none of these is $p(m-1)$, then $\bar{\mathbf{d}}(m,n) = \mathbf{d}(m,n)$. If $f(k) = p(m-1)$ for some $0 \leqslant k \leqslant n$, then $\bar{\mathbf{d}}(m,n) = e_{(p(m-1))^{n+2}}$. We define $\mathbf{c}(0) = e_{f(0)}$. For $k \geqslant 1$, if $f(k) = p(m-1)$, we define

$$\mathbf{c}(k) = e_{p(m-1)} - \left(\sum_{i=1}^{k} a(i)e_{(p(m-1))^{i+1}}\right).$$

We note that $\mathbf{c}$ is also recursive. As an example, suppose $f(4) = p(1) = 3$. Then

$$\bar{\mathbf{d}}(4,0) = e_3 + e_9$$
$$\bar{\mathbf{d}}(4,1) = e_3 - e_9 + e_{27}$$

$$\begin{aligned}
\bar{\mathbf{d}}(4,2) &= e_3 - e_9 - 2e_{27} + e_{81}\\
\bar{\mathbf{d}}(4,3) &= e_3 - e_9 - 2e_{27} - 6e_{81} + e_{243}\\
\bar{\mathbf{d}}(4,4) &= e_{729}\\
\bar{\mathbf{d}}(4,n) &= e_{3^{n+2}}, \text{ for } n \geqslant 4\\
\mathbf{c}(4) &= e_3 - e_9 - 2e_{27} - 6e_{81} - 42e_{243}.
\end{aligned}$$

Note that $\mathbf{L}(\bar{\mathbf{d}}(4,0), \ldots, \bar{\mathbf{d}}(4,3), \mathbf{c}(4)) = \mathbf{L}(e_3, e_9, e_{27}, e_{81}, e_{243})$. For fixed m define $\bar{Q}_m = \rho\bar{\mathbf{d}}(m,n)$. We note that facts (i)-(iv) are true when Q_m is replaced with $\bar{Q}_m$. Let $\delta = \bigcup_{m \in \varepsilon} \bar{Q}_m$, $\gamma = \rho\mathbf{c}$. Then $\gamma \cup \delta$ is a repère since γ is orthogonal and $\langle \mathbf{c}(k), \delta \rangle = 0$ for all k. Since γ and δ are each r.e., $W = \mathbf{L}(\delta)$ and $S = \mathbf{L}(\gamma)$ are each infinite dimensional r.e. spaces. We claim:

(1) *for each* $x \in U_F$, *we can effectively test* $\langle x, W \rangle = 0$,
(2) $W^{\perp} = S$,
(3) $W \oplus W^{\perp}$ *is not decidable.*

Re (1). Let $x \in U_F$. Then $x = \alpha_1 e_{i_1} + \ldots + \alpha_p e_{i_p}$. By looking at $e_{i_1}, \ldots, e_{i_p}$ we can effectively decompose x uniquely into a finite number of pieces

$$x = x_{j_1} + x_{j_2} + \ldots + x_{j_q}$$

such that $\emptyset \not\subseteq \operatorname{supp} x_{j_k} \not\subseteq \operatorname{supp}(\bar{Q}_{j_k})$. Note that for all j_k, n such that $j_k \neq n$, $\langle x_{j_k}, \bar{Q}_n \rangle = 0$. Then $\langle x, W \rangle = 0$ if and only if $\langle x, \delta \rangle = 0$ if and only if $\langle x_{j_k}, \bar{Q}_{j_k} \rangle = 0$, for $k = 1, 2, \ldots, q$. Each of these last q conditions can be effectively tested as follows:

Case 1. $j_k = 0$. Let $e_{t(i)}$ be the element of maximum index in $\operatorname{supp}(x_{j_k})$. Compute $\bar{\mathbf{d}}(0,0), \ldots, \bar{\mathbf{d}}(0,i)$. Then $(x_{j_k}, \bar{Q}_{j_k}) = 0$ if and only if $\langle x_{j_k}, \bar{\mathbf{d}}(0,0) \rangle = 0$ and . . . and $\langle x_{j_k}, \bar{\mathbf{d}}(0,i) \rangle = 0$. By the same reasoning as in the proof of P10 (5), this happens if and only if $x_{j_k} = 0$.

Case 2. $j_k = s > 0$. Let $e_{(p(s-1))^r}$ be the element of $\operatorname{supp}(x_{j_k})$ of largest index. Compute $\bar{\mathbf{d}}(s,0), \ldots, \bar{\mathbf{d}}(s, r-2), \bar{\mathbf{d}}(s, r-1)$.

Subcase 2.1. $\operatorname{Card}(\operatorname{supp}(\bar{\mathbf{d}}(s, r-2))) = 1$. Then $\bar{\mathbf{d}}(s, r-2) = e_{(p(s-1))^r}$ and $\langle x_{j_k}, \bar{\mathbf{d}}(s, r-2) \rangle \neq 0$.

Subcase 2.2. $\operatorname{Card}(\operatorname{supp}(\bar{\mathbf{d}}(s, r-2))) > 1$ and $\operatorname{card}(\operatorname{supp}(\bar{\mathbf{d}}(s, r-1))) = 1$. Then $x_{j_k} \in \mathbf{L}(e_{p(s-1)}, \ldots, e_{(p(s-1))^{r-1}}, e_{(p(s-1))^r})$, i.e., $x_{j_k} \in \mathbf{L}(\bar{\mathbf{d}}(s, 0), \ldots, \bar{\mathbf{d}}(s, r-2), \mathbf{c}(r-1)$. Then $\langle x_{j_k}, \bar{Q}_{j_k} \rangle = 0$ if and only if $x_{j_k} \in \mathbf{L}(\mathbf{c}(r-1))$ and this can be effectively tested.

Subcase 2.3. $\operatorname{Card}(\operatorname{supp}(\bar{\mathbf{d}}(s, r-2))) > 1$ and $\operatorname{card}(\operatorname{supp}(\bar{\mathbf{d}}(s, r-1))) > 1$. Then $x_{j_k} \in \mathbf{L}(\bar{\mathbf{d}}(s, 0), \ldots, \bar{\mathbf{d}}(s, r-2), \bar{\mathbf{d}}(s, r-1) - e_{(p(s-1))^{r+1}})$ and, as in Case 1, $\langle x_{j_k}, \bar{Q}_{j_k} \rangle = 0$ if and only if $x_{j_k} = 0$.

Re (2). If $\langle x, W \rangle = 0$, then the proof of (1) implies that $x \in S$, and thus $W^{\perp} \leqslant S$. Conversely, $\langle \mathbf{c}(k), \delta \rangle = 0$ for all k implies that $S \leqslant W^{\perp}$.

Re (3). Let p be a prime. By P17, $e_p \in U_F - (W \oplus W^{\perp})$ if and only if $\langle e_p, \bar{\mathbf{d}}(m,n) \rangle \neq 0$ for infinitely many pairs (m,n). Suppose $p = p(s-1)$. Then by construction, $e_p \in U_F - (W \oplus W^{\perp})$ if and only if $\langle e_p, \bar{\mathbf{d}}(s,n) \rangle \neq 0$ for infinitely many n. Again by construction, this can happen if and only if $p \in \alpha'$. Thus $e(\tau) \cap (U_F - (W \oplus W')) = e(\alpha')$. If $W \oplus W'$ were decidable, $e(\alpha')$ (and hence α') would be r.e., a contradiction to the choice of the function f. Q.E.D.

The reader can easily show that if $Z = \mathbf{L}(\rho\mathbf{d})$ as defined in the beginning of the previous proof, then Z is an infinite dimensional recursive space with infinite codimension and $Z^\perp = (0)$. The space W constructed in the previous proof is also an infinite dimensional recursive space with infinite codimension. In both cases, $T = \mathbf{L}(\{e_j \mid j = 0 \text{ or } j \in \tau\})$ is an r.e. complementary space. We summarize this in the following.

Proposition P19 *There exist three infinite dimensional* r.e. *spaces* W, Z, T *such that*

(i) $Z \oplus T = W \oplus T = U_F$,
(ii) $Z^\perp = (0)$ *so* $Z \oplus Z^\perp$ *is recursive,*
(iii) $W \oplus W^\perp$ *is not decidable.*

Proof: P19 and previous remarks. Q.E.D.

REFERENCES

[1] Dekker, J. C. E., "Countable vector spaces with recursive operations, Part 1," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 363-387.

[2] Hoffman, K., and R. Kunze, *Linear Algebra*, Prentice Hall, Inc., Englewood Cliffs, New Jersey (1961).

[3] Guhl, R., "Two types of recursively enumerable vector spaces," doctoral dissertation, Rutgers University, New Brunswick, New Jersey (1973), 77 pp.

[4] Guhl, R., "A theorem on recursively enumerable vector spaces," *Notre Dame Journal of Formal Logic*, vol. XVI (1975), pp. 357-362.

[5] Guhl, R., "Two notes on recursively enumerable vector spaces," *Notre Dame Journal of Formal Logic*, vol. XVIII (1977), pp. 295-298.

[6] Lang, S. *Linear Algebra*, Second edition, Addison-Wesley Publishing Co., Reading, Massachusetts (1972).

[7] Metakides, G., and A. Nerode, "Recursively enumerable vector spaces," *Annals of Mathematical Logic*, vol. 11 (1977), pp. 147-171.

[8] van der Waerden, B. L., *Algebra*, vol. 1, Frederick Ungar Publishing Co., New York (1970).

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