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BINARY CONSISTENT CHOICE ON TRIPLES

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1 *Introduction* Łoś and Ryll-Nardzewski introduced various principles of "consistent" choice with respect to symmetrical relations in [4], [5] and then showed many were equivalent to **P.I.**, the prime ideal theorem for Boolean algebras.[1] In particular, they showed that even for binary relations, consistent choice from finite sets of cardinality $\leqslant n$ equals **P.I.**, for $n = 4, 5, 6, \ldots$. Here we extend this result to include $n = 3$.

2 Let A be a collection of sets and R a binary symmetric relation. A set t is a *choice set* for A if $t \bar{\bar{\cap}} a = 1$, for all $a \in A$; if, in addition, $\{x, y\} \in R$ for all x, y in t with $x \neq y$, t is an *R-consistent* choice set for A. The collection of all choice sets for A will be denoted by $\mathrm{c}(A)$, while the collection of all R-consistent choice sets is $\mathrm{c}_R(A)$. In [4], [5], the following theorem was proved equivalent to **P.I.**

Theorem 1 *Let A be a collection of finite sets and R a binary symmetric relation, and suppose that for any finite $A_0 \subset A$, $\mathrm{c}_R(A_0) \neq \emptyset$. Then $\mathrm{c}_R(A) \neq \emptyset$.*

Let F_n denote the statement of Theorem 1 if the sets of A are further restricted to have cardinality $\leqslant n$; then, as mentioned above, Łoś and Ryll-Nardzewski even showed $F_n \leftrightarrow$ **P.I.**, $n = 4, 5, 6, \ldots$. We will prove $F_3 \leftrightarrow$ **P.I.** It is, of course, enough to show $F_3 \rightarrow$ **P.I.**

Let $\beta = \langle B, \wedge, \vee, \sim, 0, 1\rangle$ be a Boolean algebra. For any $K \subset B$, let $\tilde{K} = \{\{b, \sim b\} | b \in K\}$. If $K \subset B$ is a subalgebra, any prime ideal of K is an element of $\mathrm{c}(\tilde{K})$. Moreover, any ideal of K which belongs to $\mathrm{c}(\tilde{K})$ is a prime ideal of K. Let $\mathrm{pr}(K)$ denote the set of prime ideals of K and let $\Sigma(B) = \{K \subset B | K \text{ is a finite subalgebra of } \beta\}$. It is easy to see that any $I \in \mathrm{c}(\tilde{B})$ will be a prime ideal of β if $I \cap K$ is an ideal of K, for all $K \in \Sigma(B)$.

Theorem 2 $F_3 \rightarrow$ **P.I.**

Proof: Let $\beta = \langle B, \wedge, \vee, \sim, 0, 1\rangle$ be a Boolean algebra. For each finite

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1. Equivalent here means in **ZF** without the axiom of choice.

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subalgebra, K, and for each $t \in c(\tilde{K}) - \text{pr}(K)$, $t = \{b_1, \ldots, b_n\}$, take $3n - 2$ new sets, $b_1^t, \ldots, b_n^t, c_1^t, \ldots, c_{n-1}^t, d_1^t, \ldots, d_{n-1}^t$, the sets are to be outside of B and distinct. Let A_t be the (unordered) pairs and triples formed from the columns of the following array:

$$\begin{array}{c} d_1^t, \ldots, d_{n-2}^t, d_{n-1}^t \\ b_1^t, b_2^t, \ldots, b_{n-1}^t, b_n^t \\ c_1^t, c_2^t, \ldots, c_{n-1}^t. \end{array} \tag{1}$$

Let $A_K = \bigcup_t A_t$, $(t \in c(\tilde{K}) - \text{pr}(K))$, and $A = \tilde{B} \cup \left(\bigcup_K A_K\right)$, $(K \in \Sigma(B))$. Define a binary symmetric relation on $\bigcup A$ as follows:

$$\{x, y\} \in R \text{ iff } \{x, y\} \neq \{b_i, b_i^t\} \text{ and } \{x, y\} \neq \{c_i^t, d_i^t\}. \tag{2}$$

We claim that $c_R(A) \neq \emptyset$. Since A consists entirely of pairs and triples, it is sufficient, by F_3, to show $c_R(A_0) \neq \emptyset$, for all finite $A_0 \subset A$. Suppose, then, that $A_0 \subset A$, A_0 finite. Then $A_0 \subset D$, where $D = \tilde{B}_0 \cup \left(\bigcup_K A_K\right)$, $(K \in \Sigma')$, with $B_0 \subset B$, $\Sigma' \subset \Sigma(B)$ and both B_0 and Σ' finite. It suffices to show $c_R(D) \neq \emptyset$.

Let H be the finite subalgebra generated by $B_0 \cup \left(\bigcup_K K\right)$, $(K \in \Sigma')$. If I is a prime ideal of H, then $I \cap K \in \text{pr}(K)$, for $K \in \Sigma'$. Moreover, for any $t \in c(\tilde{K}) - \text{pr}(K)$, there exists a $b_{i_t} \in t - (I \cap K) = t - I$. Let

$$a_t = \{c_1^t, \ldots, c_{i_t-1}^t, b_{i_t}^t, d_{i_t}^t, \ldots, d_{n-1}^t\}, \text{ and let } a_K = \bigcup_t a_t, \ (t \in c(\tilde{K}) - \text{pr}(K)).$$

Then, by (1), $a_t \in c(A_t)$, and $a_K \in c(A_K)$. Also $I \cap B_0 \in c(\tilde{B}_0)$. Therefore, $(I \cap B_0) \cup \left(\bigcup_K a_K\right) \in c(D)$, $(K \in \Sigma')$. However, by (2), $(I \cap B_0) \cup \left(\bigcup_K a_K\right)$, $(K \in \Sigma')$, is an R-consistent choice set as well as $(b_{i_t} \notin I)$. Hence $c_R(D) \neq \emptyset$, and so $c_R(A_0) \neq \emptyset$.

By F_3, $c_R(A) \neq \emptyset$. If $s \in c_R(A)$ and $I_s = B \cap s$, then $I_s \in c(\tilde{B})$ and we claim that I_s is a prime ideal of β. It suffices to show that $I_s \cap K$ is an ideal of K, for every $K \in \Sigma(B)$. We show $I_s \cap K \neq t$, for every $t \in c(\tilde{K}) - \text{pr}(K)$. Suppose, then, that $t \in c(\tilde{K}) - \text{pr}(K)$, $t = \{b_1, \ldots, b_n\}$. Since $A_t \subset A$ and s is a choice set for A, s must select an element from each column of (1); since $\{d_i^t, c_i^t\} \notin R$, $1 \leq i \leq n - 1$, by (2), at least one b_i^t must be selected by s. But then $b_i \notin s$, by (2). Therefore, $b_i \notin I_s \cap K$ and $I_s \cap K \neq t$, completing the proof.

We are unable to say whether or not $F_2 \leftrightarrow$ **P.I.** Finally, an indirect proof of Theorem 2 can be based on the recent results of Läuchli [3], that **P.I.** $\leftrightarrow P_n$, $n = 3, 4, 5, \ldots$, where P_n is the theorem of DeBruijn and Erdös [2] that an infinite graph is n-colorable if every finite subgraph is—see our paper [1] for details, as well as additional theorems equivalent to **P.I.**

REFERENCES

[1] Cowen, R. H., "Generalizing König's lemma," *Notre Dame Journal of Formal Logic*, vol. XVIII (1977), pp. 243-247.

[2] De Bruijn, N. G., and P. Erdös, "A colour problem for infinite graphs and a problem in the theory of relations," *Indagationis Mathematicae*, vol. 13 (1951), pp. 369-373.

[3] Läuchli, H., "Coloring infinite graphs and the Boolean prime ideal theorem," *Israel Journal of Mathematics*, vol. 9 (1971), pp. 422-429.

[4] Łoś, J., and C. Ryll-Nardzewski, "On the application of Tychonoff's theorem in mathematical proofs," *Fundamenta Mathematicae*, vol. 38 (1951), pp. 233-237.

[5] Łoś, J., and C. Ryll-Nardzewski, "Effectiveness of the representation theory for Boolean algebras," *Fundamenta Mathematicae*, vol. 41 (1955), pp. 49-56.

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