Notre Dame Journal of Formal Logic
Volume XIX, Number 4, October 1978
NDJFAM

PROVABLY RECURSIVE REAL NUMBERS

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1 *Introduction* In this paper we shall begin development of a theory of provably recursive real numbers similar in spirit to the theory of provable recursive functions discussed by Kreisel [7, 8], Fischer [4] and Ritchie and Young [12]. For example, we say that a program i names a provably recursive real number if we can prove (in some axiomatization at least as powerful as elementary number theory) that the function defined by i is total and satisfies a recursive Cauchy criterion. Of special interest will be the contrasts between provably recursive real numbers and recursive real numbers.

Neither our base theory nor our metatheory will be specified explicitly. For our base theory we let **S** be an axiomatization of any theory which encompasses elementary number theory; we require that our metatheory be powerful enough to express the soundness of **S** for arithmetic. All of our theorems will have as an implicit hypothesis that **S** is sound for arithmetic; this hypothesis would be unnecessary if **S** were an axiomatization of elementary number theory and the metatheory were full set theory because we can prove in set theory that **S** is sound for arithmetic. We take as fixed the enumeration $\phi_0, \phi_1, \phi_2, \ldots$ of partial recursive functions of one variable described in Davis [3] (which can be proven, in **S**, to be a standard enumeration). Let $K = \{i: \phi_i(i)\downarrow\}$. K is a nonrecursive, recursively enumerable set (and a proof of this can be carried out in **S**). Let rat (see Kleene and Post [6]) be a one-to-one primitive recursive function mapping the set N of natural numbers onto the set Q of rationals in lowest terms.

Our treatment will be informal: we will usually present the intuitive idea behind a proof and leave to the reader the details of carrying the proof

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*This work includes results from the author's Ph.D. thesis and was partially supported by NSF Research Contract GJ-271-27-A1 at Purdue University, and partially supported by a research grant from Hofstra University.

Received October 25, 1974

out in **S**; details are provided in [1]. Our notation will be similarly informal; for example, we write

$$(\exists i)_N[\vdash \phi_i \text{ is a total function}]$$

rather than

$$(\exists i)_N[\vdash (\forall n)(\exists m)[\mathrm{T}(i, n, m)]]$$

where T is Kleene's T-predicate.

The recursive real numbers, first defined by Turing [14], are those real numbers which can be effectively approximated to an arbitrarily high degree of accuracy. We can make this definition precise by means of recursive functions. The following definition is equivalent (in the sense of Mayoh [9]) to the most general definition of a Cauchy index for a recursive real number as given by Moschovakis [11].

Definition 1: A natural number i is called a "Cauchy index for a recursive real number" if

1. ϕ_i is a total function;
2. $(\forall n)[|\mathrm{rat}(\phi_i(n)) - \mathrm{rat}(\phi_i(n+1))| \leq 10^{-n}]$.

(Hereafter we omit rat when no confusion results). Let

$$C = \{i: i \text{ is a Cauchy index for a recursive real number}\}.$$

For $i \in C$, we let $\bar{i} = \lim_{n\to\infty} \phi_i(n)$ and we let $\mathcal{C} = \{\bar{i}: i \in C\}$. We say that $\mathcal{C}$ is the set of recursive real numbers.

2 *Provably recursive real numbers* We now define "provably Cauchy index":

Definition 2: A natural number i is called a "provably Cauchy index for a recursive real number" if

1. $\vdash \phi_i$ is a total function;
2. $\vdash (\forall n)[|\phi_i(n) - \phi_i(n+1)| \leq 10^{-n}]$.

(Note: we write $\vdash$ instead of $\vdash_{\mathbf{S}}$ since **S** is fixed). Let

$$P = \{i: i \text{ is a provably Cauchy index for a recursive real number}\}.$$

We shall abbreviate this by writing $P = \{i: \vdash i \in C\}$. For $i \in P$ we let $\bar{i} = \lim_{n\to\infty} \phi_i(n)$ and we let $\mathcal{P} = \{\bar{i}: i \in P\}$. We say that $\mathcal{P}$ is the set of provably recursive real numbers. If **S** is sound for arithmetic then $P \subseteq C$ and thus $\mathcal{P} \subseteq \mathcal{C}$.

Examples We can show that each of the following is a provably recursive real number by proving, in **S**, that its canonical program is a Cauchy index for a recursive real number: 1. every rational number; 2. every algebraic number; 3. e, π, and cos(1).

The following definitions extend the notions of recursive enumerability and productivity to subsets of recursive real numbers.

Definition 3: A subset $\mathcal{D}$ of $\mathcal{C}$ is said to be *recursively enumerable* if there is a recursively enumerable subset D of C such that $\mathcal{D} = \{\bar{i}: i \in D\}$.

Definition 4: A subset $\mathcal{D}$ of $\mathcal{C}$ is said to be *productive* if, for every recursively enumerable subset $\mathcal{B}$ of $\mathcal{D}$, we can find a recursive real number α such that $\alpha \in \mathcal{D} - \mathcal{B}$.

Since the theorems of **S** can be effectively enumerated, P is recursively enumerable and thus $\mathcal{P}$ is also recursively enumerable; a simple diagonalization shows that $\mathcal{C}$ itself is productive (and thus not recursively enumerable). Hence we can find an $e_0 \in C$ such that $e_0 \notin \mathcal{P}$. We now use that fact to show that the decision problem for P is just as difficult as the decision problem for K, in the sense that they are one-to-one equivalent.

Theorem 1 *P is one-to-one equivalent to K.*

Proof: Take $e_0 \in C$ such that $e_0 \notin \mathcal{P}$. Define a total recursive function g by

$$\phi_{g(i)}(n) = \begin{cases} \phi_{e_0}(n), \text{ if } i \notin K^{(n)}; \\ \phi_{e_0}(n_0), \text{ if } n \geq n_0 \wedge n_0 = (\mu m)[i \in K^{(m)}]. \end{cases}$$

Now take any $i \in N$. Clearly we have

1. $i \notin K \rightarrow g(i) \notin P$.

Inversely, suppose $i \in K$. Then, since recursive predicates are numeralwise expressible in **S**, we can prove, in **S**, that $i \in K$, and thus, by the above construction, that $\phi_{g(i)}$ is eventually constant and thus defines a recursive real number. Thus $i \in K \rightarrow \vdash g(i) \in C$, that is

2. $i \in K \rightarrow g(i) \in P$.

Combining 1. and 2. we get $(\forall i)_N [i \in K \leftrightarrow g(i) \in P]$. The recursive function g is clearly one-to-one. The recursive enumerability of P immediately implies that there is a one-to-one recursive function h such that $(\forall i)_N [i \in P \leftrightarrow h(i) \in K]$. Thus P is one-to-one equivalent to K.

Corollary *P is a nonrecursive set.*

Remark: If we let $T = \{i: \vdash \phi_i \text{ is total}\}$, then an even simpler version of the above argument shows that T is one-to-one equivalent to K. In fact, every result about P in this paper has an analogous (and more easily proven) result about T. For other results about T, see Fischer [4].

Note that $\vdash[(\forall i)[i \in P] \rightarrow P$ is recursive], and so

$$\vdash[P \text{ is nonrecursive} \rightarrow (\exists i)[\neg\vdash i \in C]].$$

Also, $\vdash (\forall i)[\neg \mathrm{Cons}_{\mathbf{S}} \rightarrow \vdash i \in C]$, and so

$$\vdash[(\exists i)[\neg\vdash i \in C] \rightarrow \mathrm{Cons}_{\mathbf{S}}].^1$$

Therefore, $\vdash[P \text{ is nonrecursive} \rightarrow \mathrm{Cons}_{\mathbf{S}}]$, so that, by Gödel's Second Incompleteness Theorem,

$$\neg\vdash[P \text{ is nonrecursive}].$$

Therefore, since $\vdash[K$ is nonrecursive],

$$\neg\vdash[P \text{ is one-to-one equivalent to } K];$$

thus neither Theorem 1 nor its Corollary can be proved in **S**.

3 *Classification of provably recursive real numbers* Since the set of provably recursive real numbers was defined in terms of rational numbers, the dichotomy of provably recursive real numbers into rationals and irrationals is certainly the most natural. In this section we examine this dichotomy and show that it is deficient in at least one respect, namely that it cannot be carried out effectively (*cf.* Corollary 1 of Theorem 2).

Definition 5: An index $i \in P$ is said to be an "irrational index" if Irrational (i):

$$(\forall j)_N(\exists m)_N(\exists k)_N(\forall n)_N\left[n \geq k \rightarrow |\phi_i(n) - \mathrm{rat}(j)| > \frac{1}{m+1}\right].$$

i is a "provably irrational index" if $\vdash$Irrational (i).

Remark: We define a "rational index", a "provably rational index", and Rational (i) in the obvious way.

Definition 6: A provably recursive real α is said to be *provably irrational* if $(\exists a)[a \in P \wedge \bar{a} = \alpha \wedge a$ is a provably irrational index].

Example: Let e_0 be the canonical Cauchy program for $\sqrt{2}$ (viz. the one given by the algorithm for finding square roots). Then $e_0 \in P$ and e_0 is a "provably irrational index" since the proof that $\sqrt{2}$ is irrational can be carried out in **S** using the program e_0. Thus $\sqrt{2}$ is provably irrational.

Since every rational has a "provably rational index", there is no point in defining "provably rational". We are thus naturally led to ask:

(1) Is every irrational number in $\mathcal{P}$ provably irrational?

We provide a negative answer to this question later in the section. The following construction answers a related question by showing that some irrational indices are not provably irrational and that some rational indices are not provably rational. (Note that the set of theorems of **S** is recursively enumerable).

For each $e \in N$ we define a sequence of programs $\{g(i, e)\}$ as follows: $\phi_{g(i,e)}$ is defined to look like ϕ_e until and unless i is found to be a provably irrational index (in which case $\phi_{g(i,e)}$ becomes a nearly rational) or a provably rational index (in which case $\phi_{g(i,e)}$ becomes a nearly irrational). We can arithmetize the Extended Recursion Theorem (*cf.* Rogers [13]) to obtain a provably total function h such that

$$(\forall e)_N[\vdash\phi_{h(e)} = \phi_{g(h(e),e)}].$$

We immediately obtain

$$(\forall e)_N [e \in P \rightarrow h(e) \in P].$$

If **S** is sound for arithmetic, the above construction thus yields a uniform method whereby, given any index $e \in P$, we can find another index $h(e) \in P$ which defines the same real number as e but which is neither a provably rational nor a provably irrational index. (For a different proof of a similar result, see Goodstein [5].)

Notation: Let $\mathcal{L} = \{\bar{i}\colon \text{Rational } (i)\}$. Thus $\mathcal{L}$ is the set of provably recursive reals which are rational. Since the set of canonical programs for rationals is recursively enumerable (even recursive), $\mathcal{L}$ is easily seen to be recursively enumerable. What about $\mathcal{P} - \mathcal{L}$, the irrationals? We now use a "delaying" technique[2] to show that $\mathcal{P} - \mathcal{L}$ is productive, and thus not recursively enumerable. An immediate corollary of this theorem provides a negative answer to Question (1).

Theorem 2 *$\mathcal{P} - \mathcal{L}$ is productive.*

Proof: Let $\mathcal{B}$ be a recursively enumerable subset of $\mathcal{P} - \mathcal{L}$ and let h be a primitive recursive function such that $\mathcal{B} = \{\overline{h(i)}\colon i \in N\}$.[3] The construction proceeds as follows: We define $\phi_{e_0}(0)$, $\phi_{e_0}(1)$, . . . to be $\mathrm{rat}(0)$ until (and *unless*[4]) we discover that $\mathrm{rat}(0)$ is different from $\overline{h(0)}$. When this happens (say after n_0 steps) we define $\phi_{e_0}(n_0)$ to be some nearby rational $\mathrm{rat}(j_0)$ so that $\overline{e_0}$ will be different both from $\mathrm{rat}(0)$ and from $\overline{h(0)}$; if $\mathrm{rat}(1)$ is such a nearby rational, then $j_0 = 1$. We then define $\phi_{e_0}(n_0 + 1)$, $\phi_{e_0}(n_0 + 2)$, . . . to be $\mathrm{rat}(j_0)$ until (and unless) we discover that $\mathrm{rat}(j_0)$ is different from $\overline{h(1)}$. When this happens (say after n_1 steps) we define $\phi_{e_0}(n_1)$ to be some nearby rational $\mathrm{rat}(j_1)$ so that $\overline{e_0}$ will be different both from $\mathrm{rat}(j_0)$ and from $\overline{h(1)}$; if $\mathrm{rat}(2)$ is such a nearby rational then $j_1 = 2$. Continuing in this fashion we obtain a recursive real $\overline{e_0}$ which is different from every rational and from every member of $\mathcal{B}$. Thus all that remains is to show that $e_0 \in P$. Since we can prove by induction (in **S**) that every primitive recursive function is provably total, we have

$$\vdash h \text{ is a total function.}$$

Thus, by the above construction,

$$\vdash \phi_{e_0} \text{ is a total function.}$$

Furthermore the construction guarantees that we can prove that successive values of ϕ_{e_0} are close together. Combining these two facts, we obtain

$$\vdash e_0 \in C, \text{ that is } e_0 \in P.$$

We have thus produced a provably recursive real $\overline{e_0} \in (\mathcal{P} - \mathcal{L}) - \mathcal{B}$. Thus $\mathcal{P} - \mathcal{L}$ is productive.

Remark: In the above Theorem we could not prove uniformly that every $h(n)$ was an irrational index. Thus we could not prove that the function ϕ_{e_0} would *not* eventually "settle" on some rational, and so the program e_0 is an irrational index but not a provably irrational index.

Corollary 1 *$\mathcal{P} - \mathcal{L}$ is not recursively enumerable.*

Proof: Otherwise we could find an $\alpha \in (\mathcal{P} - \mathcal{L}) - (\mathcal{P} - \mathcal{L})$, a contradiction.

Corollary 2 *We can find a provably recursively real which is irrational but not provably irrational.*

Proof: Since we can effectively enumerate the set of all indices i such that

$$\vdash \text{Irrational } (i),$$

the set $\mathcal{D} = \{\alpha\colon \alpha \in \mathcal{P} \text{ and } \alpha \text{ is provably irrational}\}$ is recursively enumerable. Thus, by Theorem 2, we can find a provably recursive real which is irrational but not member of $\mathcal{D}$.

Results such as Corollary 1 are interesting partly because $\mathcal{P}$ itself *is* recursively enumerable. Since $\mathcal{C}$ is not recursively enumerable, it follows immediately (and uninterestingly) that $\mathcal{C} - \mathcal{L}$ is not recursively enumerable. In recursive analysis, a better analogue of Corollary 1 would be: $\mathcal{C} - \mathcal{L}$ is not listable.[5] Moschovakis proves this in [11]. More generally, it follows from the Ceitin-Moschovakis continuity theorem for recursive analysis that $\mathcal{C}$ cannot be decomposed into two disjoint, dense, listable subsets. Thus a natural question for provably recursive analysis is: Can $\mathcal{P}$ be decomposed into two disjoint, dense, recursively enumerable subsets? Clearly (by Corollary 1) the canonical decomposition into rationals and irrationals will not work. We now show that in contrast with $\mathcal{C}$, $\mathcal{P}$ can be nicely decomposed.[6]

Theorem 3 *There are subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{P}$ such that*

1. *$\mathcal{A}$ and $\mathcal{B}$ are recursively enumerable.*
2. $\mathcal{A} \cap \mathcal{B} = \phi$.
3. $\mathcal{A} \cup \mathcal{B} = \mathcal{P}$.
4. *Both $\mathcal{A}$ and $\mathcal{B}$ are dense in $\mathcal{P}$.*

Proof: The idea of the proof is to define, in stages, two recursively enumerable subsets A and B of P such that $A \cup B = P$. We will thus correspondingly define $\mathcal{A} = \{\overline{p(i)}\colon p(i) \in A\}$ and $\mathcal{B} = \{\overline{p(i)}\colon p(i) \in B\}$ (where that p is any recursive function which enumerates P).

Stage 0: Put $p(0)$ into A (and thus put $\overline{p(0)}$ into $\mathcal{A}$).

For $n = 0, 1, 2, \ldots$

Stage $3n + 1$: Suppose i is the largest integer such that $p(i)$ has already been put into A or B. For each element $p(j)$ of A, we seek exactly one element $p(k)$, where $k > i$, such that $\overline{p(k)}$ has not yet been put into $\mathcal{A}$ but such that

$$|\overline{p(j)} - \overline{p(k)}| < \frac{1}{n+1}\,.$$

Put each such $p(k)$, when and if it is found, into B.

Stage $3n + 2$: Suppose i is the largest integer such that $p(i)$ has already been put into A or B. For each element $p(j)$ of B, we seek exactly one element $p(k)$, where $k > i$, such that $\overline{p(k)}$ has not yet been put into $\mathcal{B}$ but such that $|\overline{p(j)} - \overline{p(k)}| < \frac{1}{n+1}$. Put each such $p(k)$, when and if it is found, into A.

Stage $3n + 3$: Suppose i is the largest integer such that $p(i)$ has already been put into A or B. For each $m < i$, if $p(m)$ has not yet been put into A or B, then put $p(m)$ into A if we discover that $\overline{p(m)} \notin \mathcal{B}$ before we discover that $\overline{p(m)} \notin \mathcal{A}$; if, however, we discover that $\overline{p(m)} \notin \mathcal{A}$ before we discover that $\overline{p(m)} \notin \mathcal{B}$, then put $p(m)$ into B.

We now show that the above construction does the job.

1. Since $<$ and $\neq$ are recursively enumerable predicates on $\mathcal{P}$, the searches at Stages $3n + 1$ and $3n + 2$, will succeed.
2. It follows by induction that $\mathcal{A}$ and $\mathcal{B}$ are disjoint (and thus the test at Stage $3n + 3$ will succeed).
3. Since A and B are recursively enumerable (by Church's Thesis) and $A \cup B = P$, it follows that $\mathcal{A}$ and $\mathcal{B}$ are recursively enumerable and $\mathcal{A} \cup \mathcal{B} = \mathcal{P}$.
4. Since $\mathcal{A}$ and $\mathcal{B}$ are dense in each other (by the union over all n of Stages $3n + 1$ and $3n + 2$), they are both dense in $\mathcal{P}$.

This completes the proof of the theorem.

Remark: We can use the sets A and B defined in the above theorem to obtain an apparently interesting result: Define, for every $i \in N$,

$$\phi_{f(p(i))} = \begin{cases} 0 \text{ if } p(i) \in A; \\ 1 \text{ if } p(i) \in B. \end{cases}$$

Then f is a partial recursive function which satisfies

$$(\forall i)(\forall j)\,[\overline{p(i)} = \overline{p(j)} \rightarrow \overline{f(p(i))} = \overline{f(p(j))}],$$

so we can define an "effective operator" $\mathcal{F}$ on $\mathcal{P}$ by setting

$$\mathcal{F}(\phi_{p(i)}) = \phi_{f(p(i))}\,, \text{ for every } i \in N.$$

Since the operator $\mathcal{F}$ is (everywhere) discontinuous, it would seem that we have disproved, in provably recursive analysis, the analogue of the Ceitin-Moschovakis continuity theorem of recursive analysis. Further reflection, however, reveals that the proper analogue of an effective operator on $\mathcal{P}$ should be an operator on $\mathcal{C}$ which is *provably* correct, that is, implemented by a partial recursive function f which satisfies:

$$\vdash (\forall i)(\forall j)\,[\overline{p(i)} = \overline{p(j)} \rightarrow \overline{f(p(i))} = \overline{f(p(j))}].$$

It is not apparent that the above operator is not provably correct. In fact, no $0 - 1$ valued operator can be provably correct (although, in [2] we do construct a provably correct operator which is everywhere discontinuous).

4 *Relationship between provably recursive real numbers and primitive recursive real numbers* Let E be the set of canonical programs for primitive recursive functions (viz., those given by the primitive-recursive defining equations) and let PR be the subset of E consisting of those programs which are Cauchy indices for recursive real numbers. Then $\mathcal{PR} = \{\bar{i}: i \in PR\}$ is the set of primitive recursive real numbers. Given any $i \in E$, if we define

$$\phi_{i_0}(0) = \phi_i(0),$$

and, for $n \geq 0$,

$$\phi_{i_0}(n+1) = \begin{cases} \phi_i(n+1), \text{ if } |\phi_{i_0}(n) - \phi_i(n+1)| \leq 10^{-n}; \\ \phi_{i_0}(n), \text{ else} \end{cases}$$

then $i_0 \in P$, and $\bar{i}_0 = \bar{i}$ whenever $i \in PR$. Thus $\mathcal{PR} \subseteq \mathcal{P}$. Furthermore, since $i_0 \in PR$ for every $i \in E$, and E is recursively enumerable, we conclude that $\mathcal{PR}$ is recursively enumerable.[7] In the proof that $\mathcal{P} - \mathcal{L}$ is productive, the program e_0 is, in fact, primitive recursive. Thus $\mathcal{PR} - \mathcal{L}$ is productive and hence not recursively enumerable. Since $\mathcal{PR}$ is a recursively enumerable set, it can be decomposed into two disjoint, dense recursively enumerable subsets (via the technique of Theorem 3) and thus, as in the Remark following Theorem 3, we obtain an apparent discontinuity theorem in primitive recursive analysis. But, again, the function f is deficient: now because it is *not* primitive recursive. The operator constructed in [2] is primitive recursive and establishes a discontinuity theorem in primitive recursive analysis. For other results in primitive recursive analysis, see Mazur [10].

ACKNOWLEDGMENT

The author wishes to thank Paul Young for his advice in the preparation of this paper.

NOTES

1. $\text{Cons}_{\mathbf{S}}$ is the Gödel sentence whose standard interpretation is that **S** is consistent.
2. This technique was invented by Paul Young and used to construct a discontinuous provably correct operator on the provably recursive reals. This result is presented in [2].
3. Note that for every recursively enumerable set D of natural numbers, we can effectively find a primitive recursive (and thus provably total) function which enumerates D.
4. Although $(\forall n)[\text{Irrational}(h(n))]$, we will not, in general, have $\vdash(\forall n)[\text{Irrational}(h(n))]$. Thus, since we want $\vdash e_0 \in C$, we cannot assume, in defining $\phi_{e_0}(n)$, that $h(n)$ is an irrational index.

5. A subset $\mathcal{L}$ of $\mathcal{C}$ is listable if there is a subset L of C and a recursively enumerable set W such that
   1. $\mathcal{L} = \{\bar{i} : i \in L\}$;
   2. $W \cap C = L$;
   3. $(\forall i)(\forall j)[i \in L \wedge \bar{i} = \bar{j} \rightarrow j \in L]$.

6. More generally, the construction used in the proof of Theorem 3 shows that any dense recursively enumerable subset of $\mathcal{C}$ can be "nicely" decomposed. We shall use this observation in Section 4.

7. Paradoxically, *PR* itself is neither recursively enumerable nor a subset of *P*: let s and t be primitive recursive functions which enumerate the sentences of **S** and theorems of **S**, respectively. Define, for all i and n,

$$\phi_{g(i)}(n) = \begin{cases} 0, \text{ if } s(i) \notin \{t(0), t(1), \ldots, t(n)\}; \\ n, \text{ otherwise.} \end{cases}$$

   Then $(\forall i)[g(i) \in PR \leftrightarrow s(i)$ is not a theorem of **S**], so if *PR* were recursively enumerable, then we could enumerate the unprovable sentences of **S**, which is known to be impossible. Thus *PR* is not recursively enumerable. But if *PR* were a subset of *P*, we would have $PR = P \cap E$ (since $PR \subseteq E$ and $P \cap E \subseteq C \cap E = PR$), and so *PR* would be recursively enumerable since both *P* and *E* are. Thus *PR* cannot be a subset of *P*.

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