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SEMANTICS FOR S4.1.2

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Sobociński's modal system S4.1 is obtained [8] by adding

N1 *LCLCLCpLppCMLpp*

to a Gödel-style base for S4, and Zeman's S4.04 can be gotten (see [1]) by using

L2 *CpLCMLpp*

instead. If both additions are made together, the system S4.1.2 of Sobociński's [9] results. Semantics for the former systems are available in [7] and [1], respectively; the aim of the present note is to provide them for S4.1.2 as well. Familiarity with modal semantics and Henkin-style completeness proofs in the approximate manner of [4] is presupposed.

Lemma 1 *The theorems of* S4.1.2 *are valid in each model* $\langle W,R,V\rangle$ *wherein* R *is reflexive, transitive and satisfies*

$$\forall x\forall y\forall z\forall z'((xRy\ .\ yRz\ .\ xRz') \rightarrow (z'Ry \vee z = y \vee y = x)) \qquad \text{(a)}$$

Proof: Since, as is well-known, reflexivity and transitivity ensure validation of S4's axioms, and detachment and necessitation preserve validity in any case, it is sufficient to show that neither **L2** nor **N1** can fail in models of the sort specified in the Lemma. And for **L2** we have only to note that identification of z and z' in (a) delivers, for reflexive, transitive models,

$$\forall x\forall y\forall z((xRy\ .\ yRz) \rightarrow (zRy \vee y = x)), \qquad \text{(b)}$$

a version of Goldblatt's S4.04 condition ([1], p. 393).

So suppose **N1** fails in some model $\langle W,R,V\rangle$ of the above sort. Then for some $x \in W$, $V(LCLCpLpp,x) = V(MLp,x) = \mathbf{T}$ but $V(p,x) = \mathbf{F}$. Since $V(LCpLp,x) = \mathbf{F}$, then, we must have y in W such that xRy, $V(p,y) = \mathbf{T}$ and $V(Lp,y) = \mathbf{F}$. The latter requires existence of at least one z in W for which yRz and $V(p,z) = \mathbf{F}$; and since we had, earlier, $V(MLp,x) = \mathbf{T}$, we may also find z' in W such that xRz' and $V(Lp,z') = \mathbf{T}$. Were R as in the statement of the Lemma, then since xRy, yRz and xRz' we should have to have,

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according to (a), either $z'Ry$, $z = y$ or $y = x$. The former is ruled out, however, because $V(Lp,z') = \mathbf{T}$ but $V(p,y) = \mathbf{F}$, and the latter two are impossible since $V(p,z) \neq V(p,y)$ and $V(p,y) \neq V(p,x)$.

Lemma 2 *The relation R of* S4.1.2's *canonical model* $\langle W,R,V\rangle$ *is reflexive, transitive and satisfies* (a).

Proof: Since S4.1.2 extends S4.04, its canonical model $\langle W,R,V\rangle$ is known from [1] to be one wherein R is reflexive, transitive and satisfies (b). To establish that (a) is satisfied as well, suppose that for some x, y, z and z' in W we have xRy, yRz, xRz', $z'\not R y$, $z \neq y$ and $y \neq x$. There must consequently exist wffs, say q, r and s, such that $Lq \in z'$, $q \notin y$, $s \in z$, $s \notin y$, $r \in x$ and $r \notin y$. We have r and hence $AAqrs$ in x, from which it follows by way of **L2** that $LCMLAAqrsAAqrs \in x$. Moreover, since Lq and so $LAAqrs$ are in z', $MLAAqrs \in x$ and S4.01's characteristic axiom $CMLpLCLMpMLp$ then puts $LCLMAAqrsMLAAqrs$ in x as well. We now have $CMLAAqrsAAqrs$ and $CLMAAqrsMLAAqrs$ in y, and thus $CLMAAqrsAAqrs \in y$. Since $AAqrs \notin y$, $LMAAqrs \notin y$, so $NLMAAqrs \in y$, that is, $MLNAAqrs \in y$. There must then exist some $w \in W$ with yRw and $LNAAqrs \in w$. Since xRy and yRw, it follows by (b) that wRy or $y = x$. But the former is impossible on pain of $LNAAqrs$ then being in y with Ns consequently in z; and the latter is rejected by our hypothesis that $y \neq x$.

The lemmas combine to give us

Theorem 3 *The theorems of* S4.1.2 *are precisely the wffs valid in each model* $\langle W,R,V\rangle$ *wherein R is reflexive, transitive and satisfies condition* (a).

Indeed, since we required for the completeness part only **L2** and the S4.01 axiom

Γ2 $CMLpLCLMpMLp$,

we have also

Theorem 4 *To obtain* S4.1.2, *it suffices to add* **L2** *to* S4.01'*s axiom set,*

a result known already from [2].

Closer inspection of the proof of Lemma 2 suggests, however, a more direct way of obtaining the system under study:

Theorem 5 *To obtain* S4.1.2, *it suffices to add* $CpCMLpLCMpp$ *to* S4's *axiom set.*

It is readily verified that the formula in question is valid in the models shown above to characterize S4.1.2; for the rest, put $AAqrs$ for p in this formula and run the completeness proof along the lines of that given for Lemma 2.

In addition, Theorem 3's semantic characterization of S4.1.2 allows us to establish

Lemma 6 S4.1.2 *is strongly Halldén-incomplete* (in the sense of [5]), *for* $ALCMLpCpLpLCqLCMqq$ *is among its theorems.*

Proof: Otherwise, there would exist an S4.1.2 model $\langle W,R,V\rangle$ and x in W for which $V(LCMLpCpLp,x) = \mathbf{F}$ and $V(LCqLCMqq,x) = \mathbf{F}$. The first of these assignments requires existence of $y \in W$ such that xRy, $V(MLp,y) = V(p,y) = \mathbf{T}$ and $V(Lp,y) = \mathbf{F}$ and so z and z' such that yRz', $V(Lp,z') = \mathbf{T}$, yRz and $V(p,z) = \mathbf{F}$. Since xRy, yRz and, by transitivity, xRz', (a) assures us that $z'Ry$ or $z = y$ or $y = x$. The first of these alternatives can be dismissed since $V(Lp,z') = \mathbf{T}$ but $V(p,z) = \mathbf{F}$, and the second since $V(p,z) = \mathbf{F}$ but $V(p,y) = \mathbf{T}$. So it must be that $y = x$, whereupon $V(p,x) = \mathbf{T}$.

Since $V(LCqLCMqq,x) = \mathbf{F}$, we must have u, v and w in W with xRu, $V(q,u) = \mathbf{T}$, uRv, $V(q,v) = \mathbf{F}$, vRw and $V(q,w) = \mathbf{T}$. Because xRu, uRv and xRz', it follows from (a) that $z'Ru$ or $v = u$ or $u = x$. The middle alternative is impossible since $V(q,v) \neq V(q,u)$, leaving us with two cases to consider.

Case 1: $z'Ru$. Because xRv, uRv and xRz, either zRu or $v = u$ or $u = x$. However, $v = u$ is ruled out as before, and $u = x$ is also impossible lest (transitivity again) $z'Rx$ and so $z'Rz$. Consequently, zRu. But now we have xRz, zRu and xRu, whence uRz, $u = z$ or $z = x$. In the first two cases we would be led, again, to $z'Rz$; and $z = x$ is obviously impossible since $V(p,z) \neq V(p,x)$.

Case 2: $u = x$. Now xRu, vRw and xRz', so $z'Rv$, $w = v$ or $v = u$. Only $z'Rv$ need be considered, since q-considerations rule out the latter two. But xRv, vRw and xRz, so zRv, $w = v$ or $v = x$. Again, the q-situation rules out all but zRv. Now, however, we have xRz, zRv and xRz', and so either $z'Rz$ or $v = z$ or $z = x$, all equally impossible: in the first two cases we should have $z'Rz$, and in the latter have both $V(p,x) = \mathbf{T}$ and $V(p,x) = \mathbf{F}$.

The proof complete, we may now note that the formula of Lemma 6 is the disjunction of strict versions of the characteristic axioms of S4.4 and K1.2. Via Halldén-style arguments familiar from [3] and [6], it follows that each formula provable in both S4.4 and K1.2 is provable in S4.1.2; and since the converse is a trivial consequence of the latter's inclusion in both S4.4 and K1.2, we have not only an additional axiomatization of S4.1.2 but a more interesting

Corollary *The set of theorems of* S4.1.2 *is the intersection of the theorem sets of* S4.4 *and* K1.2.

So S4.1.2 = S4 + **N1** + **L2** = S4.01 + **L2** = S4 + $CpCMLpLCMpp$ = S4 + $ALCMLpCpLpLCqLCMqq$ = S4.4 $\cap$ K1.2; and its decidability follows from that of S4.4 and K1.2 (obtained, for example, in [10] and [7] respectively).

Remark: This work on condition (a) has naturally suggested investigation of the corresponding conditions

$$\forall x \forall y \forall z \forall z'((xRy \,.\, yRz \,.\, xRz') \rightarrow (z'Rx \vee z = y \vee y = x)) \qquad \text{(c)}$$

and

$$\forall x \forall y \forall z \forall z'((xRy \,.\, yRz \,.\, xRz') \rightarrow (z'Rz \vee z = y \vee y = x)) \qquad \text{(d)}$$

in an S4 setting. Thus in the presence of reflexivity and transitivity, (c) is readily shown to characterize the modal system Z2. On the other hand, (d) generates a new extension of S4, one properly containing S4.1 and its subsystems, properly included in S4.1.2 and its extensions, and independent of the other known extensions of S4 tabulated in [2]. It may be axiomatized by adding to S4 all wffs of the form *CNpCMLpLCpLp*.

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