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ON DE MORGAN'S ARGUMENT

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*In a recent article, R. G. Wengert[1] has suggested that the common formulation of the conclusion of a traditional argument is defective. That argument, which is emphasized by De Morgan,[2] is "Every man is an animal; therefore, every head of a man is the head of an animal." The premise of the argument is formulated easily by

(1) $(x)(Fx \supset Gx)$.

In most logic books, the conclusion is formulated as

(2) $(x)[(\exists x)(Fx \cdot Hyx) \supset (\exists x)(Gx \cdot Hyx)]$.

Wengert suggests, however, that this formulation is defective, since it does not make clear that whatever animal of which y is the head is the same as the man of which y is the head. To achieve this effect Wengert suggests the formulation

(3) $(x)(y)(Fx \cdot Hyx .\supset. Gx \cdot Hyx)$.

Both (2) and (3) follow from (1); but while (2) follows from (3), (3) does not follow from (2). In (3) the desired effect is obtained by using the same variable "x" in both the antecedent and the consequent of the conditional, rather than having separate quantifications in the antecedent and consequent, as was the case in (2).

Wengert's proposal raises two sorts of issues. The first issue arises from Wengert's claim that (3) is the proper formulation of the conclusion of (1), at least in the context of that argument. However, while he convincingly distinguishes between (2) and (3), he does not support the claim that (3) is preferable, apparently taking this as obvious. His only attempted

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arguments for the preferability of (3) rest on the fact that (2) is weaker than (3). For instance, (2) but not (3) can be derived from

(4) $(y)[(\exists x)Hyx . \supset . (\exists x)(Fx \cdot Hyx) \cdot (\exists x)(Gx \cdot Hyx)]$.

However, this suffices only to distinguish (2) from (3), and not to show that the stronger assertion (3) is what is meant in this context. While sympathetic with Wengert's intuitions here, I do not believe that he has provided an argument to justify them.

The other issue, and the one to which this article will be devoted, concerns the interpretation which De Morgan had in mind in proposing this example. While it is not clear that Wengert considers this issue directly, he does write of "the obvious intent of De Morgan's conclusion" and "De Morgan's interpretation." On one level, this issue seems to be only of biographical interest. In fact, an examination of the relevant texts in De Morgan's writings does not reveal a clear statement on the choice between (2) and (3). On a deeper level, however, the issue is of considerable interest, for it is not so much a matter of intentions as it is a matter of the power of De Morgan's symbolism. I will maintain that, questions of intention aside, the limited power of De Morgan's symbolism (at least until 1860) would not allow him to give to the statement in question the interpretation which Wengert suggests.

1 It is interesting to note that De Morgan's original formulation of this inference was slightly different from that given by Wengert. Whereas Wengert's formulation is

(5) <u>Every man is an animal</u>
Therefore, every head of man is the head of an animal,

De Morgan's original formulation in **FL** is

(6) <u>Man is animal</u>
Therefore, the head of a man is the head of an animal.

While the premises of (5) and (6) do not differ importantly, their conclusions might be interpreted in such a way as to give quite different readings. While the conclusion of (5) seems like a straightforward universally quantified proposition, the conclusion of (6) looks as if it contains two definite descriptions, thus suggesting the problem of existence and uniqueness conditions for such propositions. In fact, Bertrand Russell treats it in this way, as Wengert notes.[3] Russell considers a version of the argument found in Jevons,[4] which is "Because a horse is an animal, the head of a horse is the head of an animal." The natural formulation of this would be

(7) <u>A horse is an animal</u>
Therefore, the head of a horse is the head of an animal.

However, this formulation does not quite catch the nuance which Russell seems to feel in the premise, since he seems to take "a horse" to refer to some particular horse, which he calls "the horse in question." His symbolic formulation of a conditionalized form of this inference apparently uses the free individual variable "*y*" to stand for that which has the head, and there is no class term for "horse" in the formulation at all. Under this formulation and assuming the theory of descriptions, there is a clear existence and uniqueness assumption, so that Russell's symbolic formulation may be roughly translated as

(8) If there is exactly one head of this horse and this horse is an animal, then the head of this horse is a member of the class of heads of animals.

On this reading, then, the inference (7) would require the additional assumption that this horse has exactly one head.

It is obvious from De Morgan's later versions of this argument that he did not have Russell's version in mind, for he often used the version (5). Interestingly enough, however, problems concerning existence assumptions still arise, and De Morgan's solution of these problems is of considerable interest. In the only major review of **FL**, H. L. Mansel[5] complained that De Morgan's innovations involved the introduction of material elements into formal logic, and he took De Morgan's treatment of this argument as an instance of this complaint.

Mansel claims that the argument is not formally valid, because the conclusion assumes that there is such a thing as a man's head, which is a fact of natural history and not of logic. This is shown by the fact that the similar argument

(9) A guinea-pig is an animal
Therefore, the tail of a guinea-pig is the tail of an animal

would fail because guinea-pigs have no tails. He concludes that "the consequence is therefore a special inference, *gained from our material knowledge of the thing thought about*, not a general inference *necessitated by the universal laws of thinking*."[6]

As Mansel's sole objection to this argument, this is somewhat peculiar. The problem which it raises could be met in several ways: by denying that the conclusion has existential import, by rephrasing the conclusion in a hypothetical form, or by adding a premise asserting the existence of men's heads. Under any of these modifications the argument could still not be validated by syllogistic means.

De Morgan did not reply to this criticism in his first general reply to Mansel in 1858;[7] however, he did reply in his article on "Logic" which appeared in the *English Cyclopaedia* of 1860.[8] His reply there takes place in the context of a general discussion of the distinction between form and matter. He proposed that an assertion is true by its form if and only if "it can be refused admission by impossibility in its matter . . . and by impossibility we mean incompatibility with the conditions of the universe understood." Thus, the impossible will vary with the universe of discourse which is being considered. If it is "the whole sphere of possible thought," then the impossible is that which contradicts itself; if it is actually existing things on the earth, it is that which does not exist on the earth, such as a rational quadruped or a guinea-pig with a tail; if it is the domain of animal life, the impossible would include such things as a stone. Thus "*A* is *A*" is true by its form because we can only refuse to assert it when *A* is impossible within our universe of discourse.

De Morgan applies this view not only to formal truths, but to formal consequences as well. Thus, if *X* is an animal, the tail of *X* is the tail of an animal, so long as *X* has a tail. Thus the argument form

$$(10)\quad \frac{X \text{ is } F}{\text{Therefore, the } R \text{ of } X \text{ is the } R \text{ of an } F}$$

is formally valid because, no matter what *X*, *R* and *F* may be, the truth of the premise will necessitate the truth of the conclusion, as long as the subject term of the conclusion denotes something. In De Morgan's words, ". . . we know the *consequence* to be necessitated by the laws of thinking, because we must go to impossible matter, we must make the tail of *X* a non-existence, before we can refuse to assert it."

What are we to say when this existence condition fails? Is the conclusion of (9) trivially true or trivially false? If we were to interpret this statement as a statement about a particular tail of a particular guinea-pig, and utilize the theory of descriptions, we would be forced to say that it is false. If, on the other hand, we were to interpret it as a statement about all things that are tails of guinea-pigs, and adopt either the reading given by (2) or by (3), we would be forced to say that it is true. Interestingly enough, however, De Morgan refuses to say that it is either true or false; or, more accurately, he says that it should be neither asserted nor denied.[9] He says, instead, that "A guinea-pig, for instance, puts this proposition out of the pale of assertion, and equally out of that of denial; the tail of a non-tailed animal is beyond us." This view, which is remarkably similar to the presuppositional view of Strawson's, would lead to the following definition of formal validity: if the premise is true, then the conclusion must be true, if it has a truth-value at all.

It appears, then, that neither (2) nor (3) can be said to formulate the proposition in question, since they are both true when their antecedents are vacuous, while De Morgan claims that the proposition can neither be asserted nor denied in this case.

2 Let us leave the question of existential import and move on to the question of whether (3) is a better formulation than (2) of what De Morgan meant in proposing this example. I will argue that De Morgan could not have consistently had (3) in mind, at least in his earlier work, because his entire method and symbolism would not allow him to formulate it in this sense.

De Morgan's analysis of this proposition is tied to a fundamentally conservative analysis of inferences in which it occurs. This analysis is based on a principle which he came to call the *dictum de majore et minore*, and which includes the traditional *dictum de omni et nullo* as a part. This principle is formulated by De Morgan in several different ways, but perhaps the best formulation is found in "On the Syllogism: II" (1850).[10]

> A little consideration suggests as a necessary rule of inference, the right to substitute a larger term used particularly for a smaller one, however used, and a smaller, used in either way, for a larger used universally. What we may affirm or deny of *some* or *all* men, we may affirm or deny of some *animals*: what we may affirm or deny of *all* animals, we may affirm or deny of *all* or *some* men.

As the second sentence from this passage suggests, the most obvious applications of this principle are to syllogistic inferences. For instance, the following argument, which would normally be represented as a sorites, can be validated using the two parts of this principle:

(11)
All tigers are vicious creatures
All Bengal tigers are tigers
All vicious creatures are creatures with hot tempers
____
All Bengal tigers are creatures with hot tempers.

The second half of the principle allows us to move from the larger term "tigers" used universally to the smaller term "Bengal tigers" used universally; and the first part of the principle allows us to move from the smaller term "vicious creatures" used particularly to the larger term "creatures with hot tempers" used particularly. The rules would have also allowed us to infer "Some Bengal tigers are creatures with hot tempers." With the additional premise that all tigers are felines, we could have also inferred that some felines are creatures with hot tempers.

From this point of view we may look upon inference as essentially a matter of the substitution of genus for species, and of species for genus, under certain stated conditions. While the application of this method to validate syllogistic inferences is quite straightforward, its application to relational inferences is most difficult. Yet, it is through the application of this *dictum* to such inferences that De Morgan is able to claim that the *dictum* will validate arguments which cannot be validated syllogistically. He says that (5) is a valid inference, being "the substitution, in a compound phrase, of the name of the genus for that of the species, when the use of the name is particular."[11] This formulation is puzzling, since he does not indicate the compound phrase in which the substitution is taking place. However, he seems to have in mind an inference which would be explicitly stated as:

(12)
Every man is an animal
Every head of a man is the head of a man
____
Therefore, every head of a man is the head of an animal.

The conclusion would then have been reached by replacing the species term used particularly by a genus term used particularly.

With this background in mind, let us now examine the interpretation of our proposition. The crucial factor which distinguishes (3) from (2) is the fact that the variable "y" occurs in both the antecedent and the consequent of (3), allowing us to state that whatever animal of which x is the head must be the same as a man of which x is the head. However, this type of cross-referencing between subject and predicate cannot be exhibited in the fundamentally conservative manner in which De Morgan treats this proposition. It seems clear that in this framework, what is asserted is that "head of an animal" is a predicate which applies to everything to which "head of a man" applies, but not that the animal and the man must be the same thing. The traditional doctrine of the categorical proposition, even as modified by De Morgan, does not contain the resources to exhibit the type of cross-referencing that is required. On the other hand, (2) does not require these resources.

Thus, I do not believe that it is correct to state, as Wengert apparently does, that (3) is the interpretation De Morgan had in mind when he proposed this example of a non-syllogistic inference. The interpretation (2) seems to reflect it more accurately, with the understanding that vacuous satisfaction may still pose a problem. Interestingly enough, however, the logic of relations which De Morgan developed in "On the Syllogism: IV" (1860)[12] may provide the means for stating (3) which were absent in his earlier work. It is not possible to claim this with much assurance, since his logic of relations is not developed in a very rigorous way; however, it appears that with some ingenuity it might be done as follows.

To begin with, De Morgan adopts the expression "$A))B$" to indicate that the class A is contained within the class B. He then extends this notation to relations, so that "$L))N$" means that the relation L is contained in the relation N. The relation of containment applies when all pairs which are related by L are also related by N. The fact that we are now dealing with sets of pairs allows for the type of cross-referencing which Wengert's (3) requires, since "$L))N$" is just equivalent to "$(x)(y)(xLy \supset xNy)$." In fact given some natural assumptions, we may also validate the inference (5). De Morgan often utilized relational inferences involving the relative product, and this may be adapted to our needs in this case. At several points he uses inferences of the form

$$(13) \quad \frac{L))N}{RL))RN}$$

where "L", "N" and "R" are relation terms, and where "RL" is the relative product term "R of an L of".[13] Thus, if the relation "lover" is contained in the relation "servant," the compound relation "brother of a lover" will be contained in the compound relation "brother of a servant." This type of inference sounds not unlike the inference (5) with which we began, suggesting that it be represented as

$$(5a) \quad \frac{M))A}{HM))HA} .$$

Unfortunately, however, this formula does not make sense as it stands, since "M" and "A" are class terms, not relation terms. De Morgan does not give a meaning to mixed terms such as "HM", but the natural reading would be that this represents the class of all things which are the head of some man.[14] On this reading, the conclusion of (5a) would be (2), not (3). However, we can convert these class terms into relation terms by setting

$$(14) \quad \begin{aligned} xM'y &=_{Df}. \; x = y \cdot My \\ xA'y &=_{Df}. \; x = y \cdot Ay \end{aligned}$$

so that (5) becomes

$$(5b) \quad \frac{M'))A'}{HM'))HA'}.$$

The conclusion of (5b) is equivalent to (3). Thus, it seems that (3) can be represented, albeit somewhat artificially, in De Morgan's later logic of relations, and that (2) can be represented if we define the "product" of a relation and a class. However, we must not ignore the fact that when De Morgan *originally* proposed (5), his notation did not allow for its conclusion to have the interpretation (3).

NOTES

1. See: R. G. Wengert, "Schematizing De Morgan's argument," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 165-166.
2. For this argument or variants on it, see A. De Morgan, *Formal Logic* [to be referred to as **FL**] (1847), pp. 114-115 and *On the Syllogism and Other Logical Writings* [to be referred to as **OS**], ed. Peter Heath, Yale University Press, New Haven (1966), pp. 29, 52, 216, 225 and 253.
3. See ***37.62** of B. Russell, *Principia Mathematica.*
4. W. Stanley Jevons, *Principles of Science*, Macmillan, New York (1887), p. 18.
5. H. L. Mansel, "Recent extensions of formal logic," *North British Review* (May 1851), pp. 90-121.
6. *Ibid.*, p. 106.
7. **OS**, pp. 78-83.
8. *Ibid.*, pp. 252-253.
9. It might be denied that the refusal to assert or deny a proposition is the same as the refusal to ascribe it a truth-value. However, in the absence of an explicit denial, this would be the natural assumption to make.
10. **OS**, pp. 28-29.
11. **FL**, p. 14.
12. **OS**, pp. 220-237.
13. See **OS**, p. 224.
14. Just as the reading for "$xRLy$" is $(\exists z)(xRz \cdot zLy)$," so the natural reading for "xHM" would be "$(\exists z)(xHz \cdot Mz)$."

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