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DOUBLY TRANSITIVE SETS

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Since we are interested in linearly ordered sets, we shall adopt the convention throughout this paper that unless the contrary is stated explicitly, "set" means "linearly ordered set".* Let S, T be two sets, and let $f: S \rightarrow T$ be an order-preserving map. If f is bijective, then f is called an "isomorphism" and S, T are said to be similar ("$S \simeq T$"). An isomorphism $f: S \simeq S$ is called an "automorphism", and the group (under composition) of all automorphisms of S is denoted by "$\mathsf{A}(S)$". We can impose a partial order on $\mathsf{A}(S)$ by setting $f \leq g$ whenever $f(x) \leq g(x)$ for every $x \in S$, and under this order $\mathsf{A}(S)$ becomes a lattice ordered group. A study of $\mathsf{A}(S)$ for general sets S seems to have started with Holland in [1], although results pertaining to specific classes of S had been obtained previously.

If for any two $x, y \in S$ there exists $f \in \mathsf{A}(S)$ with $f(x) = y$, then S is called "transitive" (or "homogeneous"): furthermore, if there is exactly one such f, then S is called "uniquely transitive". Okhuma in [4] has shown that every uniquely transitive set is similar to a subgroup of the additive group of real numbers. If for all $x, y, u, v \in S$ with $x < y$ and $u < v$ there is $f \in \mathsf{A}(S)$ such that $f(x) = u$ and $f(y) = v$, then S is called "2-transitive", and it is with such sets that this paper is concerned. Clearly a 2-transitive set S is finite if and only if $|S| \leq 2$, and so we exclude the finite case from future considerations. A 2-transitive set is obviously transitive but not uniquely transitive. A nonempty subset R of a set S is called a "segment" of S if for all $x, y, z \in S$ with $x \leq y \leq z$ and $x, z \in R$ we have $y \in R$. A segment R of S is called an "interval" of S if there exist $x, z \in S$ such that R has one of the following forms:

(a) $\{y \in S: x < y < z\}$; (b) $\{y \in S: x \leq y \leq z\}$;
(c) $\{y \in S: x \leq y < z\}$; (d) $\{y \in S: x < y \leq z\}$.

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These forms are denoted by "(x, z)", "$[x, z]$", "$[x, z)$", and "$(x, z]$" respectively. An interval I is called "trivial" if $|I| = 1$, and unless the contrary is stated we shall henceforth use the word "interval" to mean "nontrivial interval". A segment is said to be bounded if it is contained in some interval.

One final piece of notation will prove useful before we commence our study of 2-transitive sets. Let S be any set, and take $x, y, u, v \in S$ with $x < y$ and $u < v$. Then we define the set A_{uv}^{xy} to consist of all those $f \in A(S)$ for which $f(x) = u$ and $f(y) = v$: moreover, whenever the symbol "A_{uv}^{xy}" appears, it will be tacitly assumed that $x < y$ and $u < v$.

To classify all countable 2-transitive sets (remembering that we are excluding the finite cases) is an easy task, since up to similarity there is only one. This follows at once from our first result.

Theorem 1 *Every 2-transitive set is dense and without endpoints.*

Proof: Let S be a 2-transitive set and suppose that for some $x, y \in S$ with $x < y$ there is no $z \in S$ such that $x < z < y$. Since S is infinite, we may assume without loss of generality that the set $T = \{t \in S;\ y < t\}$ is infinite, and so we may choose $u, v, w \in T$ such that $u < v < w$. Take $f \in A_{uw}^{xy}$; then we must have $x < f^{-1}(v) < y$. This contradiction shows that S is dense.

Now assume that S has a right endpoint w, and take $x, y, z \in S$ with $x < y < z < w$. Take $f \in A_{zw}^{xy}$. Since f is order-preserving and $y < z$, we must have $w = f(y) < f(z)$, which is absurd. Thus S has no right endpoint, and in a similar manner we can show that S has no left endpoint.

Corollary *Let S be a countable 2-transitive set. Then S is (similar to) the set of rational numbers under the usual ordering.*

A set S is called "symmetric" if $S \simeq S^*$, where S^* is of course the converse set to S. All the familiar 2-transitive sets, such as the set of rationals, the set of reals, and so on, are symmetric. To show that there are asymmetric 2-transitive sets, Longyear in [3] defined a specific set H, and since this set will figure in some of our future arguments, we repeat the definition here. We recall firstly, however, that if we are given two sets S, T, then the ordered product $S \times T$ is the set of all ordered pairs $\langle s, t\rangle$ with $s \in S$, $t \in T$, such that $\langle s, t\rangle < \langle s', t'\rangle$ whenever either $t < t'$ or else $t = t'$ and $s < s'$. This is sometimes called the "antilexicographic product" and coincides with general usage but is the converse of the ordered product defined in [3].

Let Q be the set of rationals. Then we define H to be the ordered product $Q \times \omega_1$, where of course ω_1 is the first uncountable ordinal. In order to see that H is 2-transitive, we simply note that for any $x, y, u, v \in H$ with $x < y$ and $u < v$ there is some open interval I of H such that $x, y, u, v \in I$, and that every open interval of H is similar to Q. On the other hand, H cannot be symmetric as $\mathrm{cf}(H) = \omega_1$ and $\mathrm{cf}(H^*) = \omega$. [For any set S, the cofinality $\mathrm{cf}(S)$ of S is defined to be the least ordinal α for which there is an order-preserving map $f\colon \alpha \to S$ such that for each $x \in S$ there is $\beta < \alpha$ with $x \leqslant f(\beta)$.]

Longyear in [3] claims that a 2-transitive set S is symmetric if and only if $\mathrm{cf}(S) = \mathrm{cf}(S^*)$ and S contains some symmetric interval. This claim is unfortunately false, as is shown by Holland in [2]. We can, however, salvage the claim to some extent by placing certain conditions upon S.

Theorem 2 (Longyear) *Let S be a 2-transitive set, and assume that either S is continuous or else that $\mathrm{cf}(S) = \omega$. Then S is symmetric if and only if $\mathrm{cf}(S) = \mathrm{cf}(S^*)$ and S contains a symmetric interval.*

Proof: Assume that S is symmetric; then obviously $\mathrm{cf}(S) = \mathrm{cf}(S^*)$. Furthermore it is easy to see that there is an order-inverting bijection $f: S \to S$ such that $f = f^{-1}$. Define R, T by $R = \{x \in S;\ x < f(x)\}$, $T = \{x \in S;\ f(x) < x\}$; then S is the ordered union $R \dot{\cup} J \dot{\cup} T$, where $J = \{x \in S;\ x = f(x)\}$. Clearly $|J| \leqslant 1$, and R, T are both infinite. Take any $t \in T$ such that $x < t$ for some $x \in T$, and put $K = \{y \in S;\ f(y) < y < t\}$. Then K is a symmetric interval of S.

Now assume that S has some symmetric interval K (which we may clearly assume to be open) and that $\mathrm{cf}(S) = \mathrm{cf}(S^*)$. Let us suppose firstly that $\mathrm{cf}(S) = \omega$. Then there is a subset $V = \{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$ of S of order-type $\omega^* + \omega$ that is both coinitial and cofinal with S. Now it is obvious that in any 2-transitive set, all open intervals are similar. Thus for each natural number n, the interval (v_{-1-n}, v_{-n}) is similar to K and hence symmetric, and thus for each such n there is an isomorphism $f_n: \langle v_n, v_{n+1}\rangle \to \langle v_{-1-n}, v_{-n}\rangle^*$. We can now define a map $f: S \to S$ as follows. Take $x \in S$. If $x = v_j$ for some integer j, then $f(x) = v_{\bar{j}}$. If $x = v_j$ for no integer j, then there must exist an integer i such that $x \in \langle v_i, v_{i+1}\rangle$. If $i \geqslant 0$, then $f(x) = f_i(x)$; if $i < 0$, then $f(x) = f^{-1}_{-i-1}(x)$. It is easily seen that f is an order-inverting bijection, and so in this case we have shown that S is symmetric.

Now let us suppose that S is continuous, and put $\mathrm{cf}(S) = \kappa$. Then there is an increasing κ-sequence $(v_\xi)_{\xi<\kappa}$ of elements of S that is cofinal with S: since S is continuous, $\lim_{\xi<\rho} v_\xi$ exists for each nonzero limit ordinal $\rho < \kappa$, and we may without loss of generality assume that $v_\rho = \lim_{\xi<\rho} v_\xi$ for each such ρ. It follows that for each $x \in S$ with $x \geqslant v_0$, either $x = v_\alpha$ for some $\alpha < \kappa$, or else $x \in \langle v_\beta, v_{\beta+1}\rangle$ for some $\beta < \kappa$. Similarly we have a decreasing κ-sequence $(u_\xi)_{\xi<\kappa}$ of elements of S that is coinitial with S and such that $u_\rho = \lim_{\xi<\rho} u_\xi$ for each nonzero limit ordinal $\rho < \kappa$: we may of course assume that $u_0 = v_0$.

We can now construct an order-inverting bijection $f: S \to S$ in a manner similar to that described in the previous case.

Before proceeding further, we present a simple criterion on 2-transitivity, part of which was used in the proof of the preceding theorem.

Theorem 3 *Let S be a set. Then S is 2-transitive if and only if S has no endpoints and any two open intervals of S are similar.*

Proof: The "only if" direction being clear, we concentrate on the converse and assume that S has no endpoints and that any two open intervals of S are similar. Take any $x, y, u, v \in S$ such that $x < y$ and $u < v$. We assume that $x < u$ and $y < v$; the other possibilities are dealt with in similar vein.

Since S has no endpoints, we may choose $t, w \in S$ such that $t < x$ and $v < w$. Let f_0: $\langle t, x\rangle \to \langle t, u\rangle$, f_1: $\langle x, y\rangle \to \langle u, v\rangle$ and f_2: $\langle y, w\rangle \to \langle v, w\rangle$ be isomorphisms. Define f: $S \to S$ by the following conditions.

(a) $f(x) = u$, $f(y) = v$, and $f(z) = z$ for $z \notin \langle t, w\rangle$;
(b) $f(z) = f_i(z)$ for $z \in \mathrm{dom}(f_i)$, $i = 0, 1, 2$.

It is clear that $f \in A_{uv}^{xy}$. Thus S is 2-transitive.

Obviously it follows from the above theorem that if the set S has no endpoints and any two bounded "open" segments of S are similar, then S is 2-transitive. The converse, however, is not necessarily true; we shall exhibit later a 2-transitive set containing non-similar bounded open segments.

The 2-transitive sets thus far encountered have been reasonably easy to visualize. We now wish to make use of Theorem 3 to prove a result that will give us many—a proper class, in fact—2-transitive sets that, by contrast, could well be called "visually pathological". These sets, so to speak, double back on themselves infinitely often.

We recall that a prime component is a nonzero ordinal α such that $\beta + \alpha = \alpha$ for each $\beta < \alpha$; the ordinal α is a prime component if and only if $\alpha = \omega^{\tau}$ for some ordinal τ. Let S be a set and η an ordinal. Then by "S^{η^*}" we denote the set consisting of all functions g: $\eta^* \to S$ ordered anti-lexicographically. For $\eta = 2$, this coincides with the ordered product $S \times S$. Let S be a set and take $x \in S$. We find it convenient to denote the set $\{y \in S;\ x < y\}$ by "(x, ∞)". The sets (∞, x), $[x, \infty)$ and $(\infty, x]$ are defined in the obvious manner.

Theorem 4 *Let S be a 2-transitive set, and let τ be a prime component. Then S^{τ^*} is 2-transitive.*

Proof: Put $T = S^{\tau^*}$, and let $x = (\ldots, s_\tau, \ldots, s_2, s_1, s_0)$ be any element of T. Take $s \in S$ with $s < s_0$ and define $y \in T$ by $y = (\ldots, s, \ldots, s, s, s)$. Then $y < x$ and thus T has no left endpoint. Similarly we can show that T has no right endpoint. Therefore by Theorem 3 it suffices to show that any two open intervals of T are similar.

Let us note firstly that for any $s, s' \in S$, we have $(\infty, s) \simeq (\infty, s')$, for there is certainly $f \in A(S)$ such that $f(s) = s'$. Similarly we have $(s, \infty) \simeq (s', \infty)$. Therefore we can denote by "γ" the order-type of any segment of S of the form (∞, s), and by "δ" the order-type of any segment of S of the form (s, ∞). Furthermore, we let τ be the order-type of T.

Let $I = (g, h)$ be an open interval of T, and let α be the last ordinal at which g and h differ. Define $J \subseteq I$ by

$$J = \{c \in T;\ g(\alpha) < c(\alpha) < h(\alpha)\ \&\ \forall\beta > \alpha(c(\beta) = g(\beta) = h(\beta))\}.$$

[With regard to this definition we recall that we are working with η^*, not with η.] Clearly J is a segment of I. Now for any $c \in J$ and for any $\xi \in \eta^*$ with $\xi < \alpha$, the choice of $c(\xi)$ is unrestricted. Since η is a prime component, it follows at once that the order-type of J is $\tau\theta$, where θ is the order-type of the open interval $(g(\alpha), h(\alpha))$ of S.

Now take any $\xi \in \eta^*$ with $\xi < \alpha$. We define K_ξ, $L_\xi \subseteq T$ by

$$K_\xi = \{c \in T;\, g(\xi) < c(\xi) \ \&\ \forall \zeta > \xi(c(\zeta) = g(\zeta))\},$$

and

$$L_\xi = \{c \in T;\, h(\xi) > c(\xi) \ \&\ \forall \zeta > \xi(c(\zeta) = h(\zeta))\}.$$

The same reasoning as above shows that K_ξ and L_ξ have respective order-types $\tau\delta$ and $\tau\gamma$. It is easy to see, however, that the interval I is the ordered union. Then I has the form:

$$\ldots \dot{\cup}\, K_\xi \,\dot{\cup} \ldots \dot{\cup}\, K_{\alpha+1} \,\dot{\cup}\, J \,\dot{\cup}\, L_{\alpha+1} \,\dot{\cup} \ldots \dot{\cup}\, L_\xi \,\dot{\cup} \ldots$$

Hence the order-type of I is $\tau\delta\eta^* + \tau\theta + \tau\gamma\eta$, that is, $\sigma^{\eta^*}(\delta\eta^* + \theta + \gamma\eta)$. Since all open intervals of S are similar, the order-type θ is independent of the choice of I. We have therefore shown that all open intervals of T have the same order-type and hence are similar. Thus T is 2-transitive. In computing the order-type of I above, we have made essential use of our tacit assumption that S is infinite. For if we take $s = 2$ and $\eta = \omega$, then we obtain the famous Cantor set, and even if we remove the endpoints, the resulting set is still not 2-transitive.

At this stage we wish to consider a question concerning symmetry of 2-transitive sets. Let S be a 2-transitive set; Longyear in [3] has posed the following question:

If no interval of S can be inverted, must every interval contain a copy of H?

At first sight this question appears a little ambiguous, for the word "contain" could be interpreted in two distinct ways. Let I be an interval of S. Then by "I contains a copy of H" we could mean

(i) There is a segment J of I with $J \simeq H$; or else
(ii) There is an order-preserving map $f\colon H \to I$.

Since, as we shall see shortly, interpretation (i) leads to a contradiction, it must be that interpretation (ii) is the one intended by Longyear.

Suppose that the 2-transitive set S is such that for every interval I of S there is a segment J of I with $J \simeq H$. Let I_0 be a given interval of S, and let J_0 be some such segment of I_0. Now every open interval of H is similar to the set Q of rationals; hence there exists an interval I_1 of J_0 with $I_1 \simeq Q$. But J_0, being a segment of I_0, is a segment of S, and so I_1 is an interval of S. Therefore, by assumption, there is a segment J_1 of I_1 such that $J_1 \simeq H$. Since I_1 is countable and H is uncountable, this is a contradiction.

Therefore we must consider Longyear's question under interpretation (ii). Let us say at once that we do not know the answer to this question; we wish to show, however, that if Longyear is suggesting this condition as some kind of criterion for the total asymmetry of a 2-transitive set, then it fails rather badly. We do this by constructing a 2-transitive set W with the property that W is symmetric and every interval of W contains a copy of H (in sense (ii)). [Every open interval of W is symmetric by Theorem 2.]

Theorem 5 *There exists a symmetric 2-transitive set W such that if I is any interval of W then there is an order-preserving map f: $H \to I$.*

Proof: Let R be the set $\{x;\, x \text{ is real } \&\ 0 \leq x < 1\}$ under the usual ordering, and let S be the set $(R \times \omega_1) - \{\langle 0, 0\rangle\}$. Consider the ordered union $W^\circ = S^* \dot{\cup} \{\langle 0, 0\rangle\} \dot{\cup} S$. W° is of course symmetric, and it is easily seen that every open interval of W° is similar to the real line. Thus by Theorem 3 W° is 2-transitive, and hence by Theorem 4 so is the set $W = W^{\circ\,\omega^*}$.

Let $h\colon W^\circ \to W^\circ$ be an order-inverting bijection: we define a map $h^\#\colon W \to W$ as follows. For each $g \in W$ and each $n \in \omega^*$, we set $h^\#(g)(n) = h(g(n))$. It is a simple matter to show that $h^\#$ is an order-inverting bijection; thus W is symmetric. Let I be an open interval of W. In the proof of Theorem 4 we showed that the order-type of I is $\mu(\delta\omega^* + \theta + \gamma\omega)$, where μ is the order-type of W, θ is that of any open interval of W°, γ is that of any segment of W° of the form (∞, x), and δ is that of any segment of W° of the form (x, ∞). Thus, in particular, there is a segment J of I such that $J \simeq W$, and hence in order to show that there is an order-preserving map $H \to I$, we have only to show that there is an order-preserving map $f\colon H \to W$.

Now clearly there is an order-preserving map $f_0\colon H \to W^\circ$. Thus it suffices to construct an order-preserving map $f_1\colon W^\circ \to W$. Take $x \in W^\circ$, and define $f_1(x)\colon \omega^* \to W^\circ$ by $f_1(x)(0) = x$ and $f_1(x)(n) = w$ for $n \neq 0$, where w is some fixed element of W°, independent of x. Clearly f_1 preserves order. This proves our result.

Thus far we have not produced a 2-transitive set in which all intervals are asymmetric, and such sets would not seem to be particularly plentiful. Longyear in [3] defines a generalized ordered product, and uses this product to construct a certain set, stating without proof that this set satisfies the above requirements. Later in this paper we shall show that the set H^{ω^*}, which is slightly simpler in its construction than the set given in [3], also has no symmetric intervals.

Let Q be the set of rationals. Q is 2-transitive, and if I is any interval of Q, then $|I| = |Q|$. On the other hand, if J is any interval of the 2-transitive set H, then $|J|^+ = |H|$. Here of course we are using "l^+" to denote the successor cardinal of the cardinal l. We wish to show that these are the only two possibilities open to 2-transitive sets.

Theorem 6 *Let S be a 2-transitive set, and let I be an interval of S. Then* $|I| \leq |S| \leq |I|^+$.

Proof: Obviously $|I| \leq |S|$. Thus we assume that $|S| > |I|^+$ and derive a contradiction. Let κ and λ be the initial ordinals whose respective cardinalities are $|S|$ and $|I|^+$; thus $\kappa > \lambda$. Now put $\eta = \mathrm{cf}(S)$, and let $(t_\xi)_{\xi<\eta}$ be an increasing cofinal η-sequence in S. For each positive ordinal $\alpha < \eta$, define the segment T_α of S by $T_\alpha = \{s \in S;\, s < t_\alpha \ \&\ \forall \beta < \alpha (s \geq t_\beta)\}$. Similarly, if we put $\rho = \mathrm{cf}(S^*)$ and let $(r_\xi)_{\xi<\rho}$ be a decreasing coinitial ρ-sequence in S such that $r_0 = s_0$, we can define the segment R_α for each positive ordinal

$\alpha < \rho$. Putting $T = \bigcup\{T_\alpha;\ 0 < \alpha < \eta\}$ and $R = \bigcup\{R_\alpha;\ 0 < \alpha < \rho\}$, we see that S is the unordered union $R \cup T$, and so we must have either $|S| = |T|$ or $|S| = |R|$. Without loss of generality we may assume the former.

Now each T_α is a bounded segment of S, and hence $|T_\alpha| \leq |I|$. Since $|T| = |S| > |I|^+$, it follows that $\eta \geq \kappa$. Therefore $\eta > \lambda$, and so the segment $T^\circ = \bigcup\{T_\alpha;\ 0 < \alpha < \lambda\}$ is well-defined and bounded in S. Thus we have $|T^\circ| \leq |I|$. On the other hand we have $t_\alpha \in T^\circ$ for each $\alpha < \lambda$, and since $(t_\xi)_{\xi<\eta}$ is increasing, it follows that $|T^\circ| \geq |\lambda| = |I|^+$.

The following result is very simple, but as it will be used a couple of times later on, we present it explicitly.

Theorem 7 *Let S be a set without endpoints. Then S is 2-transitive if and only if every open interval of S is 2-transitive.*

Proof: Suppose that every open interval of S is 2-transitive, and take $x, y, u, v \in S$ with $x < y$ and $u < v$. Since S has no endpoints, there is an open interval I of S such that $x, y, u, v \in I$. Take $f \in \mathsf{A}(I)_{uv}^{xy}$ and extend f to $g \in \mathsf{A}(S)$ by setting $g(z) = z$ for $z \notin I$. Then $g \in \mathsf{A}(S)_{uv}^{xy}$, and so S is 2-transitive. Now assume that S is 2-transitive, and let I be an open interval of S. Let J, J' be open intervals of I. Then J, J' are open intervals of S, and so by Theorem 3 $J \simeq J'$. Thus all open intervals of I are similar, and so by Theorem 3 again, I is 2-transitive.

This result enables us to say something about the cofinality of a continuous 2-transitive set.

Theorem 8 *Let S be a continuous 2-transitive set, and let I be an open interval of S. Then* $\mathrm{cf}(I) = \mathrm{cf}(I^*) = \omega$.

Proof: Let I be an open interval of S, and choose $x, y \in I$ with $x < y$. By Theorem 7 I is 2-transitive, and so there exists $f \in \mathsf{A}(I)$ such that $f(x) = y$. Put $T = \{z \in I;\ \exists n(f^n(z) \geq x \vee f^n(x) \geq z)\}$. T is an open segment of I, and hence an open bounded segment of S. Moreover, $\mathrm{cf}(T) = \mathrm{cf}(T^*) = \omega$. Now any bounded segment of a continuous set is in fact an interval of that set. Therefore T is an open interval of S, and so by Theorem 3 $T \simeq I$. Thus $\mathrm{cf}(I) = \mathrm{cf}(I^*) = \omega$.

Theorem 9 *Let S be a continuous 2-transitive set. Then* $\mathrm{cf}(S), \mathrm{cf}(S^*) \leq \omega_1$.

Proof: We shall show that $\mathrm{cf}(S) \leq \omega_1$; since the converse of a continuous 2-transitive set is also a continuous 2-transitive set, the full result will follow. Suppose that $\mathrm{cf}(S) = \eta > \omega_1$, and let $(t_\xi)_{\xi<\eta}$ be an increasing cofinal η-sequence in S. Since S is continuous, we may assume that $t_\rho = \lim_{\xi<\rho} t_\xi$ for each nonzero limit ordinal $\rho < \eta$. Put $T = (t_0, t_{\omega_1})$; then T is an open interval of S with $\mathrm{cf}(T) = \omega_1$. Since this contradicts Theorem 8, we must have $\mathrm{cf}(S) \leq \omega_1$.

The set S constructed in the proof of Theorem 5 shows that the bound given in the preceding result is the best possible.

If S is a dense set, then S can be embedded in a continuous set $\overline{S}$ in such a way that S is dense in $\overline{S}$; moreover, $\overline{S}$ is uniquely determined (up to similarity) by these conditions. This of course is well known, and is most easily achieved by the method of Dedekind cuts.

It is natural to ask whether the completion of a 2-transitive set is itself 2-transitive. The answer is "No"; to show this, we consider the set $M = H^{\omega^*}$. Let I be an open interval of M; we know that I contains a segment J such that $J \simeq M$. Then $\mathrm{cf}(J) = \mathrm{cf}(M)$, and it is easy to see that $\mathrm{cf}(M) = \mathrm{cf}(H) = \omega_1$. Thus $\mathrm{cf}(J) = \omega_1$. By Theorem 4, M is 2-transitive. Consider its completion, $\overline{M}$. Naturally the completion $\overline{J}$ of J is (similar to) an open bounded segment of $\overline{M}$, and thus is an open interval of $\overline{M}$. But J is dense in $\overline{J}$, and so $\mathrm{cf}(\overline{J}) = \mathrm{cf}(\overline{J}) = \omega_1$. Hence by Theorem 8, $\overline{M}$ is not 2-transitive.

This answers a question posed earlier in this paper; are all open bounded segments of a 2-transitive set similar? Let K be any open bounded segment of M; then $\overline{K}$ is an open interval of $\overline{M}$. Now if all such segments K were similar, then all open intervals of $\overline{M}$ would be similar (since M is dense in $\overline{M}$), and by Theorem 3 $\overline{M}$ would be 2-transitive. Therefore M contains non-similar open bounded segments.

Before we dispose of M, we note that M is a 2-transitive set in which all open intervals are asymmetric. For let I be an open interval of M; we know that I contains a segment J with $J \simeq M$; thus $\mathrm{cf}(J) = \omega_1$. Hence if $I \simeq I^*$, then I must contain a segment R with $\mathrm{cf}(R^*) = \omega_1$. However, it is easily seen that $\mathrm{cf}(R^*) = \omega$ for every open segment R of M.

Let m be a positive integer. We make the obvious generalization of 2-transitivity and say that a set S is m-transitive if for any two increasing m-sequences $(x_k)_{k<m}$, $(y_k)_{k<m}$ in S, there is $f \in \mathrm{A}(S)$ such that $f(x_k) = y_k$, $k < m$.

Obviously a 1-transitive set is just a transitive set. By considering the set of all integers under the usual ordering, we can see that there are transitive sets that are not 2-transitive. This is as far as it goes, however; for any two integers $m, n \geqslant 2$, a set is m-transitive if and only if it is n-transitive. We conclude this paper with a demonstration of this fact.

Theorem 10 *Let S be a set, and let $m, n \geqslant 2$ be two integers. Then S is m-transitive if and only if S is n-transitive.*

Proof: Clearly if S is p-transitive for some positive integer p, then S is q-transitive for every positive integer $q \leqslant p$. We may assume that $m \leqslant n$, and it therefore suffices to show that if S is m-transitive, then S is n-transitive. Hence we assume that S is m-transitive, then it follows that S is 2-transitive. By Theorem 7, therefore, every open interval of S is 2-transitive.

Let $(x_k)_{k<n}$, $(y_k)_{k<n}$ be two increasing n-sequences in S; we must construct $f \in \mathrm{A}(S)$ such that $f(x_k) = y_k$ for $k = 0, \ldots, n-1$. Since S is 2-transitive, we know that there is $g \in \mathrm{A}(S)$ such that $g(x_k) = y_k$ for $k < 2$. We proceed by induction and assume that for some p with $2 \leqslant p < n$, there is $h \in \mathrm{A}(S)$ such that $h(x_k) = y_k$ for $k < p$. Now if $h(x_p) = y_p$, we are through; hence we may assume, without loss of generality, that $y_p < h(x_p)$. S is dense and without endpoints, and so we may choose $u, v \in S$ such that $y_{p-1} < u < y_p < h(x_p) < v$. Let I be the open interval (u, v). I is 2-transitive, and thus there is $h^\circ \in \mathsf{A}(I)$ with $h^\circ(h(x_p)) = y_p$. Define $h^\#: S \to S$ as follows. For any $z \in S$, if $h(z) \in I$, then $h^\#(z) = h^\circ(h(z))$, and $h^\#(z) = h(z)$ otherwise. It is routine to show that $h^\# \in \mathsf{A}(S)$ and that $h^\#(x_k) = y_k$, $k \leqslant p$. This proves our theorem.

REFERENCES

[1] Holland, C., "The lattice ordered group of automorphisms of an ordered set," *Michigan Mathematical Journal*, vol. 10 (1963), pp. 399-408.

[2] Holland, C., Review #4876 *cf.* [3] in *Mathematical Reviews*, vol. 49 (1975), p. 399.

[3] Longyear, J. Q., "The structure of linear homogeneous sets," *Journal für die reine und angewandte Mathematik*, vol. 266 (1974), pp. 132-135.

[4] Okhuma, T., "Sur quelques ensembles ordonnés linéairement," *Fundamenta Mathematicae*, vol. 43 (1954), pp. 326-337.

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