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AWKWARD AXIOM-SYSTEMS

BOLESŁAW SOBOCIŃSKI

As far as I know a notion of the awkward axiom-systems, which is introduced in this note, was never previously discussed in the appropriate literature in which the theory of the well-formed axiomatizations is investigated. I define the notion mentioned above as follows:

Let T be an axiomatizable theory with the rules of procedure $\mathfrak{R}$. Let $\mathfrak{T}$ be a finite or infinite axiom-system of T based on the rules $\mathfrak{R}$ which satisfies the following conditions:

(1) The axioms belonging to $\mathfrak{T}$ are mutually independent.
(2) The set of the axioms belonging to $\mathfrak{T}$ can be divided into two subsets Γ and Δ having no common elements and such that

$$\mathfrak{T} = \bigcup\{\Gamma, \Delta\}$$

where Γ is a finite set, and Δ can be either finite or infinite.
(3) There is a finite set θ of formulas which are the consequences of the set of axioms Γ obtained from Γ by the applications of the rules $\mathfrak{R}$, but without the use of the axioms belonging to Δ and such

(a) that θ is a proper subsystem of Γ,

and

(b) that the theses belonging to the set $\bigcup\{\theta, \Delta\}$ are mutually independent.

Then, if on the base of the rules $\mathfrak{R}$ system $\bigcup\{\theta, \Delta\}$ is inferentially equivalent to $\bigcup\{\Gamma, \Delta\}$, the axiom-system $\mathfrak{T}$ is awkward.

Clearly, although from the points of view of the other requirements concerning the well-formed axiomatizations, the awkward axiom-systems are entirely correct, it is self-evident that in such axiomatizations there are accepted some axioms (belonging to Γ) which are stronger than needed and, on the other hand, the deductive power of the other axioms (belonging to Δ) is not used. Hence, the awkward axiom-systems can be considered as not especially elegant axiomatizations. There are several theories, mostly

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in the field of the extended Boolean systems, e.g., the closure algebras and the cylindric algebras, whose classical axiomatizations are awkward, *cf.* [5], [6], [7], and [8].

In order to present the discussed notion in an elementary way I shall use a very simple example. In [1] Flagg has proved that the following set of the mutually independent theses:

A1 *CpCqp*
A2 *CCpCqrCCpqCpr*
B1 *CNpCpq*
B2 *CCpqCCNpqq*

together with the rules of substitution and detachment for implication can be accepted as an axiom-system of the complete classical propositional calculus. It will be shown below that Flagg's axiomatic is awkward.

Proof: In all deductions given below only the rules of substitution and detachment for implication will be used. Obviously, the set $\{A1, A2, B1, B2\}$ can be divided into two subsets $\Gamma = \{A1, A2\}$ and $\Delta = \{B1, B2\}$ which have no common elements. Now, we shall proceed as follows:

1 Let us assume *A1* and *A2*, i.e., the well-known axiom-system of the positive propositional calculus, *cf.*, e.g., [3], p. 217, and the deductions presented there. Then:

A3 *CsCCpCqrCCpqCpr* [*A1*, *p*/*CCpCqrCCpqCpr*, *q*/*s*; *A2*]
A4 *CCqrCCpqCpr*
[*A2*, *p*/*Cqr*, *q*/*CpCqr*, *r*/*CCpqCpr*; *A3*, *s*/*Cqr*; *A1*, *p*/*Cqr*, *q*/*p*]
A5 *CrCpCqp* [*A1*, *p*/*CpCqp*, *q*/*r*; *A1*]
A6 *CCCpqrCqr* [*A2*, *p*/*CCpqr*, *q*/*CqCpq*, *r*/*Cqr*; *A4*, *p*/*q*, *q*/*Cpq*; *A5*, *p*/*q*, *q*/*p*, *r*/*CCpqr*]
A7 *CCpCqrCqCpr* [*A4*, *p*/*CpCqr*, *q*/*CCpqCpr*, *r*/*CqCpr*; *A6*, *r*/*Cpr*; *A2*]
C1 *CCpqCCqrCpr* [*A7*, *p*/*Cqr*, *q*/*Cpq*, *r*/*Cpr*; *A4*]
D1 *Cpp* [*A7*, *q*/*CpCqp*, *r*/*p*; *A1*, *q*/*CpCqp*; *A1*]
A8 *CtCCpqCCqrCpr* [*A1*, *p*/*CCpqCCqrCpr*, *q*/*t*; *C1*]
A9 *CCpqCtCCqrCpr* [*A7*, *p*/*t*, *q*/*Cpq*, *r*/*CCqrCpr*; *A8*]
D2 *CCpqCCqrCtCpr*
[*C1*, *p*/*Cpq*, *q*/*CtCCqrCpr*, *r*/*CCqrCtCpr*; *A9*; *A7*, *p*/*t*, *q*/*Cqr*, *r*/*Cpr*]

Hence, the theses *A1*, *C1*, *D1*, and *D2* are the consequences of the axioms *A1* and *A2*.

2 The matrix, *cf.* [4], p. 47,

$\mathfrak{M}1$

	C	0	1	2
	0	1	1	0
*	1	0	1	2
	2	1	1	1

in which 1 is the designated value, verifies the theses *A1*, *C1*, *D1*, and *D2*, but falsifies *A2* for *p*/0, *q*/0, and *r*/2: *CC0C02CC00C02* = *CC00C10* = *C10* = 0.

Therefore, obviously, it follows from the sections 1 and 2 of the Proof that each of the sets {*A1*, *C1*} and {*D1*, *D2*} is a subsystem of {*A1*, *A2*}.

3 In order to establish the mutual independency of the theses *A1*, *C1*, *B1*, and *B2*, and the mutual independency of the theses *D1*, *D2*, *B1*, and *B2* we use the following matrices, *cf.* [1], Lemmas 10-13:

$\mathfrak{M}2$

	C	0	1	2	*N*
	0	2	1	2	2
*	1	2	1	2	2
	2	1	1	1	1

$\mathfrak{M}3$

	C	0	1	2	3	*N*
	0	1	1	2	1	2
*	1	0	1	2	3	3
	2	0	1	1	3	0
	3	1	1	1	1	1

$\mathfrak{M}4$

	C	0	1	*N*
	0	1	1	1
*	1	0	1	1

$\mathfrak{M}5$

	C	0	1	*N*
	0	1	1	0
*	1	0	1	0

In each of these matrices, 1 is the designated value.

Since: (a) $\mathfrak{M}2$ verifies *C1*, *D2*, *B1*, and *B3*, but falsifies *A1* for *p*/0 and *q*/0: *C0C00* = *C02* = 2, and falsifies *D1* for *p*/0; *C00* = 2; (b) $\mathfrak{M}3$ verifies *A1*, *D1*, *B1*, and *B2*, but falsifies *C1* for *p*/0, *q*/3, and *r*/2: *CC03CC32C02* = *C1C12* = *C12* = 2, and falsifies *D2* for *p*/0, *q*/3, *r*/2, and *t*/1: *CC03CC32C1C02* = *C1C1C12* = *C1C12* = *C12* = 2; (c) $\mathfrak{M}4$ verifies *A1*, *D1*, *C1*, *D2*, and *B2*, but falsifies *B1* for *p*/1 and *q*/0: *CN1C10* = *C10* = 0; $\mathfrak{M}5$ verifies *A1*, *D1*, *C1*, *D2*, and *B1*, but falsifies *B2* for *p*/0 and *q*/0: *CC00CCN000* = *C1CC000* = *C1C10* = *C10* = 0, the mutual independency of *A1*, *C1*, *B1*, and *B2*, and the mutual independency of *D1*, *D2*, *B1*, and *B2* are proven.

4 Now, let us assume *A1*, *C1*, *B1*, and *B2*. Then:

C2 *CCCpqrCqr* [*C1*, *p*/*q*, *q*/*Cpq*; *A1*, *p*/*q*, *q*/*p*]
C3 *CCCCqrCprsCCpqs* [*C1*, *p*/*Cpq*, *q*/*CCqrCpr*, *r*/*s*; *C1*]
C4 *CCpCqrCCsqCpCsr* [*C3*, *q*/*Cqr*, *r*/*Csr*, *s*/*CCsqCpCsr*; *C3*, *p*/*s*, *s*/*CpCsr*]
C5 *CCCpqrCNpr* [*C1*, *p*/*Np*, *q*/*Cpq*, *B1*]
C6 *Cpp* [*B2*, *q*/*Cpp*; *A1*, *q*/*p*; *B1*, *q*/*p*]
C7 *CCNppp* [*B2*, *q*/*p*; *C6*]
C8 *CCCpqpp* [*C1*, *p*/*CCpqp*, *q*/*CNpp*, *r*/*p*; *C5*, *r*/*p*; *C7*]
C9 *CCpCpqCpq* [*C3*, *q*/*Cpq*, *r*/*q*, *s*/*Cpq*; *C8*, *p*/*Cpq*]
C10 *CCCqrqCCqrr* [*C3*, *p*/*Cqr*, *s*/*CCqrr*; *C9*, *p*/*Cqr*, *q*/*r*]
C11 *CpCCpqq* [*C2*, *p*/*Cpq*, *q*/*p*, *r*/*CCpqq*; *C10*, *q*/*p*, *r*/*q*]
C12 *CCpCqrCqCpr* [*C4*, *p*/*q*, *q*/*Cqr*, *s*/*p*; *C11*, *p*/*q*, *q*/*r*]
C13 *CpCNpq* [*C12*, *p*/*Np*, *q*/*p*, *r*/*q*; *B1*]

Since it is proved above that the theses *A1*, *C1*, *B1*, and *B2* infer *C1*, *C7*, and *C13*, i.e., Łukasiewicz's celebrated axiom-system of the classical propositional calculus, *cf.* [2], it is self-evident that

$$\{A1, A2, B1, B2\} \rightleftarrows \{A1, C1, B1, B2\}$$

and, therefore, it follows from this result and the discussions presented in sections **1**, **2**, and **3** of the Proof that the system $\{A1, A2, B1, B2\}$ is awkward. On the other hand, it is clear, *cf.* sections **3** and **4** above, that the system $\{A1, C1, B1, B2\}$ can be accepted as an axiom-system of the classical propositional calculus.

Remark: In section **3** above it was proved that the theses belonging to the set $\{D1, D2, B1, B2\}$ are mutually independent. In fact, these theses constitute an adequate axiomatization of the classical propositional calculus, but the required deductions are rather difficult. Namely, let us assume *D1*, *D2*, *B1*, and *B2*. Then:

D3	*CtCpp*	[*D2*, *q/p*, *r/p*; *D1*, *D1*]
D4	*CCNppp*	[*B2*, *q/p*; *D1*]
D5	*CCCpqrCtCNpr*	[*D2*, *p/Np*, *q/Cpq*; *B1*]
D6	*CCCCNpqqrCtCCpqr*	[*D2*, *p/Cpq*, *q/CCNpqq*; *B2*]
D7	*CCpqCtCNNpq*	[*D6*, *r/CtCNNpq*, *t/Cpp*; *D5*, *p/Np*, *r/q*; *D1*]
D8	*CCCtCNNpqrCvCCpqr*	[*D2*, *p/Cpq*, *q/CtCNNpq*, *t/v*; *D7*]
D9	*CCpqCNNpq*	[*D8*, *t/NCNNpq*, *r/CNNpq*, *v/Cpp*; *D4*, *p/CNNpq*; *D1*]
D10	*CCCNNpqrCtCCpqr*	[*D2*, *p/Cpq*, *q/CNNpq*; *D9*]
D11	*CCNpqCtCCpqq*	[*D6*, *p/Np*, *r/CtCCpqq*, *t/Cpp*; *D10*, *r/q*; *D1*]
D12	*CCpCpqCpq*	[*D11*, *p/Cpq*, *t/Cpp*; *B1*; *D1*]
D13	*CCpqCCqrCpr*	[*D2*, *p/Cpq*, *q/CCqrCCqrCpr*, *r/CCqrCpr*, *t/Cpp*; *D2*, *t/Cqr*; *D12*, *p/Cqr*, *q/Cpr*; *D1*]
D14	*CCCCqrCprsCCpqs*	[*D13*, *p/Cpq*, *q/CCqrCpr*, *r/s*; *D13*]
D15	*CCpCqrCCsqCpCsr*	[*D14*, *q/Cqr*, *r/Csr*, *s/CCsqCpCsr*; *D14*, *p/s*, *s/CpCsr*]
D16	*CCNpqCCpqq*	[*D13*, *p/CNpq*, *q/CCpqCCpqq*, *r/CCpqq*; *D11*, *t/Cpq*; *D12*, *p/Cpq*]
D17	*CNNpCCpqq*	[*D5*, *p/Np*, *r/CCpqq*, *t/Cpp*; *D16*; *D1*]
D18	*CNpCCNpqq*	[*D5*, *r/CNpqq*, *t/Cpp*; *B2*; *D1*]
D19	*CCsCNpqCNpCsq*	[*D15*, *p/Np*, *q/CNpq*, *r/q*; *D18*]
D20	*CNpCtNp*	[*D19*, *q/Np*, *s/t*; *D3*, *p/Np*]
D21	*CpNNp*	[*D16*, *p/Np*, *q/CpNNp*; *D20*, *p/Np*, *t/p*; *B1*, *q/NNp*]
D22	*CpCCpqq*	[*D13*, *q/NNp*, *r/CCpqq*; *D21*; *D17*]
D23	*CCpCqrCqCpr*	[*D15*, *p/q*, *q/Cqr*, *s/p*; *D22*, *p/q*, *q/r*]
D24	*CpCNpq*	[*D23*, *p/Np*, *q/p*, *r/q*; *B1*]

Since *D13*, *D24*, and *D4* are the consequences of *D1*, *D2*, *B1*, and *B2*, the proof is complete, *cf.* [2]. It is worth-while to remark that among many axiom-systems of the classical propositional calculus, which I know and which are not artificially constructed, the system $\{D1, D2, B1, B2\}$ is the only one in which the thesis *Cpp* occurs as an axiom.

Final Note: It should be noticed that there can be the awkward axiom-system of the given theory T which in some respect are more elegant than some axiomatizations of T which are not awkward. For example, consider the following axiom-systems of the classical propositional calculus:

System **A**. As the axioms assume *A1*, *A2* and

E1 *CCNpNqCqp*,

i.e., the well-known Łukasiewicz's axiomatization of the considered theory, *cf.* [9], p. 136, footnote 8.

System **B**. Let us assume *E1* and

F1 *CCCpqrCsCCqCrtCqt*

Since *F1* is Meredith's single axiom of the positive implicational calculus, *cf.* [11], it is self-evident that $\{A1;\ A2\} \rightleftarrows \{F1\}$. Therefore, systems **A** and **B** are inferentially equivalent. Moreover, it is easy to prove that each of these systems is not awkward.

System **C**. Let us assume *E1* and

G1 *CCCpqrCCrpCsp*,

i.e., Łukasiewicz's single axiom of the implicational calculus, *cf.* [10] and additionally [14] and [13]. Obviously, $\{G1\} \rightarrow \{F1\}$, but not otherwise. Hence, $\{E1;\ F1\} \rightleftarrows \{E1;\ G1\}$, but, it is self-evident, that system **C** is awkward.

On the other hand, since $\{E1;\ G1\}$ is shorter than $\{E1;\ F1\}$ and the latter system is shorter than $\{A1;\ A2;\ E1\}$, from the point of view that a shorter axiomatization is better than a longer one, system **C** is more elegant than **B** and system **B** than **A**.

Finally, let us consider system **D** of the bi-valued propositional calculus based on single axiom of Meredith, *cf.* [12],

H1 *CCCCCpqCNrNsrtCCtpCsp*

System **D** is shorter than each of the systems **A**, **B** and **C**, and, on the other hand, clearly it is not awkward.

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University of Notre Dame
Notre Dame, Indiana