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WHEN DO *CONTINUOUS EXTENSIONS EXIST?

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In non-standard analysis we frequently need a non-standard extension of some standard function $f: X \to Y$, that is, an internal function $g: {}^*X \to {}^*Y$ other than $^*f$ such that $g|_X = f$. Often X and Y are topological spaces and we want g to be *continuous, so that for each *open $U \subseteq {}^*Y$, $g^{-1}(U) \subseteq {}^*X$ is *open. H. Gonshor showed in [3] that if X is normal, then any function $f: x \to \mathbb{R}$ has a *continuous extension. Obviously, the problem of which pairs (X, Y) have this property has some uninteresting solutions: (X, X) is such a pair where X is a discrete space. However, by introducing additional conditions on X and Y we can produce an interesting class of such pairs. First we remind the reader of the following definitions.

Definition 1 A topological space X is said to be Urysohn (or functionally Hausdorff) iff for any two points $x, y \in X$ there is a continuous function $g: X \to \mathbb{R}$ such that $g(x) = 1$ and $g(y) = 0$.

Definition 2 A topological space Y is said to be pathwise connected iff for any two points $x, y \in Y$ there is a continuous function $h: \mathbb{I} \to Y$ such that $h(1) = x$ and $h(0) = y$.

It is evident that any space Y is pathwise connected iff for any two points $x, y \in Y$ there is a continuous function $h: \mathbb{R} \to Y$ such that $h(1) = x$ and $h(0) = y$.

Theorem 1 *Let X be a Urysohn space and let Y be a pathwise connected space in a model $\mathfrak{M}$. Then for any enlargement $^*\mathfrak{M}$ of $\mathfrak{M}$, each function $f: X \to Y$ has a *continuous extension $g: {}^*X \to {}^*Y$. Moreover, if X is not a Urysohn space, then there is a function $f_1: X \to \mathbb{R}$ with no *continuous extension, and if Y is not pathwise connected, then there is a function $f_2: \mathbb{R} \to Y$ with no *continuous extension.*

Proof: Let X be a Urysohn space, let Y be a pathwise connected space and let $f: X \to Y$ be an arbitrary function. We shall show that the binary relation R defined by: $\langle x, y\rangle\, Rg$ iff $g: X \to Y$ is continuous and $g(x) = y$ is concurrent on $\{\langle x, f(x)\rangle : x \in X\}$.

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For any n distinct points $x_1, \ldots, x_n$ in X let $y_i = f(x_i)$ for $1 \leq i \leq n$. If $n = 1$, then the function $g \equiv y_1$ trivially satisfies $\langle x, y_1 \rangle\, Rg$, so we can assume that $n > 1$. For each $1 \leq i, j \leq n$ let $g_{i,j} : X \to \mathbb{R}$ be a continuous function such that $g_{i,j}(x_i) = 1$ and $g_{i,j}(x_j) = 0$ if $i \neq j$, and let $g_{i,i} \equiv 1$ for $1 \leq i \leq n$. Now define $g' : X \to \mathbb{R}$ by:

$$g'(x) = \sum_{i=1}^{n} i \prod_{j=1}^{n} g_{i,j}(x),$$

so that g' is continuous and $g'(x_1) = 1, \ldots, g'(x_n) = n$.

For each i, $1 \leq i \leq n - 1$, there exists a continuous function $h_i : [i, i + 1] \to Y$ such that $h_i(i) = y_i$ and $h_i(i + 1) = y_{i+1}$. Define $h' : \mathbb{R} \to Y$ by:

$$h'(r) = \begin{cases} y_1 & \text{if } r < 1 \\ h_1(r) & \text{if } 1 \leq r < 2 \\ \vdots & \\ h_i(r) & \text{if } i \leq r < i + 1 \\ \vdots & \\ h_n(r) & \text{if } n - 1 \leq r < n \\ y_n & \text{if } n \leq r \end{cases}$$

so that h' is continuous and $h'(i) = y_i$ for $1 \leq i \leq n$. The function $h' \circ g' : X \to Y$ is continuous and for each x_i, $h' \circ g'(x_i) = h'(i) = y_i$, so R is concurrent on $\{\langle x, f(x) \rangle : x \in X\}$.

If X is not Urysohn, then there exist points $x, y \in X$ for which there is no continuous $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0$, so there can be no *continuous extension of any function $f_1 : X \to \mathbb{R}$ such that $f_1(x) = 1$ and $f_1(y) = 0$.

If Y is not pathwise connected, then there exist points $x, y \in Y$ for which there is no continuous $f : \mathbb{R} \to Y$ such that $f(1) = x$ and $f(0) = y$, so there can be no *continuous extension of any function $f_2 : \mathbb{R} \to Y$ such that $f_2(1) = x$ and $f_2(0) = y$.

Corollary 1 *A topological space X is Urysohn iff each function $f : X \to \mathbb{R}$ has a *continuous extension $g : {}^*X \to {}^*\mathbb{R}$.*

Corollary 2 *A topological space Y is pathwise connected iff each function $f : \mathbb{R} \to Y$ has a *continuous extension $g : {}^*\mathbb{R} \to {}^*Y$.*

If $(X, \mathfrak{T})$ is a topological space and ${}^*(X, \mathfrak{T}) = ({}^*X, {}^*\mathfrak{T})$ is an enlargement, then ${}^*\mathfrak{T}$ is closed under *finite intersections and internal unions, and contains $\emptyset$ and ${}^*X$. The topology on ${}^*X$ for which ${}^*\mathfrak{T}$ is a base is called the Q-*topology* and will be denoted by ${}^*\overline{\mathfrak{T}}$. In [1] we showed that for any topological spaces X and Y, if $f : {}^*X \to {}^*Y$ is internal, then it is *continuous iff it is Q-continuous.

Corollary 3 *Let X be a Urysohn space, let Y be pathwise connected, and let $f : X \to Y$ be arbitrary. Then f has a* Q-*continuous extension $g : {}^*X \to {}^*Y$.*

The following example shows that the remainder of Theorem 1 does not hold for Q-continuous extensions. Let Y be a totally pathwise disconnected space, so that each continuous f: I $\to Y$ is constant. For any two points $x, y \in Y$, the function f: I $\to Y$ defined by $f(r) = y$ for $r \in$ I$/\{1\}$, $f(1) = x$ has no *continuous extension. In [2] we showed that $\mu(1)$ is Q-clopen, so the function

$$g(r) = \begin{cases} x \text{ for } r \in \mu(1) \\ y \text{ for } r \notin \mu(1) \end{cases}$$

is a Q-continuous extension of f.

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