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Notre Dame Journal of Formal Logic
Volume XVIII, Number 2, April 1977
NDJFAM

THE MODAL PREDICATE LOGICS PF*F

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1 *Introduction* This paper introduces the five modal predicate logics **PF*F** that correspond to the five modal propositional logics **F*F** (**F** = **Ł**, **W**, **S**, **D**, **E**) of [1]. The notation used here follows Smullyan ([2] Chs. IV, V) except where indicated and Wilson [1], and hence many details are omitted. In section 2 the semantics **PF*** are discussed with emphasis on the semantics of quantified expressions. In section 3 the formal systems **PF** are given based again on Smullyan's 'analytic tableaux' [2]. In section 4 the semantical consistency and completeness proofs are illustrated with specific reference to **PS*S**. In the final section, section 5 further points and problems are mentioned, the key one being the possible bearing of these logics **PF*F** on 'traditional predication theory', at least when set against the back-cloth of Angelelli's observations in [3]. Throughout this paper, except where indicated, the index i ranges over 1, 2, 3, 4 and the index j ranges over 1, 2. (Similarly for **i** and **j**).

2 *The Semantics* PF*

2.1 For the syntax of **PF**, we add to the syntax of **F** (omitting propositional variables and the functor 'F'), the symbols '$\forall$' and '$\exists$', and denumerable lists of individual variables $x, y, z, \ldots$; individual parameters, $a, b, c, \ldots$ (the set Π); and for each positive integer n, n-ary predicates $P, Q, R, \ldots$ (in all cases with or without subscripts). We now give some definitions. An atomic formula of **PF** is defined as an $(n + 1)$-tuple $Pv_1 v_2 \ldots v_n$ where P is any n-ary predicate and v_i, $i = 1, 2, \ldots, n$, are any individuals (i.e., variables or parameters).

We can then define a wff in **PF** by making use of the formation rules for **F**, see [1], together with the new rule: If A is a wff in **PF** and x is a variable then $\forall xA$ and $\exists xA$ are wff. The definition of wff in **PF** can be made explicit in the usual way. Signed wff (swff) in **PF** are analogous to swff in **F**. We now define a closed wff (cwff) in **PF** as follows:

A is a cwff of **PF** if A is a wff of **PF** and for every variable x and every parameter a, $A^x_a = A$.

Received October 10, 1971

2.2 *Interpretations* Let U be any non-empty set of individual constants. By an interpretation $\mathcal{I}$ of a set $\mathcal{S}$ of pure cwff of **PF** with constants in U, we mean a mapping which assigns to each n-ary predicate P an n-place relation P' of elements of U. Unlike the case of 'classical' predicate logic we need here to distinguish interpretations from semantical interpretations. For **PF*F**, every interpretation immediately gives rise to *two* different kinds of semantical interpretations: **PF**C_1- and **PF**C_2-semantical interpretations (or abbreviated **PF**C_j-interpretations).

Definition A **PF**C_j-interpretation $\mathcal{I}_j$ of a set $\mathcal{S}$ of pure cwff is an interpretation $\mathcal{I}$ together with a mapping of $\mathcal{S}$ onto C_j-truth-values (those relevant to the particular logic **PF*F**).

One example, for **PS*S** and **PF**C_1-interpretations of a set $\mathcal{S}$, the C_1-truth-values are t_1, f_1, i_1.

For cwff with parameters, by an interpretation $\mathcal{I}$ of a set $\mathcal{S}$ of cwff for a universe U we mean an interpretation of the predicates in $\mathcal{S}$ together with a mapping φ that assigns to each parameter 'a' that occurs in any element of $\mathcal{S}$ a constant k in U (i.e., $\varphi(a) = k$). To define C_j-truth-value of a cwff of **PF** under a **PF**C_j-interpretation, we proceed as follows.

We can define a **PF**C_j-valuation tree for a cwff of **PF** with constants in U (*cf.* Smullyan's U-formula in [2], pp. 46-48). The semantic rules for quantified cwff are given on the basis of the following rules (i.e., those rules applicable to the particular logic **PF*F** under interpretation). The semantic rules for the functions N, C, A, K, E, M, L, T are as in [1], Tables I-VII, except that for **E*E**, we introduce a new functor 'T' (see below).

Semantic Rules for Quantifiers ($\forall$, $\exists$)

SA_j-Rules

SA_j1. $\forall xPx$ is t_j iff for every $k \in U$ Pk is t_j.
SA_j2. $\forall xPx$ is f_j iff for some $k \in U$ Pk is f_j.
SA_j3. $\exists xPx$ is t_j iff for some $k \in U$ Pk is t_j.
SA_j4. $\exists xPx$ is f_j iff for every $k \in U$ Pk is f_j.

SB_j-Rules

SB_j1. $\forall xPx$ is t_j iff for every $k \in U$ Pk is t_j.
SB_j2. $\forall xPx$ is f_j iff for some $k \in U$ Pk is f_j.
SB_j3. $\forall xPx$ is i_j iff for some $e \in U$ Pe is i_j and for every $k \in U$ Pk is t_j or i_j.
SB_j4. $\exists xPx$ is t_j iff for some $k \in U$ Pk is t_j.
SB_j5. $\exists xPx$ is f_j iff for every $k \in U$ Pk is f_j.
SB_j6. $\exists xPx$ is i_j iff for some $l \in U$ Pl is i_j and for every $k \in U$ Pk is f_j or i_j.

These semantic rules for '$\forall$' and '$\exists$' bring out the intuitive meaning of '$\forall$' as 'for every' and '$\exists$' as 'for some'. For example, in **PS*S** and **PS**C_1-interpretations we make use of SB_1 1-6 and for **PS**C_2-interpretations we make use of SA_2 1-4. Referring to the C_j-truth-tables given in [1] for the functors 'K' ('and') and 'A' ('or') we can see that these interpretations of '$\forall$' and '$\exists$' in **PF*F**, for a denumerable universe U, amount to:

'$\forall xPx$' is interpreted as 'Pk_1 and Pk_2 and . . .' where $k_i \in U$, $i = 1, 2, 3 . . . .$
'$\exists xPx$' is interpreted as 'Pk_1 or Pk_2 or . . .' where $k_i \in U$, $i = 1, 2, 3, ...$

The semantical interpretations of the functors N, C, A, K, E, M, L, T are as in [1], Tables I-VII, except that for **E*E** we introduce a new functor 'T' which seems more appropriate, with the following C_2-table:

(E*E)	C_2	T
	t_2	t_2
	f_2	f_2
	i_2	i_2

'Tp' is interpreted as '$'p'$ is or will be C_2-true' or in the autonomous mode[1] of [1], 'p is or will be C_2-true'.
The C_1-table for 'Tp' is as in [1].

In the formal systems **PF** we make use of **PF**-tableaux with signed cwff as points. We can extend the definitions of **PF**C_j-interpretations to sets of **PF**C_j-scwff. As in [1], although cwff in **PF** are given both **PF**C_1-interpretations and **PF**C_2-interpretations, **PF**C_j-scwff are given only **PF**C_j-interpretations.

2.3 *Satisfiability, validity, and semantic models* We give the relevant definitions:

1. A cwff A of **PF** is **PF**C_j-satisfiable if A is C_j-true under at least one **PF**C_j-interpretation (i.e., in at least one universe U of constants).

2. A cwff A of **PF** is **PF**C_j-valid if A is C_j-true under every **PF**C_j-interpretation in every universe U.

Here **PF**C_1-, **PF**C_2-satisfiability and validity are said to be fundamental.

As in [1] (*cf.* $\mathbf{F}_i$-tautology), we define two derived kinds of satisfiability and validity.

3. A is **PF**C_1/C_2-satisfiable (valid) if A is **PF**C_1-satisfiable (valid) or A is **PF**C_2-satisfiable (valid).

4. A is **PF**C_1C_2-satisfiable (valid) if A is **PF**C_1-satisfiable (valid) and A is **PF**C_2-satisfiable (valid).

If 1 does not hold for a cwff A we say that A is not **PF**C_j-satisfiable. If 2 does not hold we say that A is **PF**C_j-invalid. Hence A is **PF**C_j-invalid if there is a **PF**C_j-interpretation in some universe U such that A is not C_j-true. As in [1] we can introduce four sub-logics $\mathbf{PF}\underset{i}{*}\mathbf{F}_i$ for **PF*F**, each associated with $\mathbf{PF}_i$-validity and $\mathbf{PF}_i$-invalidity. Here $\mathbf{PF}_i$-, $i = 1, 2, 3, 4$ correspond to **PF**C_1-, **PF**C_2-, **PF**C_1/C_2-, and **PF**C_1C_2- respectively.

1. Autonomous Mode: R. Carnap, *The Logical Syntax of Language*, Paul, Trench, Trubner & Co., London (1949). Thomas Aquinas, *In Aristotelis libros* Peri hermeneias et Posteriorum analyticorum expositio, Lecture 5, number 6. Marietti, Torino (1955).

5. *Semantic models* We say a set $\mathcal{S}$ of cwff (or PFC_j-scwff) is PFC_j-satisfiable if there is at least one PFC_j-interpretation such that every cwff (or PFC_j-scwff) in $\mathcal{S}$ is C_j-true.

A PFC_j-interpretation for which a given set $\mathcal{S}$ is PFC_j-satisfiable is called a PFC_j-semantic model or simply a PFC_j-model for the set $\mathcal{S}$.

3 *The Formal Systems* PF

3.1 We list here groups of rules for tableaux construction comparable to [2], Ch. V, p. 52, and [1] section **4**, and then indicate which rules apply to each of the five systems **PF** (**F** = **Ł**, **W**, **S**, **D**, **E**). The schematic variables used here are 'X', 'Y' and 'Z' for wff. Also 'a' denotes any parameter and 'b' any new parameter not already introduced in the branch.

A_j-*Tableaux Rules*

I. *Rules for non-modal functors* (N, C, A, K, E)

$A_j1.\ \dfrac{T_jNX}{F_jX}$ $A_j2.\ \dfrac{F_jNX}{T_jX}$ $A_j3.\ \dfrac{T_jCXY}{F_jX \mid T_jY}$ $A_j4.\ \dfrac{F_jCXY}{\begin{array}{c}T_jX\\ F_jY\end{array}}$

$A_j5.\ \dfrac{T_jAXY}{T_jX \mid T_jY}$ $A_j6.\ \dfrac{F_jAXY}{\begin{array}{c}F_jX\\ F_jY\end{array}}$ $A_j7.\ \dfrac{T_jKXY}{\begin{array}{c}T_jX\\ T_jY\end{array}}$ $A_j8.\ \dfrac{F_jKXY}{F_jX \mid F_jY}$

$A_j9.\ \dfrac{T_jEXY}{\begin{array}{c|c}T_jX & F_jX\\ T_jY & F_jY\end{array}}$ $A_j10.\ \dfrac{F_jEXY}{\begin{array}{c|c}T_jX & F_jX\\ F_jY & T_jY\end{array}}$

II. *Rules for modal functors* (M, L, T)

$A_111.\ \dfrac{T_1MX}{T_1X \mid F_1X}$ $A_112.\ \dfrac{F_1LX}{T_1X \mid F_1X}$ $A_211.\ \dfrac{T_2TX}{T_2X}$ $A_212.\ \dfrac{F_2TX}{F_2X}$

$A_213.\ \dfrac{T_2MX}{T_2X}$ $A_214.\ \dfrac{F_2MX}{F_2X}$ $A_215.\ \dfrac{T_2LX}{T_2X}$ $A_216.\ \dfrac{F_2LX}{F_2X}$

III. *Rules for quantifiers* ($\forall$, $\exists$)

$A_j17.\ \dfrac{T_j\forall xZx}{T_jZ^x_a}$ $A_j18.\ \dfrac{F_j\exists xZx}{F_jZ^x_a}$ $A_j19.\ \dfrac{T_j\exists xZx}{T_jZ^x_b}$ $A_j20.\ \dfrac{F_j\forall xZx}{F_jZ^x_b}$

Condensed A-*Rules*

$A1.\ \dfrac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}}$ $A2.\ \dfrac{\beta}{\beta_1 \mid \beta_2}$ $A3.\ \dfrac{\gamma}{\begin{array}{c|c}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{array}}$ $A4.\ \dfrac{\lambda}{\lambda_1 a}$ $A5.\ \dfrac{\mu}{\mu_1 b}$

B_j-*Tableaux Rules*

I. *Rules for non-modal functors* (N, C, A, K, E)

$B_j1.$ $$\frac{T_jNX}{F_jX}$$

$B_j2.$ $$\frac{F_jNX}{T_jX}$$

$B_j3.$ $$\frac{I_jNX}{I_jX}$$

$B_j4.$ $$\frac{T_jCXY}{\begin{array}{c|c|c} F_jX & T_jY & I_jX \\ & & I_jY \end{array}}$$

$B_j5.$ $$\frac{F_jCXY}{\begin{array}{c} T_jX \\ F_jY \end{array}}$$

$B_j6.$ $$\frac{I_jCXY}{\begin{array}{c|c} I_jX & T_jX \\ F_jY & I_jY \end{array}}$$

$B_j7.$ $$\frac{T_jAXY}{\begin{array}{c|c} T_jX & T_jY \end{array}}$$

$B_j8.$ $$\frac{F_jAXY}{\begin{array}{c} F_jX \\ F_jY \end{array}}$$

$B_j9.$ $$\frac{I_jAXY}{\begin{array}{c|c|c} I_jX & F_jX & I_jX \\ F_jY & I_jY & I_jY \end{array}}$$

$B_j10.$ $$\frac{T_jKXY}{\begin{array}{c} T_jX \\ T_jY \end{array}}$$

$B_j11.$ $$\frac{F_jKXY}{\begin{array}{c|c} F_jX & F_jY \end{array}}$$

$B_j12.$ $$\frac{I_jKXY}{\begin{array}{c|c|c} I_jX & T_jX & I_jX \\ T_jY & I_jY & I_jY \end{array}}$$

$B_j13.$ $$\frac{T_jEXY}{\begin{array}{c|c|c} T_jX & F_jX & I_jX \\ T_jY & F_jY & I_jY \end{array}}$$

$B_j14.$ $$\frac{F_jEXY}{\begin{array}{c|c} T_jX & F_jX \\ F_jY & T_jY \end{array}}$$

$B_j15.$ $$\frac{I_jEXY}{\begin{array}{c|c|c|c} T_jX & I_jX & F_jX & I_jX \\ I_jY & T_jY & I_jY & F_jY \end{array}}$$

II. *Rules for modal functors* (M, L, T)

$B_116.$ $$\frac{T_1MX}{\begin{array}{c|c|c} T_1X & F_1X & I_1X \end{array}}$$

$B_117.$ $$\frac{F_1LX}{\begin{array}{c|c|c} T_1X & F_1X & I_1X \end{array}}$$

$B_118.$ $$\frac{I_1TX}{\begin{array}{c|c|c} T_1X & F_1X & I_1X \end{array}}$$

$B_216.$ $$\frac{T_2MX}{\begin{array}{c|c} T_2X & I_2X \end{array}}$$

$B_217.$ $$\frac{F_2MX}{F_2X}$$

$B_218.$ $$\frac{T_2LX}{T_2X}$$

$B_219.$ $$\frac{F_2LX}{\begin{array}{c|c} F_2X & I_2X \end{array}}$$

$B_220.$ $$\frac{T_2TX}{T_2X}$$

$B_221.$ $$\frac{F_2TX}{F_2X}$$

$B_222.$ $$\frac{I_2TX}{I_2X}$$

III. *Rules for quantifiers* $(\forall, \exists)$

$B_j23.$ $$\frac{T_j\forall xZx}{T_jZ^x_a}$$

$B_j24.$ $$\frac{F_j\forall xZx}{F_jZ^x_b}$$

$B_j25.$ $$\frac{I_j\forall xZx}{\begin{array}{c} I_jZ^x_b \\ \begin{array}{c|c} T_jZ^x_a & I_jZ^x_a \end{array} \end{array}}$$

$B_j26.$ $$\frac{T_j\exists xZx}{T_jZ^x_b}$$

$B_j27.$ $$\frac{F_j\exists xZx}{F_jZ^x_a}$$

$B_j28.$ $$\frac{I_j\exists xZx}{\begin{array}{c} I_jZ^x_b \\ \begin{array}{c|c} F_jZ^x_a & I_jZ^x_a \end{array} \end{array}}$$

Condensed B-*Rules*

B1. $$\frac{\alpha}{\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array}}$$

B2. $$\frac{\beta}{\begin{array}{c|c} \beta_1 & \beta_2 \end{array}}$$

B3. $$\frac{\gamma}{\begin{array}{c|c} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{array}}$$

B4. $$\frac{\delta}{\begin{array}{c|c|c} \delta_1 & \delta_2 & \delta_3 \\ \delta_4 & \delta_5 & \delta_6 \end{array}}$$

B5. $$\frac{\epsilon}{\begin{array}{c|c|c|c} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_5 & \epsilon_6 & \epsilon_7 & \epsilon_8 \end{array}}$$

B6. $$\frac{\lambda}{\lambda_1 a}$$

B7. $$\frac{\mu}{\mu_1 b}$$

B8. $$\frac{\nu}{\begin{array}{c} \nu_1 b \\ \begin{array}{c|c} \nu_2 a & \nu_3 a \end{array} \end{array}}$$

The tableaux rules applicable to the different systems are:

PŁ : A_i-Rules, omitting $A_2$11-12.
PS : B_1-Rules, A_2-Rules.
PE : B_j-Rules.
PW : B_1-Rules, A_2-Rules omitting $A_2$11-12, $B_1$18.
PD : B_j-Rules, omitting $B_1$18, $B_2$20-22.

3.2 The definitions of open and closed branches and tableaux are analogous to [1]. A branch is closed if it is broken or incompatible. In **PS*S**, a branch is broken if it contains at least one scwff of the form F_1MX, I_1MX, T_1LX, I_1LX, T_1TX, or F_1TX. Also, the definitions of complete branch and completed tableaux are analogous to [1] and [2]. (For example in **PS**, if ν is a point in a complete branch τ_c then for some $b \in \Pi$, $\nu_1 b$ is in τ_c, and for every $a \in \Pi$, $\nu_2 a$ or $\nu_3 a$ is in τ_c.)

3.3 *Provability and rejection in* **PF** We illustrate for **PS**: A cwff X in **PS** is **PS**C_1-provable iff there exist closed **PS**-tableaux $\mathcal{T}(F_1X)$, $\mathcal{T}(I_1X)$ and we write **PS**$C_1 \vdash X$ if such tableaux exist, otherwise we write **PS**$C_1 \dashv X$ and say that X is **PS**C_1-rejectable. A cwff X in **PS** is **PS**C_2-provable iff there exist a closed **PS**-tableau $\mathcal{T}(F_2X)$ and we write **PS**$C_2 \vdash X$ if such a tableau exists, otherwise we write **PS**$C_2 \dashv X$ and say X is **PS**C_2-rejectable. **PS**C_1C_2- and **PS**C_1/C_2-provability and rejection are defined analogously (*cf.* [1], sections **4.2** and **4.3**). For **PF** we can introduce **PF**$_i$-provability and rejection, i = 1, 2, 3, 4 corresponding to **PF**C_1-, **PF**C_2-, **PF**C_1C_2- and **PF**C_1/C_2- respectively.

4 *Semantical Consistency and Completeness Proofs for* **PF*F**

4.1 Properties hold for the unifying notation $\alpha, \beta, \ldots, \nu$, analogous to the properties P1-P6 of [1], section **5.1**, and the properties F1-F4 of [2], p. 52. These properties for λ, μ, and ν derive from the semantical rules SA_j1-4 and SB_j1-6 (see section **2.2**). We label these properties PA1-4 (see A-Rules) and PB1-6 (see B-Rules). We illustrate property PB6 for **PS*S**.

In **PS*S**, under any **PS**C_1-interpretation in a universe U, for any **PS**C_1-scwff of type ν, ν is C_1-true iff for some $k \in U$, $\nu_1 k$ is C_1-true and for every $k \in U$ either $\nu_2 k$ is C_1-true or $\nu_3 k$ is C_1-true. These properties are used in the proofs and illustrated for **PS*S**.

4.2 Theorem 1 (Semantical consistency for **PS*S**) *If* **PS**$_i \vdash X$, *then* X *is* **PS**$_i$*-valid.*

Proof: Case $i = 1$. Let $\mathcal{T}_1(S_1X)$ be a **PS**C_1-tableaux. We show first that the immediate extension $\mathcal{T}_2(S_1X)$ of $\mathcal{T}_1(S_1X)$ is **PS**C_1-satisfiable under every **PS**C_1-interpretation $\mathcal{I}_1$ in every universe U, for which $\mathcal{T}_1(S_1X)$ is **PS**C_1-satisfiable.

Suppose $\mathcal{T}_1(S_1X)$ is **PS**C_1-satisfiable under $\mathcal{I}_1$ in a universe U, then it contains a **PS**C_1-satisfiable branch $\tau(S_1X)$ say. Suppose now $\mathcal{T}_2(S_1X)$ is formed by applying one operation, derived from one of the B-Rules B1-8, to some branch τ_1 of $\mathcal{T}_1$. As in [1], section **5**, Theorem 1, we need consider only the case where τ_1 is identical with τ. If τ was extended by B1, B2, B3 (see [1], section **5**, Theorem 1), B6, B7 (see [2], p. 53), B4 or B5, then the extended branch is $\mathbf{PS}C_1$-satisfiable, because of the properties PB1-7 mentioned above. Finally, if τ was extended by B8, then some ν occurs in τ and τ is extended to τ - $\nu_1 b$ or $\tau \begin{smallmatrix} \nearrow \nu_3 a \\ \searrow \nu_2 a \end{smallmatrix}$, a, $b \in \Pi$. In the former case τ - $\nu_1 b$ is $\mathbf{PS}C_1$-satisfiable (*cf.* proof of S_4 in [2], p. 53). Also by PB8, either τ - ν_2 or τ - ν_3 is $\mathbf{PS}C_1$-satisfiable. Hence $\mathcal{T}_2(S_1X)$ is $\mathbf{PS}C_1$-satisfiable. Thus by induction, if the origin S_1X or $\mathcal{T}_1(S_1X)$ is C_1-true under $\mathcal{I}_1$ then $\mathcal{T}_1(S_1X)$ is $\mathbf{PS}C_1$-satisfiable under $\mathcal{I}_1$. Suppose now $\mathbf{PS}C_1 \vdash X$. Then there exist closed tableaux $\mathcal{T}(F_1X)$ and $\mathcal{T}(I_1X)$. But neither of these tableaux can be $\mathbf{PS}C_1$-satisfiable, since each and every branch in $\mathcal{T}(F_1X)$ and $\mathcal{T}(I_1X)$ must contain at least one scwff which is C_1-false under $\mathcal{I}_1$. Hence, by the above, X cannot be C_1-false or C_1-indeterminate. Hence X is C_1-true. But $\mathcal{I}_1$ and U are arbitrary. Hence X is $\mathbf{PS}C_1$-valid.

Case $\mathbf{i} = \mathbf{2}$ is similar to case $\mathbf{i} = \mathbf{1}$, and cases $\mathbf{i} = \mathbf{3}, \mathbf{4}$ follow from cases $\mathbf{i} = \mathbf{1}, \mathbf{2}$.

4.3 In Smullyan [2], the completeness proof for first-order logic makes use of 'Hintikka sets' for arbitrary universes, and here for **PF*F** we make use of the analogous notion of $\mathbf{PF}C_j$-model sets for arbitrary universes of constants. For **PS*S**, by a $\mathbf{PS}C_j$-model set for a universe U, we mean a set $\mathcal{S}$ of $\mathbf{PS}C_j$-scwff with constants in U (*cf.* U-formula in [2], pp. 46-48) such that the following conditions hold:

M0. There are no scwff of $\mathcal{S}$ which are broken or incompatible (in the terminology of tableaux given above).

M1. If $\alpha \in \mathcal{S}$, then α_1 and $\alpha_2 \in \mathcal{S}$.

M2. If $\beta \in \mathcal{S}$, then β_1 or $\beta_2 \in \mathcal{S}$.

M3, M4, M5, M6, M7 correspond to type γ, δ, ϵ, λ and μ.

M8. If $\nu \in \mathcal{S}$ then for some $k \in U$, $\nu_1 k \in \mathcal{S}$ and for every $k \in U$, $\nu_2 k \in \mathcal{S}$ or $\nu_3 k \in \mathcal{S}$.

Lemma 1 *Every* $\mathbf{PS}C_j$*-model set for a universe* U *is* $\mathbf{PS}C_j$*-satisfiable,* i.e., *has a* $\mathbf{PS}C_j$*-semantic model in* U.

Proof: Case $j = 1$. Let $\mathcal{S}_1$ be a $\mathbf{PS}C_1$-model set. Let U be a non-empty universe of constants. We need to give a $\mathbf{PS}C_1$-interpretation $\mathcal{I}_1$ in U for every atomic scwff in $\mathcal{S}_1$, such that every element in $\mathcal{S}_1$ is t_1 under this $\mathcal{I}_1$. We assign C_1-truth-values to every atomic cwff $Zu_1u_2 \ldots u_n$ as follows:

(1) If $T_1Z \in \mathcal{S}_1$, assign Z the value t_1.

(2) If $F_1Z \in \mathcal{S}_1$, assign Z the value f_1.

(3) If $I_1Z \in \mathcal{S}_1$, assign Z the value i_1.

(4) If neither T_1Z, F_1Z or $I_1Z \in \mathcal{S}_1$ then assign Z the value t_1. (This is always possible for $\mathcal{S}_1$ since M0 holds.)

We now show by induction on the degree of a scwff that every element $S_1Y \in \mathcal{S}_1$ is t_1 under this $\mathcal{I}_1$. By (1), (2), (3) it follows that every element of degree 0 is t_1. Suppose now S_1Y is of positive degree and that every element of lower degree than S_1Y is t_1. We show S_1Y is t_1. By M0, S_1Y must be some $\alpha_1\beta, \ldots,$ or ν. Suppose S_1Y is a ν. Then, by M8, for some $k \in U$, $\nu_1 k \in \mathcal{S}_1$ and for every $k \in U$, $\nu_2 k \in \mathcal{S}_1$ or $\nu_3 k \in \mathcal{S}_1$. Hence by hypothesis, since $\nu_1 k$, $\nu_2 k$ and $\nu_3 k$ are of lower degree than ν, then for some k, $\nu_1 k$ is t_1 and for every k, $\nu_2 k$ is t_1 or $\nu_3 k$ is t_1. Hence, by PB8, ν is t_1. Similarly if S_1Y is an $\alpha, \beta, \ldots,$ or μ, then, by PB1, PB2, . . ., or PB7 and M1-M7, it follows that S_1Y is t_1. Hence by induction $\mathcal{S}_1$ is PSC_1-satisfiable under $\mathcal{I}_1$, and the above PSC_1-interpretation in U therefore constitutes a PSC_1-semantic model in U.

Case $j = 2$ is analogous to case $j = 1$.

Lemma 2 *Every complete open branch τ, say, of any PSC_j-tableau is PSC_j-satisfiable.*

Proof: By definition, the set of PSC_j-scwff in τ, $\mathcal{S}$ say, constitute a PSC_j-model set for the universe Π of parameters (or a denumerable universe U of constants under a mapping). Hence, by Lemma 1, the result follows.

Lemma 3 *For any scwff S_jX of* **PS**, *there exists a completed PSC_j-tableau.*

Proof: Unlike completed **F**-tableaux in [1] which need involve only a finite number of points, in **PF**-tableaux generally, complete branches can be infinite (due to the presence of λ-type or ν-type scwff). The proof of this lemma depends on the fact that for a scwff S_jX of finite degree, any possible PSC_j-tableaux will involve only a denumerable number of occurrences of λ-type and ν-type scwff and applications of A4, B6 or B8 can then lead only to a denumerable number of successors for λ- and ν-completion. In principle then, it is possible to define a procedure for completing any PSC_j-tableaux, although in general we are unable (as in first-order logic, [2], p. 63) to decide at any finite stage of the construction whether the completed tableaux will close or remain open. We use Lemmas 2 and 3 to prove

Theorem 2 (Semantical completeness for **PS*S**) *If X is PS_i-valid, then* $\mathsf{PS}_i \vdash X$.

Proof: Case $i = 1$. Suppose X is PSC_1-valid. Let $\mathcal{T}(F_1X)$ and $\mathcal{T}(I_1X)$ be completed **PS**-tableaux which exist by Lemma 3. If either tableau contains an open branch τ_1, then by Lemma 2, τ_1, would be PSC_1-satisfiable. Hence either F_1X or I_1X would be C_1-true, i.e., X would be C_1-false in both cases. Thus $\mathcal{T}(F_1X)$ and $\mathcal{T}(I_1X)$ are closed. Hence $\mathsf{PSC}_1 \vdash X$.

Case $i = 2$ is similar to case $i = 1$. Cases $i = 3, 4$ follow.

From Theorems 1 and 2 we get:

Theorem 3 (Semantical consistency and completeness for **PS*S**) $\mathsf{PS}_i \dashv X$ *iff X is PS_i-invalid.*

We now state the main theorem for **PF*F**:

Main Theorem (Semantical consistency and completeness for $\mathsf{PF*F}$)

(1) $\mathsf{PF}_i \vdash X$ *iff* X *is* PF_i*-valid,*
(2) $\mathsf{PF}_i \dashv X$ *iff* X *is* PF_i*-invalid,*

where $\mathsf{F} = \text{Ł}, \mathsf{W}, \mathsf{S}, \mathsf{D}, \mathsf{E}$, $\mathsf{i} = 1, 2, 3, 4$.

5 *Further Points and Controversial Problems*

5.1 *The 3-valued logic* In [1] we suggested a possible semantical interpretation for Słupecki's[2,3] tertium function '*Tp*', but these interpretations require '*T*' to be considered as a modal functor rather than as a non-modal one. Also, on the basis of these interpretations for '*T*' it follows that the correct (non-modal) 3-valued logic is the one initially formulated by Łukasiewicz without the functor '*T*', and axiomatised by Wajsberg.[4] An interesting question remains in connection with the full 3-valued logic: is it possible to provide a non-modal semantical interpretation for the functor '*T*'?

5.2 C_1*-indeterminateness* We suggested in [1] that in mathematics, C_1-truth, or more generally C_1-truth-value of a proposition, bears some relationship to the intuitionist view of mathematical truth and mathematical propositions. In this connection the third truth-value C_1-indeterminateness, i_1, is elevated to a new important logical status. We can roughly divide the C_1-indeterminate propositions into four classes:

(1) Absolutely undecidable propositions.
(2) Propositions relating to future contingent events.
(3) The logical propositions '*TX*' of $\mathsf{F*F}$ and $\mathsf{PF*F}$, where, in the former case '*X*' is a wff of F, and in the latter '*X*' is a cwff of PF.
(4) Propositions which are C_1-indeterminate but which may or may not become C_1-determinate.

Of these four classes (1) and (3) can be taken as examples of propositions that are intrinsically C_1-indeterminate; (3) on purely logical grounds and (1) on more than logical grounds. (2) provides examples of propositions that will become C_1-determinate. The interesting and problematic class is (4). An examination of Heyting [4] indicates that the troublesome cases for the law of excluded middle arise from propositions that should be placed in this class (4). (See for example [4], pp. 17-18, and p. 24.) Thus, concerning Heyting's 'intuitionism' (and certain other varieties) and also the new approach to the foundations suggested in [5], a common problem is to provide a sharpened characterisation of those mathematical propositions that fall into this class (4).

5.3 *Ontological presuppositions and consequent logical possibility and necessity* Several recent authors have been concerned with such questions

2. J. Słupecki, "The full three-valued propositional logic." See footnote 3.

3. S. McCall, editor, *Polish Logic, 1920-1939*, Clarendon Press, Oxford (1967).

4. M. Wajsberg, "The three-valued logic." See footnote 3.

as ontological presuppositions and logic with or without ontology. The logics **F*F**, *cf.* [1], and **PF*F** here clearly belong to the class of logics with ontology, where it is hoped that the ontological presuppositions have been made explicit. The essential ontological presupposition posited by these logics **F*F** and **PF*F**, as stated in [1], section **2**, is that of the postulate of a Supreme Being or the Divine. A crucial question immediately suggests itself here: is it possible in these logics **F*F**, **PF*F**, and the semantics in particular, to dispense with this presupposition? I think not. The idea of the categories C_1 and C_2 depends in turn upon the idea of two qualitatively different kinds of intelligibility—human intelligibility and Divine intelligibility. Although it is true that one could refer to the transcendental idea of C_2-truth, without making this postulate of the Divine, unless we make such a postulate the concepts of the absolute category and C_2-truth would become merely blank cyphers.

Of opinions on the meaning of the phrases 'logical possibility' and 'logical necessity' we can single out the predominant view associated with the analytic/synthetic division of propositions, *cf.* e.g., Carnap.[5] The notions of logical possibility and necessity underlying these logics **F*F**, in [1], involve quite different ideas. Recalling the semantics **F*** in [1], section **3**, 'possibility' and 'necessity' are ambiguous, the ambiguity arising from the need to consider both under each of the two categories C_1 and C_2. For these logics, logical possibility under C_1 is a wider notion, while under C_2, it is a narrower one than in Carnap. The reverse is the case for logical necessity: under C_1, logical necessity is a narrower concept, while under C_2, it is a wider concept than in Carnap. We illustrate with a few examples for **F*F**.

The propositions '$2 + 2 = 5$', 'the number of planets $\neq 9$', '$2 + 2 = 4$', and 'the number of planets $= 9$' we can (nominally) affirm as $\mathfrak{f}_1\mathfrak{f}_2$, $\mathfrak{f}_1\mathfrak{f}_2$, $\mathfrak{t}_1\mathfrak{t}_2$, and $\mathfrak{t}_1\mathfrak{t}_2$ respectively. Under C_1, 'it is possible that '$2 + 2 = 5$' is C_2-true' is affirmed $\mathfrak{t}_1$. Under C_2, 'it is possible that 'the number of planets $\neq 9$' is C_2-true' is affirmed $\mathfrak{f}_2$. Under C_1, 'it is necessary that '$2 + 2 = 4$' is C_2-true' is affirmed $\mathfrak{f}_1$. Under C_2, 'it is necessary that 'the numbers of planets $= 9$' is C_2-true' is affirmed $\mathfrak{t}_2$.

We associate here the notions of consequent logical possibility and necessity with each of the logics **F*F** and **PF*F**, (**F** = **Ł**, **W**, **S**, **D**, **E**), and mean by this: possibility and necessity consequent on a given logic, including of course the full semantics, formal properties and presuppositions, ontological or otherwise, for the given logic. In [6] the phrase 'logical possibility' and derivatives should be understood in this sense of consequent logical possibility. This point is important because, in particular, the argument in [6] on the 'logical possibility of freedom' for example, **S*S** and **E*E** only remain valid when we take into account the 'loaded' semantics of these logics.

5.4 *Additional functors and modal logics* **F*F**, **PF*F** In [1] we also introduced the functor '*F*' not included in this presentation of the logics **PF*F**.

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5. R. Carnap, *Meaning and Necessity*, University of Chicago, Chicago Press (1947).

The rich semantics of these logics should allow scope for introducing more, and even perhaps some new, logical functors. We can also consider whether there are further logics that naturally fall into the class **F*F** and **PF*F**. One further possibility is **G*G** with the following tables for '*M*', '*L*', and '*T*' and a new functor '*U*' not appearing in [1]. '*M*', '*L*' and '*T*' (see above) have the same C_1-tables as in **E*E**, and the same semantical interpretations. '*U*' has the same C_1-table as '*T*'.

C_2-Tables for **G*G**

	M	L	T	U
t_2	t_2	t_2	t_2	i_2
f_2	t_2	f_2	i_2	i_2
i_2	t_2	f_2	i_2	i_2

'*Up*' is interpreted as 'The C_2-truth-value of p will change.' Although **G*G** appears at first sight to be a very strange logic indeed it does seem to have a location in the metaphysical tradition—Peter Damiani[6] (1007-1072). As Weinberg[7] notes, Damiani assigns an elevated position to 'the divine omnipotence to such an extent that not only the ordinary uniformities of nature, but also the principles of logic as applied to natural events are dependent on the absolutely omnipotent will of God.' Also, from a purely formal point of view **G*G** is of interest in that this functor '*U*' enables one to accommodate the full 3-valued logic, under C_2. We can refer to **G*G** as the Damiani system. Concerning these possibilities in this section however we must be wary not to overload the treasure-ship at least until we see how our ship fares in stormy passages.

5.5 *Traditional predication theory* Angelelli's observations on the 'ontological square' and 'traditional predication theory' in [3] seem of relevance here, and particularly his remarks given in the first chapter 'Ontology':

> Traditional ontology has two-dimensions (to use an algebraic metaphor): (1) substance—accident, (2) singular—universal. This is briefly described as the *ontological square*.
>
> An important consequence of the existence of these two dimensions is that the term "property" becomes ambiguous, as does the expression "relation between individual and property." A property of a thing may be an accident; then the property is not a universal but an individual. On the other hand, a property of a thing may be a universal ("property" in the sense usual today). The same applies to other terms like "attribute", "predicate", etc. (Angelelli [3], p. 9)

Angelelli's account of the ontological square can be linked with these predicate logics **PF*F** on the following basis:

6. P. Damiani, *De Divina Omnipotentia E Altri Opuscoli*, Vallecchi, Florence (1943).

7. J. R. Weinberg, *A Short History of Medieval Philosophy*, Princeton University Press, Princeton (1964).

01. For the 'accident-substance' distinction compare: the 'predicate considered under C_1-predicate considered under C_2' distinction. (See the semantics of **PF*** in section **2** above.)
02. For the 'singular-universal' distinction compare: the familiar 'individual-predicate' distinction.

From what has been said here (and Angelelli's work indicates many further ramifications and confirmations) it is likely that these modal predicate logics **PF*F** could have an important bearing on traditional predication theory. However, at this time, historical excursions should be controlled until we have secured the measuring rod.

5.6 *Two questions on Leibniz and the weak Lewis systems* Kripke's[8] semantics for the Lewis Systems S2-S5 articulate Leibniz's notion of 'possible worlds', and are based on the postulate of *many* worlds, and *one* logical aspect. Purtill[9] discusses the application of four-valued tables to the Lewis Systems, and suggests that four values can be viewed as arising from *two* worlds, and again, only *one* logical aspect. In contrast, these logics **F*F**, **PF*F** are based on the view of *one* world and *two* logical aspects. This is clear from [1] and in this paper from the fact that in the syntax of **PF** we postulate *one* class of predicate variables and *one* class of individuals, and in the semantics **PF***, *two* logical aspects.

Concerning Leibniz and the Lewis Systems two questions suggest themselves:

L1. Does Leibniz's 'possible worlds' and treatment of predication touch on the possibility of different logical aspects?
L2. Is it possible to provide an alternative semantics for the weak Lewis systems (S2,S3) based on the following ideas: one world, two logical aspects, different factual realms, and different kinds of relations holding between factual realms?

For L1, Angelelli's remarks on Leibniz's treatment of relations in [3], pp. 19-21, seem relevant. There are two clues that may relate to the second question L2:

1. Halldén's incompleteness property holds for S2,S3 (see [1]).
2. Kripke's semantics splits the possible worlds into two groups—normal worlds and non-normal worlds.

5.7 *Practical considerations* In [1] and in this paper the emphasis has been on naturalness of presentation rather than on formal simplicity. However from the point of view of practicality it is desirable that the formal treatment of these logics **F*F**, **PF*F** be simplified, for example, by:

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8. S. Kripke, "Semantical analysis of modal logic II," published in *The Theory of Models*, edited by Addison, Henkin, and Tarski, North-Holland, Amsterdam (1965).

9. R. L. Purtill, "Four-valued tables and modal logic," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 505-511.

1. Use of definitions for the tableaux method—thus enabling one to operate with a smaller number of primitive functors.
2. Use of derived rules for the tableaux method. This again can help to simplify and shorten formal derivations.

Gentzen, *cf.* [7], in his paper entitled "The present state of research into the foundations of mathematics" (1939) presents a clear statement on the issues between the 'constructivist' (*cf.* C_1) and the 'actualist' (*cf.* C_2) interpretations of mathematics. In his final section (the possibility of reconciling the different points of view) he argues for the important *practical* significance of the 'actualist' view, well supported by other authors (e.g., Kleene, Weyl and Hilbert). In [6] the key concept for metaphysics is C_2-belief. Concerning the programme for the foundations of mathematics in [5] and the logics **S*S** and **PS*S**, C_2-belief should also be the key to practicability because **S*S** and **PS*S** can be viewed as accommodating the 'classical' propositional calculus and the 'classical' predicate calculus (i.e., under C_2). Ironically, metaphysical belief, i.e., C_2-belief (rather than C_1-belief) should also yield the greater honest practicability in the sciences.

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