Notre Dame Journal of Formal Logic
Volume XIX, Number 3, July 1978
NDJFAM

RELATIVE STRENGTH OF MALITZ QUANTIFIERS

STEVEN GARAVAGLIA

In this paper I will solve a problem concerning Malitz quantifiers which was posed in [1]. Before stating this problem I will introduce some notation which will be used in the proof. If X is a set then $c(X)$ is the cardinality of X and $[X]^n$ is the set of n-element subsets of X. S_n is the set of permutations of $\{1, 2, \ldots, n\}$. If $\mathfrak{A}$ is a structure $|\mathfrak{A}|$ denotes the domain of $\mathfrak{A}$. If $\mathcal{L}$ is a first-order language, $\mathcal{L}(|\mathfrak{A}|)$ is the result of adjoining to $\mathcal{L}$ one constant symbol for each element of $|\mathfrak{A}|$. No distinction will be made between elements of $|\mathfrak{A}|$ and the constant symbols denoting them. Variables will be denoted $x_1, x_2, \ldots, y_1, y_2, \ldots$.

Now let $\mathcal{L}$ be any first-order language. For each n and each infinite cardinal α a language $\mathcal{L}_\alpha^n$ is obtained from $\mathcal{L}$ by adjoining the quantifier Q_α^n with the following interpretation: $\mathfrak{A} \models Q_\alpha^n x_1 \ldots x_n \varphi(x_1, \ldots, x_n)$ if and only if there is a set $X \subset |\mathfrak{A}|$ such that $c(X) \geqslant \alpha$ and for all distinct $a_1, \ldots, a_n$ in X, $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$. Malitz and Magidor [2] and Badger [1] have established many deep and interesting results concerning these languages. In [1], page 91, Badger gave a list of unsolved problems about the languages $\mathcal{L}_\alpha^n$. There he raised the question whether $\mathcal{L}_\alpha^{n+1}$ is a proper extension of $\mathcal{L}_\alpha^n$. In this paper I answer this question affirmatively for all $n \geqslant 1$ and all $\alpha > \omega$ by exhibiting two structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same similarity type such that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences in $\mathcal{L}_\alpha^n$ but do not satisfy the same sentences in $\mathcal{L}_\alpha^{n+1}$.[1]

Let n be any fixed positive integer and let α be any fixed uncountable cardinal. $\mathcal{L}$ will be a first-order language with equality whose only nonlogical symbol is an $(n + 1)$-ary predicate symbol R.

Definition 1: If $\mathfrak{A}$ is an $\mathcal{L}$-structure, γ is a finite subset of $|\mathfrak{A}|$, $\sigma \in S_{n+1}$, and $t_1, \ldots, t_{n+1} \in \gamma \cup \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ then $\sigma(t_1, \ldots, t_{n+1})$ is the $(n + 1)$-tuple $(t_{\sigma(1)}, \ldots, t_{\sigma(n+1)})$ and $\sigma R(t_1, \ldots, t_{n+1})$ is the $\mathcal{L}(|\mathfrak{A}|)$-formula $R(t_{\sigma(1)}, \ldots, t_{\sigma(n+1)})$.

1. For $\alpha = \omega_1$, this result was obtained independently by Andreas Baudisch under the assumption $\diamond_{\omega_1}$.

Received June 14, 1977

Definition 2: Suppose $\mathfrak{A}$ is an $\mathcal{L}$-structure, γ is a finite subset of $|\mathfrak{A}|$, and $1 \leqslant m \leqslant n$. An m-type p over γ is a set of $\mathcal{L}(|\mathfrak{A}|)$-formulas such that

(1) all elements of p are of the form $\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(j)}, a_1, \ldots, a_{n-j+1})$ where $1 \leqslant j \leqslant m$, $\sigma \in S_m$, $\tau \in S_{n+1}$, and $a_1, \ldots, a_{n-j+1}$ are distinct elements of γ,
(2) if $1 \leqslant j \leqslant m$, $\tau \in S_{n+1}$, $\sigma \in S_m$, $\tau' \in S_{n+1}$, $\sigma' \in S_m$, and $\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(j)}, a_1, \ldots, a_{n-j+1}) \in p$ then $\tau' R(x_{\sigma'(1)}, \ldots, x_{\sigma'(j)}, a_1, \ldots, a_{n-j+1}) \in p$.

Definition 3: Suppose $\mathfrak{A}$ is an $\mathcal{L}$-structure, γ is a finite subset of $|\mathfrak{A}|$, and $1 \leqslant m \leqslant n$. A proper m-type p over γ is a set of $\mathcal{L}(|\mathfrak{A}|)$-formulas such that

(1) all elements of p are of the form $\tau R(x_1, \ldots, x_m, a_1, \ldots, a_{n-m+1})$ where $\tau \in S_{n+1}$ and $a_1, \ldots, a_{n-m+1}$ are distinct elements of γ,
(2) if $\tau \in S_{n+1}$, $\tau' \in S_{n+1}$, and $\tau R(x_1, \ldots, x_m, a_1, \ldots, a_{n-m+1}) \in p$ then $\tau' R(x_1, \ldots, x_m, a_1, \ldots, a_{n-m+1}) \in p$.

Clearly, a proper m-type p is just an m-type in which all of the variables $x_1, \ldots, x_m$ occur in every formula in p.

Definition 4: If $\mathfrak{A}$ is an $\mathcal{L}$-structure, γ is a finite subset of $|\mathfrak{A}|$, p is an m-type over γ, and $b_1, \ldots, b_m \in |\mathfrak{A}|$ then $(b_1, \ldots, b_m)$ realizes p if and only if

$$\mathfrak{A} \models \tau R(b_{\sigma(1)}, \ldots, b_{\sigma(j)}, a_1, \ldots, a_{n-j+1}) \leftrightarrow \tau R(x_{\sigma(1)}, \ldots, x_{\sigma(j)}, a_1, \ldots, a_{n-j+1}) \in p$$

for all $1 \leqslant j \leqslant m$, $\tau \in S_{n+1}$, $\sigma \in S_m$, and $a_1, \ldots, a_{n-j+1} \in \gamma$.

These notions are all borrowed from the customary model-theoretic definitions of type, realization, etc. It can be shown that types as they have been defined here correspond exactly to quantifier-free types in the usual sense with respect to a certain first-order theory. But it does not seem to simplify the exposition to use this fact so I will just ignore it.

Now I proceed to construct two $\mathcal{L}$-structures $\mathfrak{A} = \langle A, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle B, R^{\mathfrak{B}}\rangle$. First, for each $\beta < \alpha$ a structure $\mathfrak{A}_\beta = (A_\beta, R^{\mathfrak{A}_\beta})$ will be constructed. Let $A_0 = \{1, 2, \ldots, n+1\}$, $R^{\mathfrak{A}_0} = \{\sigma(1, 2, \ldots, n+1) \mid \sigma \in S_{n+1}\}$. If β is a limit ordinal let $A_\beta = \bigcup_{\delta<\beta} A_\delta$, $R^{\mathfrak{A}_\beta} = \bigcup_{\delta<\beta} R^{\mathfrak{A}_\delta}$. If $\beta = \delta + 1$ where δ is even, then let a_β be any element such that $a_\beta \notin A_\delta$ and let $A_\beta = A_\delta \cup \{a_\beta\}$ and

$$R^{\mathfrak{A}_\beta} = R^{\mathfrak{A}_\delta} \cup \{\sigma(a_1, \ldots, a_n, a_\beta) \mid \sigma \in S_{n+1}, a_1, \ldots, a_n \in A_\delta \bigwedge_{\substack{s \neq t \\ 1 \leqslant s, t \leqslant n}} a_s \neq a_t\}.$$

Now suppose that $\beta = \delta + 1$ where δ is odd. For each finite subset γ of A_δ and each n-type p over γ let $X^\beta_{\gamma,p}$ be a set of cardinality α such that $X^\beta_{\gamma,p} \cap A_\delta = \emptyset$ and if $\gamma \neq \gamma'$ or $p \neq p'$ then $X^\beta_{\gamma,p} \cap X^\beta_{\gamma',p'} = \emptyset$. Let $A_\beta = A_\delta \cup \bigcup_{\gamma,p} X^\beta_{\gamma,p}$. For each n-tuple $(b_1, \ldots, b_n)$ of distinct elements of $X^\beta_{\gamma,p}$ let

$$p(b_1, \ldots, b_n) = \{\tau(b_{\sigma(1)}, \ldots, b_{\sigma(j)}, a_1, \ldots, a_{n-j+1}) \mid \tau R(x_{\sigma(1)}, \ldots, x_{\sigma(j)}, a_1, \ldots, a_{n-j+1}) \in p\}.$$

Let

$$R^{\mathfrak{A}_\beta} = R^{\mathfrak{A}_\delta} \cup \bigcup_{\gamma,p} \bigcup \{p(b_1, \ldots, b_n) \mid b_1, \ldots, b_n \in X^\beta_{\gamma,p} \bigwedge_{\substack{s \neq t \\ 1 \leq s,t \leq n}} b_s \neq b_t\}.$$

Finally, let $\mathfrak{A} = \langle A, R^{\mathfrak{A}}\rangle$ where $A = \bigcup_{\beta<\alpha} A_\beta\, R^{\mathfrak{A}} = \bigcup_{\beta<\alpha} R^{\mathfrak{A}_\beta}$.

$\mathfrak{B}$ is constructed in a similar manner. For each $m < \omega$ a structure $\mathfrak{B}_m$ $\mathfrak{B}_m = \langle B_m, R^{\mathfrak{B}_m}\rangle$ will be constructed as follows. Set $B_0 = \{1, 2, \ldots, n + 1\}\, R^{\mathfrak{B}_0} = \{\sigma(1, 2, \ldots, n + 1) \mid \sigma \in S_{n+1}\}$. If $\mathfrak{B}_m$ has been constructed, then for each finite $\gamma \subset B_m$ and each n-type p over γ pick a set $\overline{X}^{m+1}_{\gamma,p}$ of cardinality α such that $B_m \cap \overline{X}^{m+1}_{\gamma,p} = \emptyset$ and if $\gamma \neq \gamma'$ or $p \neq p'$ then $\overline{X}^{m+1}_{\gamma,p} \cap \overline{X}^{m+1}_{\gamma',p'} = \emptyset$. Let $B_{m+1} = B_m \cup \bigcup_{\gamma,p} \overline{X}^{m+1}_{\gamma,p}$. Define $p(b_1, \ldots, b_n)$ as before and let

$$R^{\mathfrak{B}_{m+1}} = R^{\mathfrak{B}_m} \cup \bigcup_{\gamma,p} \bigcup \{p(b_1, \ldots, b_n) \mid b_1, \ldots, b_n \in \overline{X}^{m+1}_{\gamma,p} \bigwedge_{\substack{s \neq t \\ 1 \leq s,t \leq n}} b_s \neq b_t\}.$$

Finally, define $\mathfrak{B} = \langle B, R^{\mathfrak{B}}\rangle$ where $B = \bigcup_{m<\omega} B_m, R^{\mathfrak{B}} = \bigcup_{m<\omega} R^{\mathfrak{B}_m}$.

It is important to make four simple observations about these structures:

(1*) *if* $\delta < \beta < \alpha$ *then* $\mathfrak{A}_\delta \subset \mathfrak{A}_\beta$ *and if* $m < k < \omega$ *then* $\mathfrak{B}_m \subset \mathfrak{B}_k$;

(2*) *if* $\mathfrak{A} \models R(a_1, \ldots, a_{n+1})$ *then* $\mathfrak{A} \models \bigwedge_{\substack{s \neq t \\ 1 \leq s,t \leq n+1}} a_s \neq a_t$ *and if* $\mathfrak{B} \models R(b_1, \ldots, b_{n+1})$ *then* $\mathfrak{B} \models \bigwedge_{\substack{s \neq t \\ 1 \leq s,t \leq n+1}} b_s \neq b_t$;

(3*) *if* $\beta < \alpha$, $\beta = \delta + 1$ *where* δ *is odd,* γ *is a finite subset of* A_δ *and* p *is an* n*-type over* γ *then each* n*-tuple of distinct elements of* $X^\beta_{\gamma,p}$ *realizes* p, *and an analogous statement holds for* $\mathfrak{B}$;

(4*) *both* $R^{\mathfrak{A}}$ *and* $R^{\mathfrak{B}}$ *are symmetric,* i.e., *if* $\sigma \in S_{n+1}$ *and* $\mathfrak{A} \models R(a_1, \ldots, a_{n+1})$ *then* $\mathfrak{A} \models \sigma R(a_1, \ldots, a_{n+1})$ *and if* $\mathfrak{B} \models R(b_1, \ldots, b_{n+1})$ *then* $\mathfrak{B} \models \sigma R(b_1, \ldots, b_{n+1})$.

These statements are all proved by quite simple inductive arguments. Using these facts, I can now prove the following lemma which contains the easy half of the main result of this paper.

Lemma 1 $\mathfrak{A} \models Q^{n+1}_\alpha x_1 \ldots x_{n+1} R(x_1, \ldots, x_{n+1})$ *and* $\mathfrak{B} \models \neg Q^{n+1}_\alpha x_1 \ldots x_{n+1} R(x_1, \ldots, x_{n+1})$.

Proof: By construction of $\mathfrak{A}$, if $\beta = \delta + 1$ where $\delta < \alpha$ is even then $A_\beta = A_\delta \cup \{a_\beta\}$. Let $X = \{a_\beta \mid \beta < \alpha, \beta = \delta + 1, \delta \text{ even}\}$. Then $c(X) = \alpha$. Any $(n + 1)$-tuple of distinct elements of X has the form $\sigma(a_{\beta_1}, \ldots, a_{\beta_{n+1}})$ where $\beta_1 < \ldots < \beta_{n+1}$, $\beta_i = \delta_i + 1$, δ_i even for $i = 1, \ldots, n + 1$, and $\sigma \in S_{n+1}$. The set $\{a_{\beta_1}, \ldots, a_{\beta_n}\}$ is contained in $A_{\delta_{n+1}}$ since $\beta_1 < \ldots < \beta_n < \delta_{n+1}$, so by construction of $\mathfrak{A}$ $\sigma(a_{\beta_1}, \ldots, a_{\beta_{n+1}}) \in R^{\mathfrak{A}}$. This proves that $\mathfrak{A} \models Q^{n+1}_\alpha x_1 \ldots x_{n+1} R(x_1, \ldots, x_{n+1})$.

Now suppose that $X \subset B$, $c(X) = \alpha$, and for all distinct $b_1, \ldots, b_{n+1} \in X$ $\mathfrak{B} \models R(b_1, \ldots, b_{n+1})$. Since $\omega < \alpha$ there must be some m such that $X \cap (B_{m+1} - B_m)$ is infinite. Let $b_1, \ldots, b_{n+1}$ be distinct elements of $X \cap (B_{m+1} - B_m)$. By our assumption $\mathfrak{B} \models R(b_1, \ldots, b_{n+1})$. Suppose that $(b_1, \ldots, b_{n+1}) \in$

$R^{\mathfrak{B}_{m+1}}$. Then there must be some $\gamma \subset B_m$, an n-type p over γ, and distinct elements $c_1, \ldots, c_n$ in $\overline{X}_{\gamma,p}^{m+1}$ such that $R(b_1, \ldots, b_{n+1}) \in p(c_1, \ldots, c_n)$. But if $R(t_1, \ldots, t_{n+1}) \in p(c_1, \ldots, c_n)$ then at least one of the t_i must be an element of γ. This is an immediate consequence of the definition of an n-type. Since $\gamma \cap \{b_1, \ldots, b_{n+1}\} = \emptyset$, it must be that $(b_1, \ldots, b_{n+1}) \notin R^{\mathfrak{B}_{m+1}}$. But by observation (1*), $\mathfrak{B}_{m+1} \subset \mathfrak{B}$ so $(b_1, \ldots, b_{n+1}) \notin R^{\mathfrak{B}}$, i.e., $\mathfrak{B} \models \neg R(b_1, \ldots, b_{n+1})$. This is a contradiction. Therefore no such X can exist, i.e., $\mathfrak{B} \models \neg Q_\alpha^{n+1} x_1 \ldots x_{n+1} R(x_1, \ldots, x_{n+1})$. Q.E.D.

Now we move on to the more difficult part of the proof: showing that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences in $\mathcal{L}_\alpha^n$. I adjoin two O-ary predicates **T** ("true") and **F** ("false") to $\mathcal{L}$ and give them the obvious interpretation in any structure. They can be regarded as defined terms with the definitions $\mathbf{T} \equiv \forall x_1(x_1 = x_1)$ and $\mathbf{F} \equiv \exists x_1(x_1 \neq x_1)$. This expanded language is called $\mathcal{L}(\mathbf{T}, \mathbf{F})$.

Lemma 2 *To each formula $\varphi(y_1, \ldots, y_k)$ of $\mathcal{L}(\mathbf{T}, \mathbf{F})_\alpha^n$ with free variables among $y_1, \ldots, y_k$ one can effectively associate a quantifier-free formula $\psi(y_1, \ldots, y_k)$ of $\mathcal{L}(\mathbf{T}, \mathbf{F})$ with free variables among $y_1, \ldots, y_k$ such that*

$$\mathfrak{A} \models \forall y_1 \ldots \forall y_k [\varphi(y_1, \ldots, y_k) \leftrightarrow \psi(y_1, \ldots, y_k)]$$

and

$$\mathfrak{B} \models \forall y_1 \ldots \forall y_k [\varphi(y_1, \ldots, y_k) \leftrightarrow \psi(y_1, \ldots, y_k)].$$

Proof: By using induction on the length of the formula, the proof can be reduced to the consideration of two special cases.

Case 1: Suppose $\varphi(y_1, \ldots, y_k) \equiv \exists x_1 \eta(x_1, y_1, \ldots, y_k)$ where η is a conjunction of atomic formulas and negations of atomic formulas in $\mathcal{L}(\mathbf{T}, \mathbf{F})$.

(a) If $x_1 = y_j$ or $y_j = x_1$ is a conjunct in η for any j then it is easy to see that

$$\mathfrak{A} \models \forall y_1 \ldots \forall y_k [\exists x_1 \eta(x_1, y_1, \ldots, y_k) \leftrightarrow \eta(x_1/y_j, y_1, \ldots, y_k)]$$

and

$$\mathfrak{B} \models \forall y_1 \ldots \forall y_k [\exists x_1 \eta(x_1, y_1, \ldots, y_k) \leftrightarrow \eta(x_1/y_j, y_1, \ldots, y_k)]$$

where $\eta(x_1/y_j, y_1, \ldots, y_k)$ is the result of substituting y_j for every occurrence of x_1 in $\eta(x_1, y_1, \ldots, y_k)$.

(b) Suppose that there is no j such that $y_j = x_1$ or $x_1 = y_j$ is a conjunct in η. Let Δ be the smallest set of quantifier-free formulas of $\mathcal{L}(\mathbf{T}, \mathbf{F})$ satisfying the following rules:

(1) $\mathbf{T} \in \Delta$
(2) $\rho(y_1, \ldots, y_k) \in \Delta$ where $\rho(y_1, \ldots, y_k)$ is the conjunction of all the conjuncts in η which do not contain x_1
(3) if $\sigma \in S_{n+1}$, $j \geq 1$, and $\sigma R(x_1, \ldots, x_1, y_{i_1}, \ldots, y_{i_{n-j}})$ is a conjunct in η, or if $x_1 \neq x_1$ is a conjunct in η, then $\mathbf{F} \in \Delta$
(4) if $\sigma \in S_{n+1}$ and $\sigma R(x_1, y_{i_1}, \ldots, y_{i_n})$ is a conjunct in η, then $\bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n}} y_{i_s} \neq y_{i_t} \in \Delta$

(5) if $\tau_1 \in S_{n+1}$, $\tau_2 \in S_{n+1}$ and both $\tau_1 R(x_1, y_{i_1}, \ldots, y_{i_n})$ and $\neg\tau_2 R(x_1, y_{j_1}, \ldots, y_{j_n})$ are conjuncts in η, then $\bigvee_{s=1}^{n}\left(\bigwedge_{t=1}^{n} y_{i_s} \neq y_{j_t}\right) \in \Delta$.

Let $\psi(y_1, \ldots, y_k) \equiv \bigwedge_{\mu \in \Delta} \mu(y_1, \ldots, y_k)$. Then I claim that $\mathfrak{A} \models \forall y_1 \ldots \forall y_k[\exists x_1 \eta(x_1, y_1, \ldots, y_k) \leftrightarrow \psi(y_1, \ldots, y_k)]$ and $\mathfrak{B} \models \forall y_1 \ldots \forall y_k[\exists x_1 \eta(x_1, y_1, \ldots, y_k) \leftrightarrow \psi(y_1, \ldots, y_k)]$. The proofs for $\mathfrak{A}$ and $\mathfrak{B}$ are essentially identical, so I will confine my attention to $\mathfrak{A}$. So suppose that $(a_1, \ldots, a_k)$ is any k-tuple of elements of A, and suppose that $\mathfrak{A} \models \exists x_1 \eta(x_1, a_1, \ldots, a_k)$. Then for some $a \in A$, $\mathfrak{A} \models \eta(a, a_1, \ldots, a_k)$. For each $\mu \in \Delta$ I will show that $\mathfrak{A} \models \mu(a_1, \ldots, a_k)$. Any μ in Δ must be placed there according to one of the rules (1)-(5). So I consider each rule in turn. If $\mu \equiv \mathbf{T}$ then there is nothing to prove since $\mathfrak{A} \models \mathbf{T}$ is always valid. If $\mu(y_1, \ldots, y_k) \equiv \rho(y_1, \ldots, y_k)$ then since ρ is a conjunction of conjuncts in η and $\mathfrak{A} \models \eta(a, a_1, \ldots, a_k)$ we have $\mathfrak{A} \models \rho(a_1, \ldots, a_k)$. μ cannot be put into Δ according to rule (3) since in that case either $x_1 \neq x_1$ or something of the form $\sigma R(x_1, \ldots, x_1, y_{i_1}, \ldots, y_{i_{n-j}})$ $(\sigma \in S_{n+1}, j \geq 1)$ would be a conjunct in η. It would follow that either $\mathfrak{A} \models a \neq a$, which is impossible or $\mathfrak{A} \models \sigma R(a, \ldots, a, a_{i_1}, \ldots, a_{i_{n-j}})$ which is impossible by observation (2*). Next suppose μ arises via rule (4). Then $\mu \equiv \bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n}} y_{i_s} \neq y_{i_t}$ and for some $\sigma \in S_{n+1}$, $\sigma R(x_1, y_{i_1}, \ldots, y_{i_n})$ is a conjunct in η. Therefore $\mathfrak{A} \models \sigma R(a, a_{i_1}, \ldots, a_{i_n})$ and by observation (2*), this implies that $\mathfrak{A} \models \bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n}} a_{i_s} \neq a_{i_t}$. Finally suppose μ arises from rule (5). Then μ is of the form $\bigvee_{s=1}^{n}\left(\bigwedge_{t=1}^{n} y_{i_s} \neq y_{j_t}\right)$ and for some $\tau_1 \in S_{n+1}$, $\tau_2 \in S_{n+1}$ both $\tau_1 R(x_1, y_{i_1}, \ldots, y_{i_n})$ and $\neg\tau_2 R(x_1, y_{j_1}, \ldots, y_{j_n})$ are conjuncts in η. Hence $\mathfrak{A} \models \tau_1 R(a, a_{i_1}, \ldots, a_{i_n})$ and $\mathfrak{A} \models \neg\tau_2 R(a, a_{j_1}, \ldots, a_{j_n})$. By observation (4*), $R^{\mathfrak{A}}$ is symmetric, so $\{a_{i_1}, \ldots, a_{i_n}\} \neq \{a_{j_1}, \ldots, a_{j_n}\}$. By observation (2*) $a_{i_1}, \ldots, a_{i_n}$ are distinct, so in fact $\{a_{i_1}, \ldots, a_{i_n}\} \not\subset \{a_{j_1}, \ldots, a_{j_n}\}$. Therefore there is some s such that $a_{i_s} \notin \{a_{j_1}, \ldots, a_{j_n}\}$. This implies that $\mathfrak{A} \models \bigvee_{s=1}^{n}\left(\bigwedge_{t=1}^{n} a_{i_s} \neq a_{j_t}\right)$.

Conversely, suppose $\mathfrak{A} \models \bigwedge_{\mu \in \Delta} \mu(a_1, \ldots, a_k)$. I will show that $\mathfrak{A} \models \exists x_1 \eta(x_1, a_1, \ldots, a_k)$. Let $\gamma = \{a_1, \ldots, a_k\}$. Pick some $\delta < \alpha$ such that δ is odd and $\gamma \subset A_\delta$. Let $p = \{\tau R(x_{\sigma(1)}, a_{i_1}, \ldots, a_{i_n}) \mid \sigma \in S_n, \tau \in S_{n+1}$, and for some $\tau' \in S_{n+1}$ $\tau' R(x_1, y_{i_1}, \ldots, y_{i_n})$ is a conjunct in $\eta\}$. I claim that p is an n-type over γ. First, if $\tau R(x_{\sigma(1)}, a_{i_1}, \ldots, a_{i_n}) \in p$ for some $\tau \in S_{n+1}$, $\sigma \in S_n$ then by definition of p $\tau' R(x_1, y_{i_1}, \ldots, y_{i_n})$ is a conjunct in η for some $\tau' \in S_{n+1}$ and hence, again by definition of p, $\tau'' R(x_{\sigma''(1)}, a_{i_1}, \ldots, a_{i_n}) \in p$ for any $\tau'' \in S_{n+1}$ and $\sigma'' \in S_n$. This shows that p satisfies the second condition in the definition of an n-type. Now suppose again that $\tau R(x_{\sigma(1)}, a_{i_1}, \ldots, a_{i_n}) \in p$. Then for some $\tau' \in S_{n+1}$, $\tau' R(x_1, y_{i_1}, \ldots, y_{i_n})$ is a conjunct in η. Hence $\bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n}} y_{i_s} \neq y_{i_t} \in \Delta$. Then since $\mathfrak{A} \models \bigwedge_{\mu \in \Delta} \mu(a_1, \ldots, a_k)$ we have $\mathfrak{A} \models \bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n}} a_{i_s} \neq a_{i_t}$. This proves that p satisfies the first condition in the definition of an n-type.

Take any n-tuple $(c_1, \ldots, c_n)$ of distinct elements of $X_{\gamma, p}^{\delta+1}$. By observation (3*), $(c_1, \ldots, c_n)$ realizes p. I claim that $\mathfrak{A} \models \eta(c_1, a_1, \ldots, a_k)$.

I consider each conjunct in η separately. For conjuncts in η which do not contain x_1 it is sufficient to note that they are also conjuncts in ρ, and since $\rho \in \Delta$ we have by assumption, $\mathfrak{A} \models \rho(a_1, \ldots, a_k)$. Also, by hypothesis, η has no conjunct of the form $x_1 = y_j$ or $y_j = x_1$. If $x_1 \neq y_j$ or $y_j \neq x_1$ is a conjunct in η, then $\mathfrak{A} \models c_1 \neq a_j$ ($\mathfrak{A} \models a_j \neq c_1$) since $c_1 \in A_{\delta+1} - A_\delta$ but $a_j \in A_\delta$. Nothing of the form $\sigma R(x_1, \ldots, x_1, y_{i_1}, \ldots, y_{i_{n-j}})$ where $\sigma \in S_{n+1}$ and $j \geq 1$ can be a conjunct in η since if it were then $F \in \Delta$ and so we could not have $\mathfrak{A} \models \bigwedge_{\mu \in \Delta} \mu(a_1, \ldots, a_k)$. For the same reason $x_1 \neq x_1$ is not a conjunct in η. A conjunct of the form $x_1 = x_1$ is trivially satisfied. If $\tau R(x_1, y_{i_1}, \ldots, y_{i_n})$ is a conjunct in η where $\tau \in S_{n+1}$, then $\tau R(x_1, a_{i_1}, \ldots, a_{i_n}) \in p$ and since $(c_1, \ldots, c_n)$ realizes p, we have $\mathfrak{A} \models \tau R(c_1, a_{i_1}, \ldots, a_{i_n})$. If $\tau \in S_{n+1}$, $j \geq 1$ and $\neg\tau R(x_1, \ldots, x_1, y_{i_1}, \ldots, y_{i_{n-j}})$ is a conjunct in η, then $\mathfrak{A} \models \neg\tau R(c_1, \ldots, c_1, a_{i_1}, \ldots, a_{i_{n-j}})$ by observation (2*). Finally suppose $\neg\tau R(x_1, y_{i_1}, \ldots, y_{i_n})$ is a conjunct in η. I claim that $\tau R(x_1, a_{i_1}, \ldots, a_{i_n}) \notin p$. If it were in p, then there would have to be some $\sigma \in S_{n+1}$, $\tau' \in S_{n+1}$, and $j_1, \ldots, j_n$ such that $\sigma R(x_1, y_{j_1}, \ldots, y_{j_n})$ is a conjunct in η and $\tau R(x_1, a_{j_1}, \ldots, a_{j_n})$ is identical with $\tau R(x_1, a_{i_1}, \ldots, a_{i_n})$. That would imply $\{a_{j_1}, \ldots, a_{j_n}\} = \{a_{i_1}, \ldots, a_{i_n}\}$. But since both $\sigma R(x_1, y_{j_1}, \ldots, y_{j_n})$ and $\neg\tau R(x_1, y_{i_1}, \ldots, y_{i_n})$ are conjuncts in η, we would have $\bigvee_{s=1}^{n}\left(\bigwedge_{t=1}^{n} y_{j_s} \neq y_{i_t}\right) \in \Delta$ and hence $\mathfrak{A} \models \bigvee_{s=1}^{n}\left(\bigwedge_{t=1}^{n} a_{j_s} \neq a_{i_t}\right)$. That means that $\{a_{j_1}, \ldots, a_{j_n}\} \neq \{a_{i_1}, \ldots, a_{i_n}\}$. This contradiction proves that $\tau R(x_1, a_{i_1}, \ldots, a_{i_n}) \notin p$. Then since $(c_1, \ldots, c_n)$ realizes p, we have $\mathfrak{A} \models \neg\tau R(c_1, a_{i_1}, \ldots, a_{i_n})$. This covers all possibilities, so we have proved $\mathfrak{A} \models \eta(c_1, a_1, \ldots, a_k)$ and therefore $\mathfrak{A} \models \exists x_1 \eta(x_1, a_1, \ldots, a_k)$.

Case 2: Suppose $\varphi(y_1, \ldots, y_k)$ is of the form $Q_\alpha^n x_1 \ldots x_n \bigvee_{i=1}^{n} \eta_i(x_1, \ldots, x_n, y_1, \ldots, y_k)$ where each η_i is a conjunction of atomic formulas and negations of atomic formulas in $\mathcal{L}(\mathbf{T}, \mathbf{F})$. For each $1 \leq i \leq m$ I define Δ_i to be the smallest set of quantifier-free formulas in $\mathcal{L}(\mathbf{T}, \mathbf{F})$ satisfying the following rules:

(1) $\mathbf{T} \in \Delta_i$

(2) $\rho_i \in \Delta_i$ where ρ_i is the conjunction of all the conjuncts in η_i which do not contain any of the variables $x_1, \ldots, x_n$

(3) if $\sigma R(x_j, \ldots, x_j, x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_{n-h}})$ is a conjunct in η_i for any j, $\sigma \in S_{n+1}$ and $h \geq 1$, or if anything of the following forms: $x_j = x_h$ $(j \neq h)$, $y_j = x_h$, $x_h = y_j$, $x_j \neq x_j$ is a conjunct in η_i, then $\mathbf{F} \in \Delta_i$

(4) if $\tau \in S_{n+1}$, $\sigma \in S_n$, $1 \leq h \leq n$ and $\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(h)}, y_{i_1}, \ldots, y_{i_{n-h+1}})$ is a conjunct in η_i then

$$\bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n-h+1}} y_{i_s} \neq y_{i_t} \in \Delta_i$$

(5) if $\tau \in S_{n+1}$, $\sigma \in S_n$, $\tau' \in S_{n+1}$, $\sigma' \in S_n$ $1 \leq h \leq n$ and both $\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(h)}, y_{i_1}, \ldots, y_{i_{n-h+1}})$ and $\neg\tau' R(x_{\sigma'(1)}, \ldots, x_{\sigma'(h)}, y_{j_1}, \ldots, y_{j_{n-h+1}})$ are conjuncts in η_i then

$$\bigvee_{s=1}^{n-h+1}\left(\bigwedge_{t=1}^{n-h+1} y_{i_s} \neq y_{j_t}\right) \in \Delta_i.$$

Now I claim that

$$\mathfrak{A} \models \forall y_1 \ldots \forall y_k \left[Q_\alpha^n x_1 \ldots x_n \left(\bigvee_{i=1}^m \eta_i(x_1, \ldots, x_n, y_1, \ldots, y_k) \right) \leftrightarrow \bigvee_{i=1}^m \left(\bigwedge_{\mu \in \Delta_i} \mu(y_1, \ldots, y_k) \right) \right]$$

and

$$\mathfrak{B} \models \forall y_1 \ldots \forall y_k \left[Q_\alpha^n x_1 \ldots x_n \left(\bigvee_{i=1}^m \eta_i(x_1, \ldots, x_n, y_1, \ldots, y_k) \right) \leftrightarrow \bigvee_{i=1}^m \left(\bigwedge_{\mu \in \Delta_i} \mu(y_1, \ldots, y_k) \right) \right].$$

Again I give the proof only for $\mathfrak{A}$ since the proof for $\mathfrak{B}$ is virtually identical. Take any k-tuple $(a_1, \ldots, a_k)$ of elements of A and suppose first that $\mathfrak{A} \models Q_\alpha^n x_1 \ldots x_n \left(\bigvee_{i=1}^m \eta_i(x_1, \ldots, x_n, a_1, \ldots, a_k) \right)$. Then there is a set $X \subset A$ such that $c(X) = \alpha$ and for all distinct $c_1, \ldots, c_n$ in X, $\mathfrak{A} \models \bigvee_{i=1}^m (\eta_i(c_1, \ldots, c_n, a_1, \ldots, a_k)$. Let $\gamma = \{a_1, \ldots, a_k\}$. For each proper n-type p over γ let

$$X_p = \{\{c_1, \ldots, c_n\} \in [X]^n \mid \text{for all } \tau \in S_{n+1} \text{ and all } a \in \gamma\ \mathfrak{A} \models \tau R(c_1, \ldots, c_n, a) \leftrightarrow \tau R(x_1, \ldots, x_n, a) \in p\}.$$

X_p is well-defined since $R^{\mathfrak{A}}$ is symmetric, and p is closed under permutations. Furthermore, for any $\{c_1, \ldots, c_n\} \in [X]^n$ if $p = \{\tau R(x_1, \ldots, x_n, a) \mid \tau \in S_{n+1},\ a \in \gamma, \text{ and } \mathfrak{A} \models \tau R(c_1, \ldots, c_n, a)\}$ then p is a proper n-type over γ and $\{c_1, \ldots, c_n\} \in X_p$. Therefore, $\{X_p \mid p \text{ is a proper } n\text{-type over } \gamma\}$ is a finite partition of $[X]^n$ and so by Ramsey's theorem there is an infinite set $X_1 \subset X$ and a proper n-type p over γ such that for all $\{c_1, \ldots, c_n\} \in [X_1]^n$ $\{c_1, \ldots, c_n\} \in X_p$. Now for any proper $(n - 1)$-type p over γ let

$$(X_1)_p = \{\{c_1, \ldots, c_{n-1}\} \in [X_1]^{n-1} \mid \text{for all } \tau \in S_{n+1} \text{ and all } a_{i_1}, a_{i_2} \in \gamma\ \mathfrak{A} \models \tau R(c_1, \ldots, c_{n-1}, a_{i_1}, a_{i_2}) \leftrightarrow \tau R(x_1, \ldots, x_{n-1}, a_{i_1}, a_{i_2}) \in p\}$$

Then as above we obtain an infinite set $X_2 \subset X_1$ and a proper $(n - 1)$-type p over γ such that $[X_2]^{n-1} \subset (X_1)_p$. Continuing in this fashion we finally obtain an infinite set $Y \subset X$ and a sequence $p_1, \ldots, p_n$ such that each p_j is a proper j-type over γ and for all distinct $c_1, \ldots, c_j$ in Y and all $a_{i_1}, \ldots, a_{i_{n-j+1}}$ in γ and all $\tau \in S_{n+1}$

$$\mathfrak{A} \models \tau R(c_1, \ldots, c_j, a_{i_1}, \ldots, a_{i_{n-j+1}}) \leftrightarrow \tau R(x_1, \ldots, x_j, a_{i_1}, \ldots, a_{i_{n-j+1}}) \in p_j.$$

This implies that for any $1 \leq j \leq n$ and any two j-tuples $(c_1, \ldots, c_j)$, $(b_1, \ldots, b_j)$ of distinct elements of Y, any $\sigma \in S_{n+1}$ and any $a_{i_1}, \ldots, a_{i_{n-j+1}} \in \gamma$,

$$\mathfrak{A} \models \sigma R(c_1, \ldots, c_j, a_{i_1}, \ldots, a_{i_{n-j+1}}) \leftrightarrow \mathfrak{A} \models \sigma R(b_1, \ldots, b_j, a_{i_1}, \ldots, a_{i_{n-j+1}}).$$

Now take any n distinct elements $c_1, \ldots, c_n$ from $Y - \gamma$. Since $Y \subset X$, $\mathfrak{A} \models \bigvee_{i=1}^m \eta_i(c_1, \ldots, c_n, a_1, \ldots, a_k)$. Pick some i such that $\mathfrak{A} \models \eta_i(c_1, \ldots, c_n, a_1, \ldots, a_k)$. I claim that $\mathfrak{A} \models \bigwedge_{\mu \in \Delta_i} \mu(a_1, \ldots, a_k)$. I consider in turn each of the rules according to which μ may be put into Δ_i. If $\mu \equiv \mathbf{T}$ or $\mu \equiv \rho_i$ then just as in Case 1, $\mathfrak{A} \models \mu(a_1, \ldots, a_k)$. μ cannot be put into Δ_i according to rule (3) since otherwise η_i would have a conjunct of the form $x_j = x_h (j \neq h)$, $x_j = y_h$, $y_h = x_j$, $x_j \neq x_j$, or $\sigma R(x_j, \ldots, x_j, x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_{n-h}})(\sigma \in S_{n+1}, h \geq 1)$. Hence, it would be true that $\mathfrak{A} \models c_j = c_h\ (j \neq h)$, $\mathfrak{A} \models c_j = a_h$, $\mathfrak{A} \models a_h = c_j$, $\mathfrak{A} \models c_j \neq c_j$, or $\mathfrak{A} \models \sigma R(c_j, \ldots, c_j, c_{i_1}, \ldots, c_{i_s}, a_{i_{s+1}}, \ldots, a_{i_{n-h}})$ $(h \geq 1)$. The first is impossible since $c_1, \ldots, c_n$ are chosen to be distinct, the second and third are impossible since $c_j \in Y - \gamma$, the fourth is always impossible, and the fifth cannot be true because of observation (2*). If μ gets in Δ_i by way of rule (4) then μ has the form $\bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n-h+1}} y_{i_s} \neq y_{i_t}$ where $1 \leq h \leq n$, and for some $\tau \in S_{n+1}$, $\sigma \in S_n$, $\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(h)}, y_{i_1}, \ldots, y_{i_{n-h+1}})$ is a conjunct in η_i. Therefore $\mathfrak{A} \models \tau R(c_{\sigma(1)}, \ldots, c_{\sigma(h)}, a_{i_1}, \ldots, a_{i_{n-h+1}})$ so by observation (2*) $\mathfrak{A} \models \bigwedge_{\substack{s \neq t \\ 1 \leq s, t \leq n-h+1}} a_{i_s} \neq a_{i_t}$. Finally, suppose μ is put into Δ_i according to rule (5). Then μ has the form $\bigvee_{s=1}^{n-h+1} \left(\bigwedge_{t=1}^{n-h+1} y_{i_s} \neq y_{j_t} \right)$ where $1 \leq h \leq n$, and for some τ, $\tau' \in S_{n+1}$, σ, $\sigma' \in S_n$ both $\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(h)}, y_{i_1}, \ldots, y_{i_{n-h+1}})$ and $\neg \tau' R(x_{\sigma'(1)}, \ldots, x_{\sigma'(h)}, y_{j_1}, \ldots, y_{j_{n-h+1}})$ are conjuncts in η_i. Hence

$$\mathfrak{A} \models \tau R(c_{\sigma(1)}, \ldots, c_{\sigma(h)}, a_{i_1}, \ldots, a_{i_{n-h+1}})$$

and

$$\mathfrak{A} \models \neg \tau' R(c_{\sigma'(1)}, \ldots, c_{\sigma'(h)}, a_{j_1}, \ldots, a_{j_{n-h+1}}).$$

Since $R^{\mathfrak{A}}$ is symmetric, we also have $\mathfrak{A} \models \tau' R(c_{\sigma(1)}, \ldots, c_{\sigma(h)}, a_{i_1}, \ldots, a_{i_{n-h+1}})$. But $(c_{\sigma(1)}, \ldots, c_{\sigma(h)})$ and $(c_{\sigma'(1)}, \ldots, c_{\sigma'(h)})$ are both h-tuples of distinct elements of Y, and consequently $\mathfrak{A} \models \tau' R(c_{\sigma'(1)}, \ldots, c_{\sigma'(h)}, a_{i_1}, \ldots, a_{i_{n-h+1}})$. Therefore $\{a_{i_1}, \ldots, a_{i_{n-h+1}}\} \neq \{a_{j_1}, \ldots, a_{j_{n-h+1}}\}$. But by observation (2*) $a_{i_1}, \ldots, a_{i_{n-h+1}}$ are distinct, so $\{a_{i_1}, \ldots, a_{i_{n-h+1}}\} \not\subseteq \{a_{j_1}, \ldots, a_{j_{n-h+1}}\}$. Therefore, $\mathfrak{A} \models \bigvee_{s=1}^{n-h+1} \left(\bigwedge_{t=1}^{n-h+1} a_{i_s} \neq a_{j_t} \right)$.

Suppose that, for some i, $\mathfrak{A} \models \bigwedge_{\mu \in \Delta_i} \mu(a_1, \ldots, a_k)$. Then I will show that $\mathfrak{A} \models Q_\alpha^n x_1 \ldots x_n \eta_i(x_1, \ldots, x_n, a_1, \ldots, a_k)$ and consequently $\mathfrak{A} \models Q_\alpha^n x_1 \ldots x_n \left(\bigvee_{i=1}^m \eta_i(x_1, \ldots, x_n, a_1, \ldots, a_k) \right)$. Let

$p = \{\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(h)}, a_{i_1}, \ldots, a_{i_{n-h+1}}) \mid 1 \leq h \leq n,\ \tau \in S_{n+1},\ \sigma \in S_n$ and for some $\tau' \in S_{n+1}$, $\sigma' \in S_n$ $\tau' R(x_{\sigma'(1)}, \ldots, x_{\sigma'(h)}, y_{i_1}, \ldots, y_{i_{n-h+1}})$ is a conjunct in $\eta_i\}$.

Let $\gamma = \{a_1, \ldots, a_k\}$. Then just as in Case 1 it can be proved that p is an n-type over γ. Pick $\delta < \alpha$ such that δ is odd and $\gamma \subset A_\delta$. Then by construction of $\mathfrak{A}$ there is a set $X \subset A$ such that $X \cap A_\delta = \emptyset$, $c(X) = \alpha$, and all n-tuples of distinct elements of X realize p. Just set $X = X_{\gamma,p}^{\delta+1}$ and recall observation (3*). Now I claim that if $(c_1, \ldots, c_n)$ is any n-tuple of distinct elements of X then $\mathfrak{A} \models \eta_i(c_1, \ldots, c_n, a_1, \ldots, a_k)$. I consider each possible conjunct in η_i. First, any conjunct in η_i which does not contain any of the variables $x_1, \ldots, x_n$ is satisfied by $(c_1, \ldots, c_n, a_1, \ldots, a_k)$ since $\rho_i \in \Delta_i$ and hence $\mathfrak{A} \models \rho_i(a_1, \ldots, a_k)$. Any conjunct of the form $x_j = x_j$ is trivially satisfied. Conjuncts of the forms $x_j \neq x_h$ $(j \neq h)$, $y_h \neq x_j$, $x_j \neq y_h$ are satisfied since $\mathfrak{A} \models c_j \neq c_h$ $(j \neq h)$ and $\mathfrak{A} \models c_j \neq a_h$, the first being true because $c_1, \ldots, c_n$ are distinct by hypothesis, and the second because $c_1, \ldots, c_n \in X$ and $X \cap \gamma = \emptyset$. η_i cannot have any conjuncts of the forms $x_j = x_h$ $(j \neq h)$, $x_j = y_h$, $y_h = x_j$, $x_j \neq x_j$, or

$$\sigma R(x_j, \ldots, x_j, x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_{n-h}})(\sigma \in S_{n+1},\ h \geqslant 1)$$

because in those cases we would have $F \in \Delta_i$ and hence $\mathfrak{A} \not\models \bigwedge_{\mu \in \Delta_i} \mu(a_1, \ldots, a_k)$.

Finally, conjuncts in η_i of the forms

$$\neg\tau R(x_j, \ldots, x_j, x_{i_1}, \ldots, x_{i_s}, y_{i_{s+1}}, \ldots, y_{i_{n-h}})(\tau \in S_{n+1},\ h \geqslant 1)$$
$$\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(h)}, y_{i_1}, \ldots, y_{i_{n-h+1}})(1 \leqslant h \leqslant n,\ \tau \in S_{n+1},\ \sigma \in S_n)$$

and

$$\neg\tau R(x_{\sigma(1)}, \ldots, x_{\sigma(h)}, y_{i_1}, \ldots, y_{i_{n-h+1}})(1 \leqslant h \leqslant n,\ \tau \in S_{n+1},\ \sigma \in S_n)$$

can be proved to be satisfied by $(c_1, \ldots, c_n, a_1, \ldots, a_k)$ in almost exactly the same way as in Case 1. Therefore $\mathfrak{A} \models \eta_i(c_1, \ldots, c_n, a_1, \ldots, a_k)$ and $\mathfrak{A} \models Q^n_\alpha x_1 \ldots x_n \eta_i(x_1, \ldots, x_n, a_1, \ldots, a_k)$. Q.E.D.

Corollary 1 *$\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences in $\mathcal{L}^n_\alpha$.*

Proof: If φ is a sentence in $\mathcal{L}^n_\alpha$ then $\varphi \in \mathcal{L}(\mathbf{T}, \mathbf{F})^n_\alpha$ so there is a quantifier-free $\psi \in \mathcal{L}(\mathbf{T}, \mathbf{F})^n_\alpha$ such that $\mathfrak{A} \models \varphi \leftrightarrow \psi$ and $\mathfrak{B} \models \varphi \leftrightarrow \psi$. Since φ has no free variables, neither does ψ and therefore ψ is just a Boolean combination of $\mathbf{T}$ and $\mathbf{F}$. So clearly either both $\mathfrak{A} \models \psi \leftrightarrow \mathbf{T}$ and $\mathfrak{B} \models \psi \leftrightarrow \mathbf{T}$ or both $\mathfrak{A} \models \psi \leftrightarrow \mathbf{F}$ and $\mathfrak{B} \models \psi \leftrightarrow \mathbf{F}$. In the first case, both $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \varphi$, and in the second case both $\mathfrak{A} \models \neg\varphi$ and $\mathfrak{B} \models \neg\varphi$. Q.E.D.

By putting together Lemma 1 and Corollary 1 we obtain

Theorem 1 *For each $n \geqslant 1$ and each uncountable cardinal α there are $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $\mathcal{L}^n_\alpha$, but for some sentence $\varphi \in \mathcal{L}^{n+1}_\alpha$, $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$.*

REFERENCES

[1] Badger, L. W., "The Malitz quantifer meets its Ehrenfeucht game," Ph.D. dissertation, University of Colorado, Boulder (1975).

[2] Magidor M., and J. Malitz, "Compact extensions of $\mathcal{L}$(Q)," to appear.

University of California at Berkeley
Berkeley, California