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THE IDEAL OF ORDERABLE SUBSETS OF A SET

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The Ordering Principle (**OP**),* which states that every set can be linearly ordered, is not provable in Zermelo-Fraenkel (**ZF**) set theory. Let $\mathrm{O}(S)$ be the set of all orderable subsets of a set S: obviously $\mathrm{O}(S)$ will be of intrinsic interest only when S cannot be ordered, that is, only when $\mathrm{O}(S) \neq P(S)$, the power-set of S. If X, $Y \in \mathrm{O}(S)$, then $X \cup Y \in \mathrm{O}(S)$; and if $X \in \mathrm{O}(S)$ and $Y \subseteq X$, then $Y \in \mathrm{O}(S)$. Thus $\mathrm{O}(S)$ is an ideal of $\mathrm{P}(S)$ (regarded as the usual algebra), and clearly $\mathrm{O}(S) \supseteq \mathrm{P}°(S)$, where $\mathrm{P}°(S)$ is the set (ideal) of all finite subsets of S. We therefore have two extreme possibilities (when S is non-orderable): (i) the lower extreme, when $\mathrm{O}(S) = \mathrm{P}°(S)$; and (ii) the upper extreme, when $\mathrm{O}(S)$ is a maximal ideal. Both these extremes, as well as varying positions in between, can be attained.

Definition 1

(i) A set S is called "finite" if there is a bijection $f: n \rightarrow S$ for some natural number n.
(ii) A set S is called "Dedekind-finite" if there is no $T \subseteq S$ with $T \neq S$ for which there is a bijection $f: S \rightarrow T$.
(iii) A set S is called "medial" if S is infinite and Dedekind-finite.
(iv) A set S is called "quasi-minimal" if for every $X \subseteq S$, exactly one of X, $S - X$ is finite.

In (i) above, we are regarding natural numbers (and ordinals in general) to be defined in such a way that each is the set of all smaller natural numbers (ordinals). It is easily seen that every quasi-minimal (**qm**) set is medial, and that a set S is Dedekind-finite if and only if there is no injection $f: \omega \rightarrow S$. Concerning the existence of **qm** sets, we refer to §1 of [1].

Lemma 1 *If S is quasi-minimal, then* $\mathrm{P}(S)$ *is Dedekind-finite.*

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Proof: Suppose that there is an injection $f\colon \omega \to \mathrm{P}(S)$; then, in view of Definition 1 (iv), there is an injection $f_0\colon \omega \to \mathrm{P}^\circ(S)$. Since each $f_0(n)$ is finite, there exists a function $g\colon \omega \to \mathrm{P}^\circ(S)$ such that for each n we have $g(n) \subseteq g(n+1)$ and $g(n) \neq g(n+1)$. Put $X = \bigcup\{g(2n+1) - g(2n);\ n < \omega\}$, and $Y = \bigcup\{g(2n+2) - g(2n+1);\ n < \omega\}$. Then $X, Y \subseteq S$, $X \cap Y = \emptyset$, and X, Y are both infinite. This is a contradiction.

Lemma 2 *If S is quasi-minimal, then S cannot be ordered.*

Proof: Suppose that S is ordered by $<$, and take $x_0 \in S$ such that the set $\{y \in S\colon x_0 < y\}$ is infinite. Define the sequence (x_n) recursively by setting x_{n+1} equal to the immediate successor of x_n if such exists, and setting $x_{n+1} = x_n$ otherwise. Thus we have $x_n \leqslant x_{n+1}$ for each n. If we had $x_n < x_{n+1}$ for each n, then there would exist an injection $f\colon \omega \to S$, a contradiction. Thus we must have $x_n = x_{n+1}$ for some n, and so we have found $x \in S$ having no immediate successor and such that $\{y \in S\colon x < y\}$ is infinite. Similarly, we can find $z \in S$ having no immediate predecessor and such that $\{y \in S;\ z > y\}$ is infinite. It follows from Definition 1 (iv) that $x < z$. Clearly there must exist $y \in S$ with $x < y < z$. Define X, Z by $X = \{u \in S\colon x < u < y\}$ and $Z = \{u \in S;\ y < u < z\}$. Then $X, Z \subseteq S$, $X \cap Z = \emptyset$, and X, Z are both infinite, a contradiction.

An alternate proof of Lemma 2 was discovered and presented by G. Monro in his thesis (unpublished).

Theorem 1 *Let S be a set. Then S is quasi-minimal if and only if*

(1) $\mathrm{O}(S) = \mathrm{P}^\circ(S)$;
(2) $\mathrm{O}(S)$ *is a (proper) maximal ideal of* $\mathrm{P}(S)$.

Proof: Let S be **qm** and take $X \subseteq S$. By Lemma 2, X is orderable if and only if X is finite, which establishes (1), and from this and Definition 1 (iv) we obtain $X \in \mathrm{O}(S) \Leftrightarrow S - X \notin \mathrm{O}(S)$, establishing (2). Conversely, if (1) and (2) hold, then for each $X \subseteq S$, exactly one of X, $S - X$ is finite, and so S is **qm**.

We have thus obtained an easy illustration of both extremes being achieved. In view, however, of the fact that quasi-minimal sets have no "middle-order" subsets, it could be argued that Theorem 1 represents a rather pathological case, and that it would therefore be interesting to provide examples of one or other of the extremes being attained, while at the same time keeping as far away as possible from **qm** sets.

We propose to provide such examples, using the Fraenkel-Mostowski method of permutation models. In order to translate our results from **FM** set theory to the more usual **ZF** set theory, all we need do is to perform a routine application of the Jech-Sochor Embedding Theorem. Therefore we assume familiarity with the description of **FM** set theory and the construction of permutation models as given in Chapter 4 of [2], as well as with the statement of the Jech-Sochor Embedding Theorem as presented in Chapter 6 of [2]. Our first example is drawn straight from §4.4 of [2].

Theorem 2 *It is relatively consistent with* **ZF** *to assume the existence of* *an infinite set S such that $\mathsf{O}(S) = \mathsf{P}^\circ(S)$ and such that no subset of S is quasi-minimal.*

Proof: Let $\mathfrak{M}$ be a model of **FM** + **AC** containing a countably infinite set A of atoms (see Problem 1 of §4.6 of [2]). We list A as $\{a_0, b_0, a_1, b_1, \ldots\}$, and for each $n \geqslant 0$ we put $B_n = \{a_n, b_n\}$. We let I be the ideal of all finite subsets of A, and let $\mathfrak{G}$ be the group of all permutations $g: A \to A$ such that $g''B_n = B_n$ for each n. These constructions are of course carried out in $\mathfrak{M}$.

Clearly I is normal with respect to $\mathfrak{G}$, and so $\mathfrak{G}$ and I together define a permutation model $\mathfrak{N}$ as described in Chapter 4 of [2]. Putting $\mathsf{sym}(x) = \{g \in G; g(x) = x\}$ and $\mathsf{fix}(x) = \{g \in G; \forall y \in x(g(y) = y)\}$ for each $x \in M$, we see that the criterion for membership in $\mathfrak{N}$ is as follows:

(#) *Take $x \in M$. Then we have $x \in N$ if and only if $x \subseteq N$ and $\mathsf{fix}(E) \subseteq \mathsf{sym}(x)$ for some $E \in I$.*

Now $A \in N$; in fact, A is the set of atoms of $\mathfrak{N}$. We wish to show that $\mathfrak{N} \models \mathsf{O}(A) = \mathsf{P}^\circ(A)$. Thus take $X \in N$ such that X is an infinite subset of A, and suppose that $\mathfrak{N} \models$"X is ordered by $R \subseteq X \times X$". By (#) we know that $\mathsf{fix}(E) \subseteq \mathsf{sym}(R)$ for some $E \in I$. Moreover, we also know that $\mathsf{fix}(D) \subseteq \mathsf{sym}(X)$ for some $D \in I$; put $C = D \cup E$. Now X is by assumption infinite relative to $\mathfrak{N}$, and hence must also be infinite relative to $\mathfrak{M}$. On the other hand, C is finite. Therefore there must exist $n \geqslant 0$ such that $C \cap B_n = \emptyset$ and $X \cap B_n \neq \emptyset$. Without loss of generality we may assume $a_n \in X$. Let $g \in G$ be such that $g(a_n) = b_n$, $g(b_n) = a_n$, and $g(a) = a$ otherwise. Thus $g \in \mathsf{fix}(C) \subseteq \mathsf{fix}(D) \subseteq \mathsf{sym}(X)$, and so $g(X) = X$. But $a_n \in X$, and so $b_n = g(a_n) \in g(X) = X$. We have thus shown that $B_n \subseteq X$. Now R orders X relative to $\mathfrak{N}$, and so R certainly orders X relative to $\mathfrak{M}$. Since $B_n \subseteq X$, we must have either $(a_n, b_n) \in R$ or $(b_n, a_n) \in R$; we may assume the former.

Since $g \in \mathsf{fix}(C) \subseteq \mathsf{fix}(E) \subseteq \mathsf{sym}(R)$, we have $g(R) = R$, from which we obtain $(b_n, a_n) = g((a_n, b_n)) \in g(R) = R$, a contradiction. Thus $\mathfrak{N} \models X \notin \mathsf{O}(A)$, from which $\mathfrak{N} \models \mathsf{O}(A) = \mathsf{P}^\circ(A)$ follows easily.

Our next task is to show that $\mathfrak{N} \models$"A has no **qm** subsets". Now clearly $B_n \subseteq N$ for any given n, and since we have $B_n \in I$ and $\mathsf{fix}(B_n) \subseteq \mathsf{sym}(B_n)$, we see that in fact $B_n \in N$. Now consider in $\mathfrak{M}$ the set $f = \{(n, B_n); n < \omega\}$. Since $B_n \in N$ for each n, it follows that $f \subseteq N$. Moreover, for any $g \in G$ we have

$$g(f) = \{g((n, B_n)); n < \omega\} = \{(g(n), g(B_n)): n < \omega\}$$
$$= \{(n, B_n): n < \omega\} = f.$$

Thus $\mathsf{fix}(\emptyset) \subseteq \mathsf{sym}(f)$, and so $f \in N$. Therefore, within $\mathfrak{N}$ we have a function $f: \omega \to \mathsf{P}(A)$ such that (i) $|f(n)\models 2$ for each n; (ii) $f(m) \cap f(n) = \emptyset$ for $m \neq n$; (iii) $A = \bigcup\{f(n); n < \omega\}$. Keeping within $\mathfrak{N}$, take any $X \subseteq A$. If X is finite, then X is certainly not **qm**. On the other hand, if X is infinite, then $\Gamma = \{n < \omega; X \cap f(n) \neq \emptyset\}$ is a countably infinite set, from which it follows easily that there is an injection $f_0: \omega \to \mathsf{P}(X)$. Thus by Lemma 2 we conclude that neither is X **qm** in this case.

Hence we have shown that $\mathfrak{N} \models$"A has no **qm** subsets", and now finish off the proof of our theorem by applying the Embedding Theorem to the model $\mathfrak{N}$.

While the set referred to in Theorem 2 has no **qm** subsets, it is nevertheless Dedekind-finite. This is unavoidable. For let S be a Dedekind-infinite set. Then there exists an injection $f\colon \omega \to S$; put $T = f''\omega$. We can order T by setting $f(m) < f(n)$ just when $m < n$; thus $\mathsf{O}(S) \neq \mathsf{P}^{\circ}(S)$.

Theorem 3 *It is relatively consistent with* **ZF** *to assume the existence of a set S having no quasi-minimal subsets and such that $\mathsf{O}(S)$ is a maximal ideal in $\mathsf{P}(S)$.*

Proof: Once again we let $\mathfrak{M}$ be an **FMC**-model containing a countably infinite set A of atoms. This time we endow A with a linear order $<$ under which A is dense and without endpoints; thus A is order-isomorphic to the rationals. We partition A into an $\omega^* + \omega$-sequence of open intervals J_i such that

$$A = \ldots \dot{\cup}\, J_{-1} \,\dot{\cup}\, J_0 \,\dot{\cup}\, J_1 \,\dot{\cup} \ldots, \text{ where "}\dot{\cup}\text{" denotes ordered union.}$$

Thinking of A as the set of rationals we could for example define J_i to be the set $\{a \in A\colon \sqrt{2} + i < a < \sqrt{2} + i + 1\}$.

As before we let I be the ideal of all finite subsets of A. This time, however, we define $\mathfrak{G}$ as follows. Let $g\colon A \to A$ be a permutation, and put $g \in G$ if and only if

(1) For each i there exists j such that $g''J_i = J_j$;
(2) For each i and for all $a, b \in J_i$, $a < b \Rightarrow g(a) < g(b)$.

Again I is normal with respect to $\mathfrak{G}$, and so we arrive at a permutation model $\mathfrak{N}$. We claim that $\mathfrak{N} \models$ "$\mathsf{O}(A)$ is a maximal ideal in $\mathsf{P}(A)$", that is, $\mathfrak{N} \models \forall X(X \subseteq A \Rightarrow (X \in \mathsf{O}(A) \Leftrightarrow A - X \notin \mathsf{O}(A)))$.

Within $\mathfrak{M}$, put $K_n = J_{-n} \,\dot{\cup} \ldots \dot{\cup}\, J_n$ for each $n \geq 0$. The above claim will be established if we show that

(i) For $X \in N$, $\mathfrak{N} \models X \in \mathsf{O}(A)$ if and only if $\mathfrak{M} \models X \subseteq K_n$ for some n;
(ii) For $X \in N$ with $X \subseteq A$, $\mathfrak{M} \models \exists n \geq 0 (X \subseteq K_n \Leftrightarrow A - X \not\subseteq K_n)$.

Take $X \in N$ with $X \subseteq A$. Then for some $E \in I$ we must have $\mathsf{fix}(E) \subseteq \mathsf{sym}(X)$. Since E is a finite subset of A, it follows that (within $\mathfrak{M}$) $E \subseteq K_n$ for some $n \geq 0$. Suppose that we do not have $\mathfrak{M} \models X \subseteq K_n$. Thus there exists $x \in X$ such that $x \in J_i$ for some i with $|i| > n$. Take any $y \in A - K_n$ and let k be the unique integer such that $y \in J_k$; of course $|k| > n$. There exists $g \in G$ such that $g''J_i = J_k$, $g''J_k = J_i$, $g(x) = y$, and $g(a) = a$ for all $a \in A - (J_i \cup J_k)$. Then we have $y = g(x) \in g(X)$. But $g \in \mathsf{fix}(K_n) \subseteq \mathsf{fix}(E) \subseteq \mathsf{sym}(X)$, and so $g(X) = X$. Thus $y \in X$. We have therefore shown that within $\mathfrak{M}''$, if $X \not\subseteq K_n$, then $X \supseteq A - K_n$, that is, $A - X \subseteq K_n$. The converse, $X \subseteq K_n \Rightarrow A - X \not\subseteq K_n$, is obvious. This proves (ii).

Take $X \in N$ with $X \subseteq A$, and suppose that $\mathfrak{M} \models X \subseteq K_n$ for some n. For each i with $|i| \leq n$, choose $c_i \in J_i$ and put $C = \{c_i;\ |i| \leq n\}$. It is easily seen that $\mathsf{fix}(C) \subseteq \mathsf{sym}(K_n)$, and since $K_n \subseteq N$, we must have $K_n \in N$. Now define in $\mathfrak{M}$ the set $R \subseteq K_n \times K_n$ by $R = \{(a, b) \in K_n \times K_n;\ a < b\}$. We have $R \in N$, and from the definition of $\mathfrak{G}$ we see that $\mathsf{fix}(C) \subseteq \mathsf{sym}(R)$. Hence $R \in N$, from which we deduce $\mathfrak{N} \models K_n \in \mathrm{O}(A)$. But as $\mathfrak{M} \models X \subseteq K_n$ it must be the case that $\mathfrak{N} \models X \subseteq K_n$, and so $\mathfrak{N} \models X \in \mathrm{O}(A)$.

Now suppose that there is no n for which $\mathfrak{M} \models X \subseteq K_n$. From (ii) we obtain $\mathfrak{M} \models A - X \subseteq K_m$ for some m, that is, $\mathfrak{M} \models A - K_m \subseteq X$. Suppose that there is $R \in N$ such that $\mathfrak{N} \models$ "R orders X", and let $E \in I$ be such that $\mathrm{fix}(E) \subseteq \mathrm{sym}(R)$, and choose $p > 0$ such that $E, K_m \subseteq K_p$. Now choose $x \in J_{p+1}$, $y \in J_{p+2}$; there exists $g \in G$ such that $g(x) = y$, $g(y) = x$, and $g(z) = z$ for all $z \in K_p$.

Since $A - K_m \subseteq X$ and $K_m \subseteq K_p$, it follows that $x, y \in X$. Therefore we must have either $(x, y) \in R$ or $(y, x) \in R$: we may assume the former. But then $(y, x) = g((x, y)) \in g(R)$, and $g \in \mathrm{fix}(K_p) \subseteq \mathrm{fix}(E) \subseteq \mathrm{sym}(R)$, when $g(R) = R$. This contradiction tells us that $\mathfrak{N} \models X \notin \mathrm{O}(A)$.

Thus (i) and (ii) have been demonstrated, when it follows at once that $\mathfrak{N} \models$ "$\mathrm{O}(A)$ is a maximal ideal in $\mathrm{P}(A)$". It can be shown that $\mathfrak{N} \models$ "$\mathrm{P}(A)$ is Dedekind-finite", and so we cannot use the same trick employed in the previous proof to show that $\mathfrak{N} \models$ "A has no **qm** subsets", and must resort to "first principles".

Take $X \in N$ with $X \subseteq A$: we wish to show that X is not quasi-minimal. We have seen that there exists $n \geqslant 0$ such that (within $\mathfrak{M}$) either $X \subseteq K_n$ or $X \supseteq A - K_n$. In the first case we have also seen that $\mathfrak{N} \models X \in \mathrm{O}(A)$, and so Lemma 2 tells us at once that $\mathfrak{N} \models$ "X is not **qm**". Hence we may assume that $A - K_n \subseteq X$. Take $c_1 \in J_{n+1}$, $c_2 \in J_{n+2}$, and put $C = \{c_1, c_2\}$. Then we have $\mathrm{fix}(C) \subseteq \mathrm{sym}(J_{n+i})$ for $i = 1, 2$, when $J_{n+i} \in N$. Furthermore, from $\mathfrak{M} \models A - K_n \subseteq X$ we obtain $\mathfrak{M} \models J_{n+i} \subseteq X$, and thus $\mathfrak{N} \models J_{n+i} \subseteq X$, $i = 1, 2$. But clearly $\mathfrak{N} \models J_{n+1} \cap J_{n+2} = \emptyset$ and $\mathfrak{N} \models$ "J_{n+1}, J_{n+2} are infinite". Thus $\mathfrak{N} \models$ "X is not **qm**".

We have therefore shown that $\mathfrak{N} \models$ "A has no **qm** subsets", and now a straightforward application of the Embedding Theorem completes the proof of our result.

From one point of view the proof of Theorem 3 is less satisfactory than that of Theorem 2. Although the set A in the former proof has no **qm** subsets, it can be shown that there exists $D \in N$ such that $\mathfrak{N} \models A = \bigcup D$ and $\mathfrak{N} \models$ "D is **qm**". Specifically, $D = \{J_i;\ i \text{ is an integer}\}$. We have not been able to rid ourselves of this connection with quasi-minimality; but neither have we been able to show that some such connection is necessary.

By combining the constructions used in the proofs of Theorems 2 and 3, we can obtain an **FM** model containing a set A with no **qm** subsets but in which $\mathrm{O}(A)$ attains neither of the two extremes. To obtain this model we follow the construction in the proof of Theorem 3, but in the two defining conditions of the group $\mathfrak{G}$ we replace condition (1) by (1′). For each i, $g''J_i = J_i$. When this change is made and the resulting permutation $\mathfrak{N}'$ obtained, the statement (i) still holds, but (ii) does not. Specifically, if we take $X \in N'$ with $X \subseteq A$, and if we suppose that $\mathfrak{M} \models X \subseteq K_n$ for no n, then all we can prove is that there exists m such that for every k with $|k| > m$, $\mathfrak{M} \models X \cap J_k \neq \emptyset \Rightarrow J_k \subseteq X$. We omit the details.

REFERENCES

[1] Hickman, J. L., "The construction of groups in models of set theory that fail the Axiom of Choice," *Bulletin of the Australian Mathematical Society*, vol. 14 (1976), pp. 199-232.

[2] Jech, T. J., *The axiom of choice*, North-Holland Co., Amsterdam (1972).

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