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INCOMPLETE TRANSLATIONS OF COMPLETE LOGICS

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Let J and K be sets of (interpreted) logical primitives and let **LJ** and **LK** be languages based on J and K respectively, but having a common set of variables and non-logical constants. Let $\mathcal{L}J$ be a logic on **LJ**. Suppose **t** is a function which carries formulas of **LJ** into logically equivalent formulas of **LK**. It has been known since at least 1958 [6] that the completeness of the logic on **LK** ($\mathcal{L}K$), resulting from the translation (by **t**) of $\mathcal{L}J$ is not assured by the completeness of $\mathcal{L}J$.

This result may not be widely known; in 1972 Crossley [2] made a mistake by overlooking it. Crossley constructed a logic, here called $\mathcal{L}[\neg, \&, \exists]$, by translating a logic known to be complete,[1] here called $\mathcal{L}[\,^{\cdot}\,, \rightarrow, \forall]$. Crossley thought that $\mathcal{L}[\neg, \&, \exists]$ is complete, but it is not.[2] Similar examples may have motivated William Frank's recent article [3] in this *Journal* concerning the reasons why some translations do not preserve completeness. Unfortunately, there are two errors in the latter; it is the purpose of this article to set them straight. Frank's main theorem reads as follows:

> *If* $\mathrm{T}(A)$ *is the closure of a formal system in a language* $\mathcal{L}$, *with axioms* A1, . . ., AN; *and rules* R1, . . ., RM *and* **t** *a rule of translation from* $\mathcal{L}$ *to* $\mathcal{L}'$, *then* T', *the closure of* **t**(A1), . . ., **t**(AN), **t**(R1), . . ., **t**(RM), *is equal to* **t**($\mathrm{T}(A)$).

In other words, the only theorems in $\mathcal{L}'$ are translations of theorems in $\mathcal{L}$.

Let $\mathcal{L}$ have 3 sentences: a, b, and c; one axiom: a; and one rule: b/c; so only one theorem: a. Let $\mathcal{L}'$ have two sentences: A, B. Let $\mathbf{t}(a) = A$, $\mathbf{t}(b) = A$, $\mathbf{t}(c) = B$. $\mathcal{L}'$ will then have two theorems: A, B because $\mathbf{t}(a) = A$ is an axiom and $\mathbf{t}(b)/\mathbf{t}(c) = A/B$ is a rule. But B is not the translation of a theorem in $\mathcal{L}$. The problem is that the translation of a non-rule (a/b) can become a rule if the translation is not 1-1.

1. Typographical errors in axiom 5 of [2], p. 19, are assumed to be corrected.

2. For example, some instances of $A \,\&\, A \rightarrow A$ are not provable (see below).

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The second problem with Frank's article is not in the theorem itself but in its alleged applicability. In general, translations are not done the way the theorem suggests. Crossley uses axiom schemes, not axioms. The axiom schemes in $\mathcal{L}[\neg, \&, \exists]$ are the translations of the axiom schemes in $\mathcal{L}[\neg, \rightarrow, \forall]$. So, for example,

$$P(x, x) \rightarrow ((P(x, x) \ \& \ P(x, x)) \rightarrow P(x, x)) \quad \text{(unabbreviated)}$$

is an instance of an axiom scheme (hence a theorem), but is not even in the range of the translation, because of the subformula $P(x, x) \ \& \ P(x, x)$. Halmos [4] (using [5]), at whom Frank's theorem is directed, also does not fall under it. He does not use axiom schemes, but has a rule of substitution. The instances of substitution in the new system include more than just translations from the old system. For instance, from $A \vee \neg A$ (i.e., $\neg(\neg A \ \& \ \neg\neg A)$) we deduce $(A \ \& \ A) \vee \neg(A \ \& \ A)$, but this also is not in the range of the translation.

Modification of Frank's theorem to read "A is a theorem in the translated system just in case it is an instance of substitution of a translation of a theorem" is also false. In Crossley, $\neg((P \rightarrow P) \rightarrow \neg(P \rightarrow P))$ becomes $\neg\neg((P \rightarrow P) \ \& \ \neg\neg(P \rightarrow P))$, but then we deduce $(P \rightarrow P) \ \& \ \neg\neg(P \rightarrow P)$. Nothing in the range of the translation has & as its main connective.

The following is a characterization of the propositional theorems in Crossley's logic. It should indicate that although completeness is usually lost in translations, what does happen can be quite complicated, contrary to Frank's paper.

Let L^- be the class of formulas of Crossley's language that lack quantifiers. From L^-, define L^+ as follows. If ϕ is any formula of L^- and ψ is any formula of L^- not beginning with negation, add $R_{\phi,\psi}$ as a new *atomic* proposition. The purpose of this is to indicate that in Crossley's deductive system certain formulas cannot be broken down, hence they are treated as atomic. Define $\mathsf{K}: L^- \rightarrow L^+$ as follows:

$$\mathsf{K}(P(x, y)) = P(x, y)$$
$$\mathsf{K}(\neg\phi) = \neg\mathsf{K}(\phi)$$
$$\mathsf{K}(\phi \ \& \ \psi) = \mathsf{K}(\phi) \ \& \ \mathsf{K}(\psi) \text{ only if } \psi \text{ begins with negation}$$
$$\mathsf{K}(\phi \ \& \ \psi) = \mathsf{R}_{\phi,\psi} \text{ otherwise.}$$

Then we have $\vdash\phi$ iff $\mathsf{K}(\phi)$ is valid (the proof is tedious but straightforward). Notice (by the abbreviation used) that $\mathsf{K}(\phi \rightarrow \psi) = \mathsf{K}(\phi) \rightarrow \mathsf{K}(\psi)$; so the two Crossley examples above are provable, but that $\phi = (P(x, x) \ \& \ P(x, x)) \rightarrow P(x, x)$ is not provable, because $\mathsf{K}(\phi) = R_{P(x,x),P(x,x)} \rightarrow P(x, x)$ which, of course, is not valid in L^+.[3]

3. Following a suggestion of Hiż [6] and Frank [3] we note that Crossley's logic could be made complete by adding the "reverse translation schemes" $(\phi \ \& \ \psi) \longleftrightarrow \neg(\phi \rightarrow \neg\psi)$ and $\exists x\phi \longleftrightarrow \neg\forall x\neg\phi$. This would entail the usual introduction and elimination rules applying to formulas having single occurrences of the new connective or quantifier (cf. [1], section 4).

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