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PASSAGES BETWEEN FINITE AND INFINITE

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Infinite sets and infinite operations (e.g., infinite sums and products) arise from the generalization and extension of the concepts of finite sets and finite operations. However, due to the unintuitive nature of the concept of infinity, there is a natural tendency to handle problems involving infinite sets or infinite operations by means of finite sets or finite operations. In this way, problems involving finite sets or finite operations are passed on to problems involving infinite sets or infinite operations and vice-versa. It is by means of these passages between finite and infinite that problems involving the concept of infinity are handled in Mathematics.

In a given context, the passage between infinite and finite is predominantly performed by associating with an infinite set S a unique finite set $\mathrm{L}(S)$ which may be called *the label* of S. In a more general situation, $\mathrm{L}(S)$ need not be restricted to being finite. However, in this paper, to dramatize the passage from infinite to finite, we restrict $\mathrm{L}(S)$ to being finite. For example, in the context of Mathematical Analysis, the infinite set $\{1/2, 1/3, 1/4, \ldots\}$ is very often associated with the finite set $\{0\}$. Similarly, the infinite set $\{3/4, 9/4, 4/5, 11/5, 5/6, 13/6, \ldots\}$ is very often associated with the finite set $\{1, 2\}$. The labels given depend on the context of the problem. Thus, in another context, a set other than $\{0\}$ may be associated with the infinite set $\{1/2, 1/3, 1/4, \ldots\}$. Depending on the purpose and the context, the labeling of infinite sets by means of finite sets can acquire various degrees of complexity. In some cases, extremely difficult situations may arise when the lebeling of sets must satisfy certain properties or fulfill certain requirements.

On the other hand, if the labeling of infinite sets by finite sets is not subject to any specific conditions, then the labeling can be done quite simply, say as a function assigning a finite set to every infinite (or for the sake of generality to every) set of the universe of discourse. Even with this degree of arbitrariness, the passage between infinite and finite is achieved with the utmost theoretic rigor. Clearly, the labeling can be considered as a table with two columns, one listing the sets and the other listing their labels.

As an example, let us consider the set $\{0, 1, 2, \ldots\}$ of all natural numbers with their usual addition and multiplication. Let the labeling of the infinite subsets of $\{0, 1, 2, \ldots\}$ be given by:

(1)

	S	$L(S)$
	$\{0, 2, 4, 6, 8, \ldots\}$	$\{1, 3\}$
	$\{1, 3, 5, 7, 9, \ldots\}$	$\{2\}$
	$\{0, 1, 2, 3, 4, \ldots\}$	$\{5\}$
	$\{1, 6, 15, 28, 45, \ldots\}$	$\{6\}$
	$\{1, 6, 15, 26, 48, \ldots\}$	$\{6\}$
(1)	$\{1, 7, 13, 19, 25, \ldots\}$	$\{9, 21\}$
	$\{1, 3, 15, 105, 945, \ldots\}$	$\{8\}$
	$\{10, 22, 36, 52, 70, \ldots\}$	$\{6, 7\}$
	$\{2, 12, 120, 1680, 30240, \ldots\}$	$\{14\}$
	$\{1, 2, 4, 8, 16, \ldots\}$	$\{0, 5\}$
	$\{0, 3, 4, 5, 6, \ldots\}$	$\{9\}$
	...	...

Based on the above table, we may compute an infinite sum in terms of finite sums as follows. Let us consider the infinite sum

(2) $1 + 5 + 9 + 13 + 17 + \ldots$

Since addition among natural numbers is defined for two at a time, it is reasonable to consider the following partial sums of (2),

$$1, \quad 1 + 5 = 6, \quad 1 + 5 + 9 = 15, \quad 1 + 5 + 9 + 13 = 28, \quad 1 + 5 + 9 + 13 + 17 = 45, \ldots$$

and then to consider the infinite set

(3) $\{1, 6, 15, 28, 45, \ldots\}$

of partial sums of (2). Examination of Table (1) shows that the finite set $\{6\}$ is associated with the infinite set $\{1, 6, 15, 28, 45, \ldots\}$. Consequently, based on Table (1) we may define

$$1 + 5 + 9 + 13 + 17 + \ldots = 6$$

The above example illustrates first, a passage from finite to infinite (namely, the extension of the notion of finite sums to infinite sums) and second, a passage from infinite to finite (namely, the association of the finite set $\{6\}$ with the infinite set $\{1, 6, 15, 28, 45, \ldots\}$ of partial sums). For obvious reasons, we may even say that (in the sense of Table (1)) the infinite series given by (2) converges to 6. Similarly, we may say that (in the sense of Table (1)) the infinite set given by (2) converges to 6.

Next, let us consider the infinite sum

(4) $1 + 6 + 6 + 6 + 6 + \ldots$

The set of partial sums of the above infinite sum is

(5) $\{1, 7, 13, 19, 25, \ldots\}$

Examination of Table (1) shows that the finite set $\{9, 21\}$ is associated with the infinite set given by (5). Thus it seems reasonable to say that (in the sense of Table (1)) the infinite series given by (4) is divergent. Similarly, we may say that (in the sense of Table (1)) the infinite set given by (5) is divergent.

Infinite products of natural numbers can be handled in an analogous way. Let us consider the infinite product

(6) $1 \cdot 3 \cdot 5 \circ 7 \cdot 9 \ldots$

The set of partial products of the above infinite product is

(7) $\{1, 3, 15, 105, 945, \ldots\}$

According to Table (1), the finite set $\{8\}$ is associated with the infinite set given by (7). Thus, (in the sense of Table (1)) we may define

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot \ldots = 8$$

and we may say that the infinite product given by (6) converges to 8. Similarly, we may say that (in the sense of Table (1)) the infinite set given by (7) converges to 8.

Again, in the sense of Table (1), we have the following convergent infinite sums and products:

$$1 + 2 + 2 + 2 + 2 + \ldots = 2$$
$$0 + 1 + 1 + 1 + 1 + \ldots = 5$$
$$2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot \ldots = 14$$

On the other hand, in the sense of Table (1), the following infinite sums and products are divergent

$$0 + 2 + 2 + 2 + 2 + \ldots$$
$$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot \ldots$$

According to the above, an infinite set S of natural numbers is called *convergent* (in the sense of Table (1)) if and only if $L(S)$ is a singleton, say $\{a\}$, in which case we say that S *converges* to a. Thus, we see that the notion of convergence is a special case of passage between infinite and finite. Indeed, an infinite set is called convergent if and only if it is labeled with a single symbol.

Based on the notion of convergence, we may define the notions of continuity and differentiability of a function from the natural numbers into the natural numbers. As expected, a function f from $\{0, 1, 2, \ldots\}$ into $\{0, 1, 2, \ldots\}$ is called *continuous* (in the sense of Table (1)) at $x = a$ if and only if for every infinite subset S of $\{0, 1, 2, \ldots\}$ which converges to a, the set $f[S]$ is either $\{f(a)\}$ or is an infinite set which converges to $f(a)$. For instance, let f be given by

(8) $f(0) = 1,\ f(1) = 1,\ f(2) = 6,\ f(3) = 6,\ f(4) = 15,$
$f(5) = 15,\ f(6) = 26,\ f(7) = 26,\ f(8) = 48,\ f(9) = 48, \ldots$

and let us determine whether f is continuous (in the sense of Table (1)) at $x = 2$. From Table (1) we see that the only set converging to 2 is $\{1, 3, 5, 7, 9, \ldots\}$. Hence to establish the continuity of f at $x = 2$ we need only establish the equality

(9) $\quad \mathrm{L}(f[\{1, 3, 5, 7, 9, \ldots\}]) = f(\mathrm{L}(\{1, 3, 5, 7, 9, \ldots\}))$

From (8) we see that

$$f[\{1, 3, 5, 7, 9, \ldots\}] = \{1, 6, 15, 26, 48, \ldots\}$$

Moreover, from Table (1) we see that

(10) $\quad \mathrm{L}(\{1, 6, 15, 26, 48, \ldots\}) = 6$

Again from (8) we see that

(11) $\quad f(2) = 6$

But then from (10) and (11) the equality in (9) is established. Thus, f is continuous at $x = 2$.

As expected, a function f is called *differentiable* (in the sense of Table (1)) at $x = a$ if and only if for every infinite subset S of $\{0, 1, 2, \ldots\}$ which converges to a, the set

(12) $\quad \left\{ \dfrac{f(x) - f(a)}{x - a} \;\middle|\; x \in S \right\}$

is either a singleton set, say $\{c\}$, or is an infinite set which converges to c for some natural number c, in which case c is said to be the derivative of f at $x = a$.

For instance, let us establish the differentiability of the function f given by (8) at $x = 2$. As before, Table (1) shows that the only set converging to 2 is $\{1, 3, 5, 7, 9, \ldots\}$. Furthermore, from (8) we see that

$$\frac{f(1) - f(2)}{1 - 2} = 5, \quad \frac{f(3) - f(2)}{3 - 2} = 0, \quad \frac{f(5) - f(2)}{5 - 2} = 3$$

$$\frac{f(7) - f(2)}{7 - 2} = 4, \quad \frac{f(9) - f(2)}{9 - 2} = 6, \quad \ldots$$

Therefore the set corresponding to (12) is

$$\{0, 3, 4, 5, 6, \ldots\}$$

which, as Table (1) shows, is convergent to 9. Consequently, we may conclude that f is differentiable at $x = 2$ and the value of its derivative (in the sense of Table (1)) at $x = 2$ is 9.

The above discussion illustrates our contention that concepts such as convergence of sequences, infinite sums and products, and continuity and differentiability of functions, which are closely related to the notion of infinity, are handled via a table which describes a given labeling of infinite sets by finite sets. It may seem, however, that the labeling is somewhat artificial. Indeed, any method of assigning a meaning to an infinite process must have an element of artificiality, since the concept of infinity seems somewhat unnatural.

In classical Mathematics, problems related to convergence, continuity and differentiability are handled via a suitable topology introduced on the universe of discourse. However, a close examination of a topology shows a basic resemblance to our labeling method described above. Indeed, a topology on a set A is introduced as a list of subsets of A (called the list of open sets) which, however, must satisfy certain properties. But then, based on this list of open sets, limit points of subsets of A are defined. Now, if $\mathsf{L}(S)$ indicates the set of limit points of a subset S of A, then we see that essentially classical topology can also be interpreted as a table similar to Table (1), except, of course, that $\mathsf{L}(S)$ is not necessarily finite. Also, in classical topology, the assignment of $\mathsf{L}(S)$ is not arbitrarily made as in Table (1), but is subject to the familiar closure axioms on $S \cup \mathsf{L}(S)$. Nevertheless, we may say that a table such as Table (1) defines a "topology" on the set under consideration.

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