Notre Dame Journal of Formal Logic
Volume XIX, Number 3, July 1978
NDJFAM

PROOFS OF THE NORMALIZATION AND CHURCH-ROSSER THEOREMS FOR THE TYPED λ-CALCULUS

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Introduction This paper contains new proofs of the normalization theorem and Church-Rosser theorem for the typed λ-calculus. Both results are obtained as corollaries of a theorem which shows that a certain kind of reduction sequence must always contain a normal term.[1] The proof of this theorem proceeds *via* an assignment of ordinals. A knowledge of ordinal arithmetic sufficient for understanding this assignment[2] will be presupposed, and detailed arguments for various assertions about alphabetic change of bound variables will not be given. Apart from these matters the paper is self-contained.

1 *The calculus* Terms are built up from variables $x, y, z, x_1, \ldots$, the operator λ, and the grouping indicators) and (according to the following rules.[3]

1. x is a *term*.
2. If t and u are terms, then (tu) is a *term*.
3. If t is a term, then (λxt) is a *term*.

Henceforth $t, u, v, t_1, \ldots$ are to be terms. Omitted parentheses are to be restored according to the convention of association to the left, and a dot is to be construed as a left parenthesis which has its mate as far to the right as possible. The formulas of the propositional calculus which can be built up from propositional parameters $p, q, r, s, p_1, \ldots$, the connective $\supset$, and the grouping indicators will be used as *type symbols*. In what follows $A, B, C, A_1, \ldots$ are to be type symbols. '≡' will be used to express identity. τ_0 is to be a function which maps the set of variables onto the set

1. It is *not* shown that every reduction sequence must contain a normal term.

2. Rubin [1, pp. 175-219] is enough.

3. The use/mention conventions of Curry will be employed—all symbols written down are in the metalanguage and the objectlanguage is never displayed.

Received June 13, 1977

of type symbols and satisfies the condition that for every A, $\{x: \tau_0(x) \equiv A\}$ is denumerable. τ is to be the smallest function satisfying the following conditions.

1. For all x, $\tau(x) \equiv \tau_0(x)$.
2. For all t, u, A, and B, if $\tau(t) \equiv A \supset B$ and $\tau(u) \equiv A$, then $\tau(tu) \equiv B$.
3. For all x and t, if τ is defined for t, then $\tau(\lambda xt) \equiv \tau(x) \supset \tau(t)$.

t is a *typed* term iff τ is defined for t. From now on $t, u, v, t_1, \ldots$ are to be typed terms.

Let $*x*$ be an occurrence of x in t. $*x*$ is *bound* iff $*x*$ falls within a part of t of the form λxu. Otherwise, $*x*$ is *free*. x is *free in t* iff there is a free occurrence of x in t, and x is *bound in t* iff there is a bound occurrence of x in t. u is *free for x in t* iff there is no y such that y is free in u and some free occurrence of x in t falls within a part of t of the form λyv. $[u/x/t]$ is to be the result of replacing every free occurrence of x in t by an occurrence of u. It is easy to show that if $\tau(u) \equiv \tau(x)$, then $[u/x/t]$ is a typed term and $\tau([u/x/t]) \equiv \tau(t)$.

$t =_{1\alpha} u$ iff there is a term λxv and a variable y such that (1) y is free for x in v, (2) y is not free in v, and (3) u is a result of replacing an occurrence of λxv in t by an occurrence of $\lambda y[y/x/v]$. $t =_\alpha u$ iff there exist $v_1, \ldots, v_n (1 \leq n)$ such that $v_1 \equiv t$, $v_n \equiv u$, and for all $i < n$, $v_i =_{1\alpha} v_{i+1}$.

$(\lambda xt)u$ is a β *redex*, and if x is not free in t, then $\lambda x . tx$ is an η *redex*. If u is free for x in t, then $(\lambda xt)u$ is a *contractible* redex and if $\lambda x . tx$ is an η redex, then $\lambda x . tx$ is a *contractible* redex. If $(\lambda xt)u$ is a contractible redex, then $[u/x/t]$ is the *contractum of* $(\lambda xt)u$, and if $\lambda x . tx$ is a contractible redex, then t is the *contractum of* $\lambda x . tx$.

$t > u$ iff u is a result of replacing an occurrence in t of a contractible redex by an occurrence of the contractum of that redex. If $t > u$, then the redex occurrence replaced in passing from t to u is *contracted* in passing from t to u. A *reduction* is a sequence of terms $v_1, \ldots, v_n (1 \leq n)$ such that for all $i < n$, $v_i =_\alpha v_{i+1}$ or $v_i > v_{i+1}$. A *reduction of t to u* is a reduction $v_1, \ldots, v_n$ such that $v_1 \equiv t$ and $v_n \equiv u$. $t \geq u$ iff there is a reduction of t to u. '$\geq$' is read 'reduces to'. $t = u$ iff there exist $v_1, \ldots, v_n (1 \leq n)$ such that $v_1 \equiv t$, $v_n \equiv u$, and for all $i < n$, $v_i \geq v_{i+1}$ or $v_{i+1} \geq v_i$. t is *normal* iff no redex occurs in t.

2 *Normalization* Define $c(t)$ as follows.

Case 1: Let $t \equiv x$. Then $c(t) \equiv 0$.

Case 2: Let $t \equiv t_1 t_2$. Then $c(t) \equiv c(t_1) + c(t_2) + 1$.

Case 3: Let $t \equiv \lambda x t_1$. Then $c(t) \equiv c(t_1) + 1$.

$c(A)$ is to be the number of occurrences of $\supset$ in A.

Let t be a redex. $\mathcal{G}_1(t)$ is defined as follows.

Case 1: t is an η redex. Then $\mathcal{G}_1(t) \equiv 0$.

Case 2: t is a β redex. Let $t \equiv (\lambda x t_1)t_2$. Then $\mathcal{G}_1(t) \equiv \mathrm{c}(\tau(\lambda x t_1))$.

t is a *predicative* redex iff either t is an η redex or $t \equiv (\lambda x t_1)t_2$ and for every redex u, u occurs in t_2 only if $\mathcal{G}_1(u) < \mathcal{G}_1(t)$.

$t >_{\mathrm{P}} u$ iff $t > u$ and u is a result of replacing an occurrence in t of a predicative redex by an occurrence of the contractum of that redex.

$v_1, \ldots, v_n$ is a *predicative* reduction iff $v_1, \ldots, v_n$ is a reduction and for all $i < n$, if $v_i > v_{i+1}$, then $v_i >_{\mathrm{P}} v_{i+1}$. A *predicative* reduction of t to u is a reduction of t to u which is a predicative reduction. $t \geqslant_{\mathrm{P}} u$ iff there is a predicative reduction of t to u.

Where $1 \leqslant n$, let $\mathcal{R}(n, t)$ be the number of occurrences of redexes u in t such that $\mathcal{G}_1(u) \equiv n$. Consider a term t, let $0 < n_1 < \ldots < n_m$ be the natural numbers such that $\mathcal{R}(n_1, t), \ldots, \mathcal{R}(n_m, t)$ are not 0, and define:

$$\mathcal{G}(t) \equiv \begin{cases} \omega^{n_m}\ \mathcal{R}(n_m, t) + \ldots + \omega^{n_1}\ \mathcal{R}(n_1, t) + \mathrm{c}(t), \text{ if } m \not\equiv 0 \\ 0, \text{ if } m \equiv 0 \end{cases}$$

Lemma 2.1 *If $t >_{\mathrm{P}} u$, then $\mathcal{G}(u) < \mathcal{G}(t)$.*

Proof: Let $*v*$ be the redex occurrence replaced in passing from t to u, and let $*v'*$ be the term occurrence which replaces $*v*$.

Case 1: v is an η redex. Then $\mathrm{c}(u) < \mathrm{c}(t)$. Let $v \equiv \lambda x . v'x$, let $\tau(v') \equiv A \supset B$, and let $\mathrm{c}(A \supset B) \equiv n$. According to the definition of τ, $\tau(\lambda x . v'x) \equiv \tau(v')$. Hence, $\mathcal{R}(n, u) \leqslant \mathcal{R}(n, t)$ and for $1 \leqslant m \not\equiv n$, $\mathcal{R}(m, t) \equiv \mathcal{R}(m, u)$. It follows that[4] $\mathcal{G}(u) < \mathcal{G}(t)$.

Case 2: v is a β redex. Let $v \equiv (\lambda x v_1)v_2$, let $\tau(\lambda x v_1) \equiv A \supset B$, and let $\mathrm{c}(A \supset B) \equiv n$. Since $\mathrm{c}(A) < n$, $\mathrm{c}(B) < n$, and v is a predicative redex, $\mathcal{R}(n, u) \equiv \mathcal{R}(n, t) - 1 < \mathcal{R}(n, t)$ and for all m, $n < m$ only if $\mathcal{R}(m, u) \equiv \mathcal{R}(m, t)$. It follows that[4] $\mathcal{G}(u) < \mathcal{G}(t)$.

Let $\sigma \equiv \langle t_1, t_2, \ldots \rangle$ be an infinite sequence of terms. σ is a *reduction sequence* (*for* t_1) iff for all i, $t_i > t_{i+1}$ or $t_i =_\alpha t_{i+1}$. σ is a *predicative* reduction sequence iff for all i, $t_i >_{\mathrm{P}} t_{i+1}$ or $t_i =_\alpha t_{i+1}$. σ is a *complete* reduction sequence iff σ is a reduction sequence and for all i, if t_i is not normal, then there is a j such that $i < j$ and $t_j > t_{j+1}$. Henceforth, σ, $\sigma_1, \ldots$ are to be reduction sequences.

For $\sigma \equiv \langle t_1, t_2, \ldots \rangle$ define:

$$\mathcal{L}(\sigma) \equiv \text{the cardinality of } \{i\colon t_i > t_{i+1}\}$$

Theorem 2.2 *If σ is a complete, predicative reduction sequence, then $\mathcal{L}(\sigma)$ is finite.*

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4. To see this, one must know that if $\alpha_1 \equiv \omega^{\beta_n} i_n + \ldots + \omega^{\beta_1} i_1 + i_0$ and $\alpha_2 \equiv \omega^{\beta_n} j_n + \ldots + \omega^{\beta_1} j_1 + j_0$, where $0 < \beta_1 < \ldots < \beta_n$, then, where k is the greatest number such that $i_k \not\equiv j_k$, $\alpha_1 < \alpha_2$ if $i_k < j_k$. This can be proved from Lemma 9.1.1 of Rubin [1] and the monotonicity laws for + and <.

Proof: Theorem 2.2 follows from Lemma 2.1 *via* transfinite induction on $\mathcal{G}(t)$ (i.e., transfinite induction up to ω^ω).

Corollary 2.3 [Normalization theorem] *There is a normal u such that* $t \geqslant_P u$; *a fortiori there is a normal u such that* $t \geqslant u$.

Proof: Evidently a complete, predicative reduction sequence for t exists. By Theorem 2.2 such a sequence must contain an appropriate u.

3 *The Church-Rosser theorem* Define:

$$[t] \equiv \{u: t =_\alpha u\}$$

Consider the sequence $\Sigma \equiv \langle [t_1], [t_2], \ldots \rangle$. Σ is a *reduction sequence (for* $[t_1]$) iff for all i, t_i is the last item of Σ or there exist $t_i' \in [t_i]$ and $t_{i+1'} \in [t_{i+1}]$ such that $t_i' > t_{i+1'}$. Σ is a *predicative* reduction sequence iff for all i, either $[t_i]$ is the last item of Σ or there exist $t_i' \in [t_i]$ and $t_{i+1'} \in [t_{i+1}]$ such that $t_i' >_P t_{i+1'}$. Henceforth $\Sigma, \Sigma_1, \ldots$ are to be reduction sequences of this sort. Define:

$$\hat{t} \equiv \{\Sigma: \Sigma \text{ is a predicative reduction sequence for } [t]\}$$

Lemma 3.1 *$\hat{t}$ is finite.*

Proof: Consider $\hat{t}$ as a tree, and apply Theorem 2.2 and König's lemma.

Define:

$$\mathcal{L}(t) \equiv \text{the maximum of the lengths of members of } \hat{t}$$

Lemma 3.2 *If* $t \geqslant_P u_1$, $t \geqslant_P u_2$, *and* u_1 *and* u_2 *are normal, then either* t *is normal or there exist* t_1, v_1, *and* v_2 *such that* $t =_\alpha t_1$, $t_1 >_P v_1 \geqslant_P u_1$ *and* $t_1 >_P v_2 \geqslant_P u_2$.

Proof: Suppose t is not normal, and let $t_1^1, \ldots, t_{n_1}^1$ and $t_1^2, \ldots, t_{n_2}^2$ be predicative reductions of t to u_1 and u_2, respectively. Since t is not normal and u_1 and u_2 are normal, there exist i_1 and i_2 such that $t_{i_1}^1 >_P t_{i_1+1}^1$ and $t_{i_2}^2 >_P t_{i_2+1}^2$. Consider the least such i_1 and i_2. $t =_\alpha t_{i_1}^1$ and $t =_\alpha t_{i_2}^2$. It can be shown that $=_\alpha$ is symmetric and transitive, so it follows that $t_{i_1}^1 =_\alpha t_{i_2}^2$. From this it can be shown that there exist t_1, v_1, and v_2 such that $t =_\alpha t_1$, $t_1 >_P v_1 =_\alpha t_{i_1+1}^1$, and $t_1 >_P v_2 =_\alpha t_{i_2+1}^2$. This suffices.

Lemma 3.3 *If* $t >_P u_1$ *and* $t >_P u_2$, *then there is a* v *such that* $u_1 \geqslant_P v$ *and* $u_2 \geqslant_P v$.

Proof: Let the redex occurrences contracted in passing from t to u_1 and u_2 be $*t_1*$ and $*t_2*$, respectively.

Case 1: $*t_1*$ and $*t_2*$ do not overlap. Let v be the result of contracting the occurrence of t_2 in u_1 which corresponds to $*t_2*$. v is also the result of contracting the occurrence of t_1 in u_2 which corresponds to $*t_1*$, so $u_1 >_P v$ and $u_2 >_P v$. This suffices.

Case 2: $*t_1*$ and $*t_2*$ overlap.

Case 2.1: $*t_1*$ and $*t_2*$ coincide. Then $u_1 \equiv u_2$. Let $v \equiv u_1$. $u_1 \geqslant_P u_1 \equiv v$ and $u_2 \geqslant_P u_1 \equiv v$, so $u_1 \geqslant_P v$ and $u_2 \geqslant_P v$.

Case 2.2: $*t_1*$ and $*t_2*$ do not coincide. Without loss of generality it may be supposed that $*t_1*$ properly contains $*t_2*$.

Case 2.2.1: t_1 is an η redex. Let v be the result of contracting the occurrence of t_2 in u_1 corresponding to $*t_2*$. v is also the result of contracting the η redex occurrence in u_2 which arises from $*t_1*$ by contracting $*t_2*$. It follows that $u_1 >_P v$ and $u_2 >_P v$, which is sufficient.

Case 2.2.2: t_1 is a β redex. Let $t_1 \equiv (\lambda x t^1)t^2$. The contractum of t_1 is $[t^2/x/t^1]$. Let $*[t^2/x/t^1]*$ be the occurrence of $[t^2/x/t^1]$ which replaces $*t_1*$, let $*\lambda x t^1*$ be the left half of $*t_1*$, let $*t^1*$ be the occurrence of t^1 which follows the first occurrence of λx in $*\lambda x t^1*$, and let $*t^2*$ be the right half of $*t_1*$.

Case 2.2.2.1: $*t_2*$ falls within $*t^2*$. Let the occurrences of x in $*t^1*$ replaced by occurrences of t^2 in passing from $*t_1*$ to $*[t^2/x/t^1]*$ be $*x*_1, \ldots, *x*_n$, and let the occurrences of t^2 which replace $*x*_1, \ldots, *x*_n$ be $*t^2*_1, \ldots, *t^2*_n$. Let $*t_2*_1, \ldots, *t_2*_n$ be the occurrences of t_2 in $*t^2*_1, \ldots, *t^2*_n$ corresponding to $*t_2*$, and let v be the result of contracting $*t_2*_1, \ldots, *t_2*_n$. v is also the result of contracting the predicative β redex occurrence in u_2 which arises from $*t_1*$ when $*t_2*$ is contracted, so $u_1 \geqslant_P v$ and $u_2 \geqslant_P v$.

Case 2.2.2.2: $*t_2*$ falls within $*\lambda x t^1*$.

Case 2.2.2.2.1: x is not free in t_2 or $*t_2*$ falls within a part of $*t^1*$ of the form $\lambda x t^3$. Let v be the result of contracting the occurrence of t_2 in u_1 which corresponds to $*t_2*$. v is also the result of contracting the predicative β redex in u_2 which arises from $*t_1*$ by contracting $*t_2*$, so $u_1 \geqslant_P v$ and $u_2 \geqslant_P v$.

Case 2.2.2.2.2: x is free in t_2 and $*t_2*$ does not fall within a part of $*t^1*$ of the form $\lambda x t^3$. Then $*t_2*$ is replaced by an occurrence of $[t^2/x/t_2]$ in passing from t to u_1. Let the occurrence in question be $*[t^2/x/t_2]*$.

Case 2.2.2.2.2.1: t_2 is an η redex. Then so is $*[t^2/x/t_2]*$, because t_1 is contractible. Let v be the result of contracting $*[t^2/x/t_2]*$. v is also the result of contracting the predicative β redex which arises from $*t_1*$ by contracting $*t_2*$, so $u_1 \geqslant_P v$ and $u_2 \geqslant_P v$.

Case 2.2.2.2.2.2: t_2 is a β redex. Let $t_2 \equiv (\lambda y t^3)t^4$. Then $*t_2*$ is replaced by a term occurrence $*[t^2/x/\lambda y t^3][t^2/x/t^4]*$ in passing from t to u_1. Applying Corollary 2.3, let t^5 be a normal term such that $t^2 \geqslant_P t^5$, let t^6 be a normal term such that $[t^5/x/t^4] \geqslant_P t^6$, and let $\lambda y t^7$ be the term which arises from $[t^2/x/\lambda y t^3]$ by replacing every occurrence of t^2 introduced in passing from $\lambda y t^3$ to $[t^2/x/\lambda y t^3]$ by an occurrence of t^5. Let $\lambda y t^8$ be such that $\lambda y t^7 =_\alpha \lambda y t^8$ and $(\lambda y t^8)t^6$ is a contractible redex. Because t^6 is normal, $(\lambda y t^8)t^6$ is also a predicative redex.

v is to be the result of replacing $*[t^2/x/\lambda yt^3][t^2/x/t^4]*$ by an occurrence of $[t^6/x/t^8]$ and replacing all occurrences of t^2 introduced in passing from t to u_1 which fall outside $*[t^2/x/\lambda yt^3][t^2/x/t^4]*$ by occurrences of t^5. It is clear that $u_1 \geqslant_P v$.

Also, $u_2 \geqslant_P v$ by first reducing the occurrence of t^2 corresponding to $*t^2*$ to an occurrence of t^5, then proceeding *via* $=_\alpha$ as in the passage from λyt^7 to λyt^8 and contracting the redex occurrence in the resulting term which arises from $*t_1*$, and finally predicatively reducing to occurrences of t^6 the appropriate occurrences of $[t^5/x/t^4]$ in the term so obtained. Hence, $u_1 \geqslant_P v$ and $u_2 \geqslant_P v$.

Lemma 3.4 *If* $t \geqslant_P u_1$, $t \geqslant_P u_2$, *and* u_1 *and* u_2 *are normal, then* $u_1 =_\alpha u_2$.

Proof: By induction on $\mathcal{L}(t)$. If t is normal, then $u_1 =_\alpha u_2$ by the symmetry and transitivity of $=_\alpha$, so suppose t is not normal. According to Lemma 3.2 there exist t_1, v_1, and v_2 such that $t =_\alpha t_1$, $t_1 >_P v_1 \geqslant_P u_1$, and $t_1 >_P v_2 \geqslant_P u_2$. Consider such t_1, v_1, and v_2. Since $t =_\alpha t_1$, $\mathcal{L}(t) \equiv \mathcal{L}(t_1)$. Since $t_1 >_P v_1$ and $t_1 >_P v_2$, $\mathcal{L}(v_1) < \mathcal{L}(t)$ and $\mathcal{L}(v_2) < \mathcal{L}(t)$. By Lemma 3.3 there is a v such that $v_1 \geqslant_P v$ and $v_2 \geqslant_P v$. Applying Corollary 2.3, let v' be a normal term such that $v \geqslant_P v'$. By Hyp Ind $v' =_\alpha u_1$ and $v' =_\alpha u_2$. Since $=_\alpha$ is symmetric and transitive, it follows that $u_1 =_\alpha u_2$.

Lemma 3.5 *If* t *is a contractible redex and* u *is the contractum of* t, *then there is a* v *such that* $t \geqslant_P v$ *and* $u \geqslant_P v$.

Proof: If $t >_P u$ there is nothing to prove, so suppose $t \not>_P u$. Then t is a β redex which is not predicative. Let $t \equiv (\lambda xt_1)t_2$. $u \equiv [t_2/x/t_1]$. Applying Corollary 2.3, let t_2' be a normal term such that $t_2 \geqslant_P t_2'$. Then $(\lambda xt_1)t_2 \geqslant_P (\lambda xt_1)t_2' >_P [t_2'/x/t_1]$, and $u \equiv [t_2/x/t_1] \geqslant_P [t_2'/x/t_1]$. This shows that $[t_2'/x/t_1]$ is an appropriate v.

Lemma 3.6 *If* $t > u$, *then there is a* v *such that* $t \geqslant_P v$ *and* $u \geqslant_P v$.

Proof: Immediate from Lemma 3.5.

Lemma 3.7 *If* $t \geqslant u$ *and* u *is normal, then* $t \geqslant_P u$.

Proof: Let $t^1, \ldots, t^n$ be a reduction of t to u. By Lemma 3.6 there exist $v_1, \ldots, v_{n-1}$ such that $t^1 \geqslant_P v_1$ and $t^2 \geqslant_P v_1, \ldots, t^{n-1} \geqslant_P v_{n-1}$ and $t^n \geqslant_P v_{n-1}$. Applying Corollary 2.3, let $v_1', \ldots, v_{n-1}'$ be normal terms to which $v_1, \ldots, v_{n-1}$, respectively, reduce predicatively. Then for all i $(1 < i < n)$, $t^i \geqslant_P v_{i-1}'$ and $t^i \geqslant_P v_i'$. By Lemma 3.4 for all i $(1 \leqslant i < n - 1)$, $v_i' =_\alpha v_{i+1}'$. Also, $u \geqslant_P v_{n-1}'$ and u is normal, so $v_{n-1}' =_\alpha u$ by the symmetry of $=_\alpha$. Since $=_\alpha$ is transitive, it follows that $v_1' =_\alpha u$. Hence, $t \geqslant_P v_1' =_\alpha u$. It follows that $t \geqslant_P u$.

Corollary 3.8 [Church-Rosser theorem, first version] *If* $t \geqslant u_1$, $t \geqslant u_2$, *and* u_1 *and* u_2 *are normal, then* $u_1 =_\alpha u_2$.

Proof: Apply Lemmas 3.7 and 3.4.

Corollary 3.9 [Church-Rosser theorem, second version] *If* $t = u$, *then there is a* v *such that* $t \geqslant v$ *and* $u \geqslant v$.

Proof: Let $t_1, \ldots, t_n$ be such that $t_1 \equiv t$, $t_n \equiv u$, and for all $i < n$, $t_i \geqslant t_{i+1}$ or $t_{i+1} \geqslant t_i$. Applying Corollary 2.3, let $v_1, \ldots, v_n$ be normal terms such that for all i $(1 \leqslant i \leqslant n)$ $t_i \geqslant v_i$. By Corollary 3.8 for all i $(1 \leqslant i < n)$ $v_i =_\alpha v_{i+1}$. It follows that $t \geqslant v_n$ and $u \geqslant v_n$, which suffices.

REFERENCE

[1] Rubin, J. E., *Set Theory for the Mathematician,* Holden-Day, San Francisco, 1967.

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