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A NOTE ON THOMASON'S REPRESENTATION OF S5

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Introduction S. K. Thomason has proved in [3] that a formula is provable in S5 iff all its substitution instances are in H, which is a unique correct set and is Thm($\mathfrak{C}$). In order to prove this, he semantically showed that a formula $A(x_1, \ldots, x_n)$ is valid in S5 (tautology of S5 in the sense of Kripke [2], pp. 11ff.) iff $V^*(A(B_1, \ldots, B_n)) = 1$ for all $B_1, \ldots, B_n$ in $\mathcal{L}_c$ (modal language with proposition constants).

In this paper, we shall show by means other than Kripke's model that $A(x_1, \ldots, x_n)$ is provable in S5 iff $\mu^*(A(B_1, \ldots, B_n)) = 1$ for all classical formulas (without modal symbols), $B_1, \ldots, B_n$, for all μ^*, where μ^* is essentially the same as V^* above, except that μ^* is a valuation for modal formulas with proposition variables. In the last section of this paper, we shall also show a relation between Kripke's partial truth tables and μ^*-valuations.

1 *Formulation of* S5 *and truth valuation* We prepare a countable set of proposition variables, Π, logical connectives, $\vee$, $\sim$, $\Box$, and parentheses, (,). Formulas are defined as usual. For any formulas A and B, we define $A \wedge B$ as $\sim(\sim A \vee \sim B)$, $A \rightarrow B$ as $\sim A \vee B$, $A \leftrightarrow B$ as $(A \rightarrow B) \wedge (B \rightarrow A)$, and $\Diamond A$ as $\sim\Box\sim A$. If A and B are formulas, the following expressions are axioms:

(A1) $(A \vee A) \rightarrow A$.
(A2) $B \rightarrow (A \vee B)$.
(A3) $(A \vee B) \rightarrow (B \vee A)$.
(A4) $(B \rightarrow C) \rightarrow ((A \vee B) \rightarrow (A \vee C))$.
(A5) $\Box A \rightarrow A$.
(A6) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$.
(A7) $\Diamond A \rightarrow \Box\Diamond A$.

When A and B are formulas, we suppose the following rules of inference:

(R1) If $\vdash A$ and $\vdash A \rightarrow B$, then $\vdash B$.
(R2) If $\vdash A$, then $\vdash \Box A$.

For any formula A, we say that A is a classical formula iff A contains none of $\Box$ and $\Diamond$. $A(x_1, \ldots, x_n)$ denotes a formula, A, having exactly n distinct

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proposition variables, $x_1, \ldots, x_n$, in Π. When $B_1, \ldots, B_n$, and $A(x_1, \ldots, x_n)$ are formulas, then $A(B_1, \ldots, B_n)$ also represents a formula obtained by substituting $B_1, \ldots, B_n$ for $x_1, \ldots, x_n$ in $A(x_1, \ldots, x_n)$, respectively.

A truth value assignment is a mapping μ: $\Pi \rightarrow \{0, 1\}$, where 0 means false and 1 means true. Let Ω be the set of all μ's. A truth valuation is a mapping μ^* from the set of all formulas into $\{0, 1\}$, which is the unique extension of μ in the following way:

For any formulas A and B,

(a) if A is x_i in Π, $\mu^*(x_i) = \mu(x_i)$,
(b) if $\mu^*(A)$ and $\mu^*(B)$ are defined, $\mu^*(A \vee B) = \text{Max}\{\mu^*(A), \mu^*(B)\}$,
(c) if $\mu^*(A)$ is defined, $\mu^*(\sim A) = 1 - \mu^*(A)$, $\mu^*(\Box A) = \text{Min}\{\nu^*(A) \mid \nu \in \Omega\}$.

We can then easily see that

(d) if $\mu^*(A)$ is defined, $\mu^*(\Diamond A) = \text{Max}\{\nu^*(A) \mid \nu \in \Omega\}$.

When A is $A(x_1, \ldots, x_n)$, then $\mu^*(\Box A)$ and $\mu^*(\Diamond A)$ are actually determined by considering 2^n cases of $\nu^*(A)$'s for all n-tuples $(\nu(x_1), \ldots, \nu(x_n)) \in \{0, 1\}^n$, and they take uniformly either 0 or 1 for all cases. A formula A is called valid iff $\mu^*(A) = 1$ for all $\mu \in \Omega$.

2 *Representation of* S5 Let $A(x_1, \ldots, x_n)$ be a formula of the form $\Diamond C \vee \Box D_1 \vee \ldots \vee \Box D_l \vee E$, where $C, D_1, \ldots, D_l$, and E are all classical formulas. The following two lemmas are stated:

Lemma 1 *If $A(B_1, \ldots, B_n)$ is valid for every classical formula, $B_1, \ldots, B_n$, then at least one of $C \vee D_1, \ldots, C \vee D_l, C \vee E$ in $A(x_1, \ldots, x_n)$ is provable in the classical logic.*

Lemma 2 *If at least one of $C \vee D_1, \ldots, C \vee D_l, C \vee E$ in $A(x_1, \ldots, x_n)$ is provable in the classical logic, then $A(x_1, \ldots, x_n)$ is provable in* S5.

Proof of Lemma 1*:* Suppose none of $C \vee D_1, \ldots, C \vee D_l, C \vee E$ is provable in the classical logic. As for classical formulas, truth valuation, μ^*, coincides with usual valuation. Hence, $\mu_i^*(C \vee D_i) = 0$ $(i = 1, \ldots, l)$, $\mu_{l+1}^*(C \vee E) = 0$ for some μ_i^*, μ_{l+1}^* such that $\mu_i(x_j) = e_{ij}$, $\mu_{l+1}(x_j) = e_{l+1j}$ $(j = 1, \ldots, n)$, respectively. (Each of e_{ij} and e_{l+1j} is 0 or 1.) We illustrate these relations with the following truth table:

x_1	x_2	$\ldots\ldots x_n$	C	D_1	$D_2 \ldots\ldots$	D_l	E
e_{11}	e_{12}	$\ldots\ldots e_{1n}$	0	0			
e_{21}	e_{22}	$\ldots\ldots e_{2n}$	0		0		
		$\ldots\ldots$	$\vdots$		$\ddots$		
e_{l1}	e_{l2}	$\ldots\ldots e_{ln}$	0			0	
e_{l+11}	e_{l+12}	$\ldots\ldots e_{l+1n}$	0				0
		$\ldots\ldots$					

Now, let k be the integer such that $2^{k-1} < l + 1 \leq 2^k$. Take k distinct proposition variables, $y_1, \ldots, y_k$, in Π. Define $B_1, \ldots, B_n$ so as to satisfy the next truth table with 2^k rows, where for the rows from $(l + 1)$'th to 2^k'th, each B_j has the same value e_{l+1j} $(j = 1, \ldots, n)$:

y_1	y_2	......	y_k	B_1	B_2	......	B_n
0	0	......	0	e_{11}	e_{12}	......	e_{1n}
0	0	......	1	e_{21}	e_{22}	......	e_{2n}
		......				......	
		......		e_{l1}	e_{l2}	......	e_{ln}
		......		e_{l+11}	e_{l+12}	......	e_{l+1n}
		......		$\vdots$	$\vdots$		$\vdots$
1	1	......	1	e_{l+11}	e_{l+12}	......	e_{l+1n}

By the functional completeness of classical logic, $B_1, \ldots, B_n$ above can be expressed by the disjunctive normal forms having $y_1, \ldots, y_k$. Then for all $\mu \in \Omega$, $\mu^*(C(B_1, \ldots, B_n)) = 0$, i.e., $\mu^*(\Diamond C(B_1, \ldots, B_n)) = 0$. For some $\mu \in \Omega$, $\mu^*(D_s(B_1, \ldots, B_n)) = 0$ $(s = 1, \ldots, l)$, hence for all $\mu \in \Omega$, $\mu^*(\Box D_s(B_1, \ldots, B_n)) = 0$. And there exists at least one $\mu \in \Omega$, say μ_0, such that $\mu^*(E(B_1, \ldots, B_n)) = 0$. Hence $\mu_0^*(A(B_1, \ldots, B_n)) = 0$, i.e., $A(B_1, \ldots, B_n)$ is not valid. This contradicts the hypothesis.

Proof of Lemma 2*:* Assume that at least one of $C \vee D_1, \ldots, C \vee D_l, C \vee E$ is provable in the classical logic. Then it is clearly provable in S5. As for the case $\vdash C \vee D_s$, i.e., $\vdash {\sim} C \rightarrow D_s$, $(s = 1, \ldots, l)$, we have $\vdash \Box {\sim} C \rightarrow \Box D_s$, i.e., $\vdash \Diamond C \vee \Box D_s$, by rule (R2), axiom (A6), and rule (R1). Hence $\vdash A(x_1, \ldots, x_n)$. As for the case $\vdash C \vee E$, we have also $\vdash \Box {\sim} C \rightarrow \Box E$, and hence $\vdash \Box {\sim} C \rightarrow E$, i.e., $\vdash \Diamond C \vee E$ by (A5). Thus we have again $A(x_1, \ldots, x_n)$.

Theorem *A formula $A(x_1, \ldots, x_n)$ is provable in* S5 *iff for every classical formula, $B_1, \ldots, B_n$, $A(B_1, \ldots, B_n)$ is valid.*

Proof: That if $A(x_1, \ldots, x_n)$ is provable in S5 then $A(B_1, \ldots, B_n)$ is valid for every classical formula, $B_1, \ldots, B_n$, is clear by verifying that all axioms are valid and all rules of inference preserve validity.

Next, we prove that for a formula $A(x_1, \ldots, x_n)$ if $A(B_1, \ldots, B_n)$ is valid for every classical formula, $B_1, \ldots, B_n$, then $A(x_1, \ldots, x_n)$ is provable in S5. It is well-known that $A(x_1, \ldots, x_n)$ can be reduced in S5 to the modal conjunctive normal form, A', which is of the form $A_1 \wedge \ldots \wedge A_r (r \geqslant 1)$, each $A_\alpha (\alpha = 1, \ldots, r)$ being of the form $\Diamond C \vee \Box D_1 \vee \ldots \vee \Box D_l \vee E$, where $C, D_1, \ldots, D_l$, and E are all classical formulas, $l \geqslant 0$, and C or E may be missing. Let $B_1, \ldots, B_n$ be any classical formulas, and suppose $A(B_1, \ldots, B_n)$ is valid. Then $A'(B_1, \ldots, B_n)$ is valid, and so is $A_\alpha(B_1, \ldots, B_n)$, $(\alpha = 1, \ldots, r)$. By Lemma 1 and Lemma 2,[1] we have $A_\alpha(x_1, \ldots, x_n)$ is provable in S5, and so is $A'(x_1, \ldots, x_n)$. Hence $A(x_1, \ldots, x_n)$ is provable in S5.

1. If C is missing then $C \vee D_1, \ldots, C \vee D_l, C \vee E$ degenerate into $D_1, \ldots, D_l, E$, if E is missing then so is $C \vee E$, and if $l = 0$ then $C \vee D_1, \ldots, C \vee D_l$ are missing. In such special cases, these two lemmas still hold.

3 *Remark* We remark that for any (classical) formulas, $B_1, \ldots, B_n$, $A(B_1, \ldots, B_n)$ is valid, iff $A(x_1, \ldots, x_n)$ is a tautology of S5 in the sense of Kripke [2], i.e., iff $A(x_1, \ldots, x_n)$ is assigned 1 in every row of every partial truth table of $A(x_1, \ldots, x_n)$. In fact, if $A(x_1, \ldots, x_n)$ is a tautology, then for any (classical) formulas, $B_1, \ldots, B_n$, $\{(\mu^*(B_1), \ldots, \mu^*(B_n)) | \mu \in \Omega\} \subseteq \{0, 1\}^n$, hence $A(B_1, \ldots, B_n)$ is valid. Conversely, assuming any (classical) formulas, $B_1, \ldots, B_n$, $A(B_1, \ldots, B_n)$ is valid. We consider any partial truth table, Σ, with m $(1 \leqslant m \leqslant 2^n)$ rows of $A(x_1, \ldots, x_n)$. Let k be the integer such that $2^{k-1} < m \leqslant 2^k$, and take k distinct proposition variables, $y_1, \ldots, y_k$, in Π. In the same way as the proof of Lemma 1, we can construct $B_j(y_1, \ldots, y_k)$ $(j = 1, \ldots, n)$ such that $A(B_1, \ldots, B_n)$ satisfies Σ. By the assumption, $A(B_1, \ldots, B_n)$ is valid. Hence $A(x_1, \ldots, x_n)$ is assigned 1 in every row of Σ. Therefore, $A(x_1, \ldots, x_n)$ is a tautology.

We notice that in the above Theorem and Remark, $B_1, \ldots, B_n$ do not need to be classical formulas, i.e., they can be any formulas of S5.

In the proof of Theorem 2 of Thomason [3], it was shown that A is valid in S5 (tautology of S5 in the sense of Kripke [2]) iff every formula of $\mathcal{L}_c$ of the form $A(B_1, \ldots, B_n)$ is valid in $\mathfrak{C}$. This fact corresponds with the above remark.

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