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# ON SOME SUBSTITUTION INSTANCES OF **R1** AND **L1**

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A study of the epistemic correlates of the modal systems between S4 and S5, [4], has drawn my interest to certain modifications of the "factoring" axioms (*cf.* [11])[1]

**R1** $p \supset (MLp \supset Lp)$
**L1** $p \supset (LMLp \supset Lp)$

which characterize S4.4 and S4.04, respectively. The following substitution instances turned out to be particularly interesting:

**R1.1** $Mp \supset (MLMp \supset LMp)$
**R1.2** $(Lp \supset Lq) \supset (ML(Lp \supset Lq) \supset L(Lp \supset Lq))$
**R1.3** $(p \supset Lp) \supset (ML(p \supset Lp) \supset L(p \supset Lp))$
**L1.1** $Mp \supset (LMLMp \supset LMp)$
**L1.2** $(Lp \supset Lq) \supset (LML(Lp \supset Lq) \supset L(Lp \supset Lq))$
**L1.3** $(p \supset Lp) \supset (LML(p \supset Lp) \supset L(p \supset Lp))$

In this note I want to investigate the results of adding these formulae as new axioms to the base of S4 (with a primitive rule of Necessitation). It will be shown that

(i) S4 + **R1.1** is deductively equivalent to S4.2;
(ii) S4 + **R1.2** is deductively equivalent to S4.3.2;
(iii) S4 + **R1.3** is a new system properly between S4.4 and S4.1.2, or else **R1.3** is a new proper axiom of S4.1.2.
(iv) S4 + **L1.2** is a new system properly between S4.04 and S4 and properly between S4.3.2 and S4;
(v) S4 + **L1.3** is a new system properly between S4.04 and S4.02, or else **L1.3** is a new proper axiom of S4.04.

(a) It is well known (*cf.* [1], p. 252) that in the field of S4 the proper axiom of S4.2,

1. I assume the reader is familiar with the literature cited in this note, especially with [5] and [6].

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**G1** $MLp \supset LMp$

entails and is entailed by

**G2** $MLp \supset LMLp$.

Substitution $p/\neg p$ in **G2** yields

(1) $ML\neg p \supset LML\neg p$

from which

(2) $\neg LML\neg p \supset \neg ML\neg p$

i.e.

(3) $MLMp \supset LMp$

and thus

**R1.1** $Mp \supset (MLMp \supset LMp)$

follows truth-functionally. Hence S4 + **R1.1** is contained in S4.2. Conversely, **G1** is easily seen to follow from **R1.1** in conjunction with the following two S2-theorems:

(4) $MLp \supset Mp$

and

(5) $MLp \supset MLMp$.

Hence (i), i.e., **R1.1** is another new proper axiom of S4.2.

(b) That, in the field of S4, **R1.2** entails the proper axiom of S4.3.2,

**F1** $L(Lp \supset q) \vee (MLq \supset p)$,

can be seen as follows:

| | | |
|---|---|---|
| (6) | $\neg p \supset (Lp \supset Lq)$ | S1 |
| (7) | $MLq \supset ML(Lp \supset Lq)$ | S4° |
| (8) | $\neg(MLq \supset p) \supset ((Lp \supset Lq) \wedge ML(Lp \supset Lq))$ | (6), (7) |
| (9) | $(Lp \supset Lq) \supset (ML(Lp \supset Lq) \supset L(Lp \supset Lq))$ | **R1.2** |
| (10) | $\neg(MLq \supset p) \supset L(Lp \supset Lq)$ | (8), (9) |
| (11) | $L(Lp \supset Lq) \supset L(Lp \supset q)$ | S1 |
| **F1** | $L(Lp \supset q) \vee (MLq \supset p)$ | (10), (11) |

Hence S4 + **R1.2** contains S4.3.2. For the converse, note that **F1** is known to be inferentially equivalent to

**F2** $L(Lp \supset Lq) \vee L(LMLq \supset Lp)$

(*cf.* [10], p. 296), and that in S4.3.2 (which contains S4.2) **G2** is derivable. Moreover, as Zeman has pointed out in [11], in S4.2 (and hence in S4.3.2) *ML* distributes over implications. Thus in particular we have

(12) $ML(Lp \supset Lq) \supset (MLLp \supset MLLq)$. S4.2

Now:

| | | |
|---|---|---|
| (13) | $(MLLp \supset MLLq) \supset (\neg MLLq \supset \neg MLLp)$ | **PC** |
| (14) | $\neg MLq \supset \neg MLLq$ | S2 |
| (15) | $\neg LMLq \supset \neg MLq$ | **G2** |
| (16) | $\neg MLLp \supset \neg MLp$ | S4° |
| (17) | $\neg MLp \supset L(Lp \supset Lq)$ | S2° |
| (18) | $ML(Lp \supset Lq) \supset (\neg LMLq \supset L(Lp \supset Lq))$ | (12)-(17) |

Furthermore we have:

| | | |
|---|---|---|
| (19) | $Lp \supset ((Lp \supset Lq) \supset L(Lp \supset Lq))$ | S4° |

and

| | | |
|---|---|---|
| (20) | $\neg Lp \supset (LMLq \supset L(Lp \supset Lq))$. | **F2** |

(18), (19) + (20) truth-functionally entail

**R1.2** $(Lp \supset Lq) \supset (ML(Lp \supset Lq) \supset L(Lp \supset Lq))$.

Hence (ii), i.e., **R1.2** is another new proper axiom of S4.3.2.

(c) The subsequent deduction shows that

S4.1.4 = S4 + **R1.3**

is an extension of Zeman's S4.04:

| | | |
|---|---|---|
| (21) | $p \supset (\neg p \supset L\neg p)$ | **PC** |
| (22) | $MLp \supset ML(\neg p \supset L\neg p)$ | S4° |
| (23) | $LMLp \supset MLp$ | S1 |
| (24) | $p \supset (LMLp \supset (\neg p \supset L\neg p) \wedge ML(\neg p \supset L\neg p))$ | (21)-(23) |
| (25) | $(\neg p \supset L\neg p) \supset (ML(\neg p \supset L\neg p) \supset L(\neg p \supset L\neg p))$ | **R1.3** |
| (26) | $L(\neg p \supset L\neg p) \supset (LMp \supset Lp)$ | S2° |
| (27) | $LMLp \supset LMp$ | S2 |
| **L1** | $p \supset (LMLp \supset Lp)$ | (24)-(27) |

Moreover, S4.1.4 also is an extension of S4.1 = S4 +

**N1** $L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$,

as is proven by the following deduction:

| | | |
|---|---|---|
| (28) | $MLp \supset ML(p \supset Lp)$ | S4° |
| (29) | $\neg p \supset (p \supset Lp)$ | **PC** |
| (30) | $\neg(MLp \supset p) \supset ((p \supset Lp) \wedge ML(p \supset Lp))$ | (28), (29) |
| (31) | $(p \supset Lp) \wedge ML(p \supset Lp) \supset L(p \supset Lp)$ | **R1.3** |
| (32) | $\neg(MLp \supset p) \supset \neg(L(p \supset Lp) \supset p)$ | (30), (31) |
| (33) | $\neg(L(p \supset Lp) \supset p) \supset \neg L(L(p \supset Lp) \supset p)$ | S1 |
| **N1** | $L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$ | (32), (33) |

Hence we may conclude that S4.1.4 is also an extension of S4.1.2 = S4.1 + **L1** (*cf.* [7], p. 383). It is easily checked that matrix $\mathfrak{M}5$ (in [6], p. 350) varifies **R1.3**. Since $\mathfrak{M}5$ is known to reject S4.2 (*cf.* [7] and [6]), S4.1.4 must be properly included in S4.4. Hence (iii).

(d) Since

(34) $LMLMp \supset LMp$

is a well-known S4-theorem (*cf.* [3], p. 47), **L1.1** is of no further interest.

(e) However,

$$S4.03 = S4 + \mathbf{L1.2}$$

is an interesting new system. Until presently, the only system known to be contained both in S4.3.2 and in S4.04 was S4 itself. But S4.03 $\neq$ S4! Sobociński's matrix $\mathfrak{M}4$ ([6], p. 350) falsifies **L1.2** for, e.g., $p/5$, $q/2$: $(L5 \supset L2) \supset (LML(L5 \supset L2) \supset L(L5 \supset L2)) = (5 \supset 6) \supset (LML(5 \supset 6) \supset L(5 \supset 6)) = 2 \supset (LML2 \supset L2) = 2 \supset (LM6 \supset 6) = 2 \supset (L1 \supset 6) = 2 \supset (1 \supset 6) = 2 \supset 6 = 5$. Since $\mathfrak{M}4$ validates both **N1** and the proper axiom of S4.3,

**D2** $L(Lp \supset Lq) \vee L(Lq \supset Lp)$,

(*cf.* [10], p. 297, [5], p. 310), S4.03 properly contains S4 but is not contained in S4.3.1. We know from [9], p. 382, that S4.02 = S4 +

**Ł1** $L(L(p \supset Lp) \supset p) \supset (LMLp \supset p)$

is not contained in S4.3.2; since, furthermore, S4.01 = S4 +

**Γ1** $MLp \supset (LMp \supset LMLp)$

is not contained in S4.04 (*cf.* [2], p. 569), it follows that S4.03 does not contain any extension of S4 known so far (including the new system S4.021 to be defined in (f) below). Hence (iv).

(f) Consider now

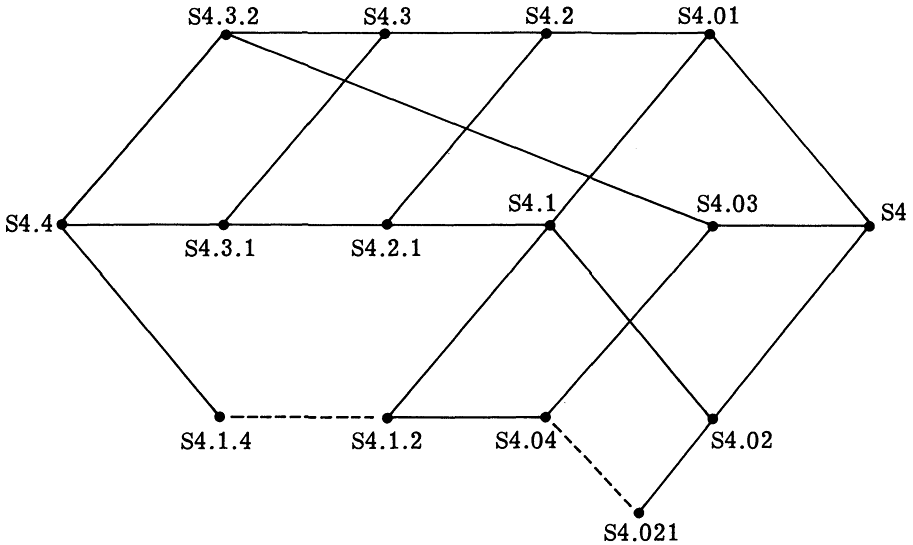
$$S4.021 = S4 + \mathbf{L1.3}$$

Since we have

(35) $LMLp \supset LML(p \supset Lp)$

as a theorem of S4°, and since **Ł1** "relates" to **N1** as **L1.3** "relates" to **R1.3**, the proof given in (c) showing that **R1.3** entails **N1** immediately transforms itself into a proof showing that analogously **L1.3** entails **Ł1**. Hence S4.021 is an extension of S4.02. It is a proper one, because matrix $\mathfrak{M}9$ ([6], p. 350) verifies **Ł1** (*cf.* [9], p. 381) but falsifies **L1.3** for $p/13$: $(13 \supset L13) \supset (LML(13 \supset L13) \supset L(13 \supset L13)) = (13 \supset 16) \supset (LML(13 \supset 16) \supset L(13 \supset 16)) = 4 \supset (LML4 \supset L4) = 4 \supset (LM12 \supset 12) = 4 \supset (L1 \supset 12) = 4 \supset (1 \supset 12) = 4 \supset 12 = 9$. Since **R1.3** does not entail **R1**, it is very probable that **L1.3** does not entail **L1** either, but I have no proof for this assumption. However, (v) is without doubt the case.

(g) The following updated diagram (see page 163):




visualizes the relations among the systems between S4.4 and S4[2]; the broken line indicates that the respective containment has not yet been proven to be proper.

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