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# ABNORMAL WORLDS AND THE NON-LEWIS MODAL SYSTEMS

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**1** The purpose of this paper is to provide semantic interpretations for the various non-Lewis modal systems comprising what Sobociński calls Family $\mathcal{K}$ (*cf.* [4] for a definition of these systems). The semantics offered are characterized by the introduction of "abnormal worlds" into Kripke-style models for S4 and some of its Lewis extensions. Although the semantics which we shall offer are in the spirit of Kripke, we shall nevertheless employ the terminology, techniques, and lemmata of Hughes and Cresswell in [3].

**2** In [3], Hughes and Cresswell define a semantic model for S4 as an ordered triple $\langle W, R, \mathbf{V}\rangle$ where $W$ is a set of objects (worlds), $R$ is a reflexive and transitive relation defined over the members of $W$, and $\mathbf{V}$ is a value-assignment satisfying the conditions specified in [3], p. 73. Now in order to construct a semantic model for modal system K1, axiomatized by appending

*K1* $CLMpMLp$

to the basis of S4, we need only introduce, as mentioned above, what I shall call "abnormal worlds" into an S4-model structure. These kinds of worlds possess two characteristic features. First, they are accessible from any other world in the model; more specifically, every K1-model structure possesses at least one of these worlds accessible from any other world. Second, modal distinctions among statements within abnormal worlds break down; in other words, these kinds of worlds do not recognize differences among actual truths, possible truths, and necessary truths. Intuitively then, a K1-model structure propounds the view that no matter what states of affairs within which we find ourselves, we are always able to conceive at least one other possible state of affairs where it would be pointless to elaborate modal distinctions among statements.

More formally, we say that $\langle W, R, \mathbf{V}\rangle$ is a K1-model if and only if (a) it is an S4-model; (b) there exists at least one *abnormal* $w_j \in W$ such that for every *normal* $w_i \in W$, $w_i R w_j$; and (c) $\mathbf{V}$ is a value-assignment not only

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satisfying the four conditions specified in [3], p. 73, but also the following additional conditions concerning the evaluation of wffs in abnormal worlds:

(5) For any abnormal $w_j \epsilon W$,

(a) there exists a wff $L\alpha$ such that $\mathsf{V}(L\alpha, w_j) = 1$ if for any wff $\alpha$, $\mathsf{V}(\alpha, w_j) = 1$;
(b) there exists a wff $L\alpha$ such that $\mathsf{V}(L\alpha, w_j) = 0$ if for any wff $\alpha$, $\mathsf{V}(\alpha, w_j) = 0$;
(c) there exists a wff $\alpha$ such that $\mathsf{V}(\alpha, w_j) = 1$ if for any wff $L\alpha$, $\mathsf{V}(L\alpha, w_j) = 1$;
(c) there exists a wff $\alpha$ such that $\mathsf{V}(\alpha, w_j) = 0$ if for any wff $L\alpha$, $\mathsf{V}(L\alpha, w_j) = 0$.

These additional conditions which the value-assignment in a K1-model must satisfy will guarantee that modal distinctions among statements in abnormal worlds will collapse. Clearly, we can now say that a wff, $\alpha$, is K1-logically true iff in every K1-model $\langle W, R, \mathsf{V}\rangle$ and for every normal $w_j \epsilon W$, $\mathsf{V}(\alpha, w_i) = 1$.

Having constructed a model for K1, we now demonstrate the soundness and completeness theorems. Clearly, in order to prove that modal system K1 is sound on interpretation, we need only show that

*K1* $CLMpMLp$

is K1-logically true. Assume for the sake of reductio that it is not; i.e., $\mathsf{V}(CLMpMLp, w_i) = 0$. Surely it then follows that both

$$(1)\quad \mathsf{V}(LMp, w_i) = 1$$
$$(2)\quad \mathsf{V}(MLp, w_i) = 0.$$

Since $R$ is reflexive, it follows from (2) that

$$(3)\quad \mathsf{V}(Lp, w_i) = 0$$

and from (3) that

$$(4)\quad \mathsf{V}(p, w_j) = 0.$$

Now it follows from (1) that

$$(5)\quad \mathsf{V}(Mp, w_j) = 1$$

and so from (5) that

$$(6)\quad \mathsf{V}(p, w_k) = 1.$$

But again from (2), since $R$ is transitive as well,

$$(7)\quad \mathsf{V}(Lp, w_k) = 0.$$

Now according to a K1-model there exists at least one abnormal world which is accessible from any other world. Hence it must be the case that $w_k$ is abnormal since it is the only world in the above model which is accessible from any other world. For example, it is accessible from $w_j$, and because $R$ is transitive it is also accessible from $w_i$. It goes without saying that it is also accessible from itself. Notice, however, that neither $w_i$ nor $w_j$ are accessible from every other world; viz., $w_i$ is accessible from only itself, and $w_j$ from only $w_i$ and itself. Clearly then, since $w_k$ is abnormal, it follows from (7) that there exists a wff, $p$, such that

$$(8)\ \mathsf{V}(p, w_k) = 0.$$

But this contradicts (6); consequently, $\mathsf{V}(CLMpMLp, w_i) = 1$.

In order to prove the completeness of K1, we must show that modal distinctions collapse within maximal consistent sets corresponding to abnormal worlds. Let there be a $\Gamma_j \epsilon \Gamma$ maximal consistent with respect to K1. Now if $\Gamma_j$ corresponds to an abnormal world in a K1-model, then it is either a subordinate or a subordinate of subordinates of any $\Gamma_i \epsilon \Gamma$. What we must show then is that

(a) if $\beta \epsilon \Gamma_j$, then $L\beta \epsilon \Gamma_j$ and $M\beta \epsilon \Gamma_j$;
(b) if $M\beta \epsilon \Gamma_j$, then $\beta \epsilon \Gamma_j$ and $L\beta \epsilon \Gamma_j$; and
(c) if $L\beta \epsilon \Gamma_j$, then $\beta \epsilon \Gamma_j$ and $M\beta \epsilon \Gamma_j$.

The lemmata utilized in this proof are taken from [3], pp. 152-154.

(a) If $\beta \epsilon \Gamma_j$, then since $MLC\beta L\beta$ is a thesis of K1, it follows (by Corollary of Lemma 2) that $MLC\beta L\beta$ is in every $\Gamma_i$ (hence also in $\Gamma_j$ for that matter). Thus (by construction of $\Gamma$) there is some subordinate $\Gamma_i$ such that $LC\beta L\beta \epsilon \Gamma_i$. But since $\Gamma_j$ is a subordinate or a subordinate of subordinates of any $\Gamma_i$, it must be the case that $C\beta L\beta \epsilon \Gamma_j$. Hence it follows (by Lemma 3) that $L\beta \epsilon \Gamma_j$. Now $C\beta M\beta$ is also a thesis of K1, hence (by Corollary of Lemma 2) $C\beta M\beta \epsilon \Gamma_j$ and so (by Lemma 3) $M\beta \epsilon \Gamma_j$.

(b) If $M\beta \epsilon \Gamma_j$, then since $MLCM\beta\beta$ is a thesis of K1, it follows (by Corollary of Lemma 2) that $MLCM\beta\beta$ is in every $\Gamma_i$. Thus (by construction of $\Gamma$) there is some subordinate $\Gamma_i$ such that $LCM\beta\beta \epsilon \Gamma_i$. But $\Gamma_j$ is a subordinate or a subordinate of subordinates of $\Gamma_i$, consequently $CM\beta\beta \epsilon \Gamma_j$ and so (by Lemma 3) $\beta \epsilon \Gamma_j$. As shown in (a), $C\beta L\beta \epsilon \Gamma_j$ and so (again by Lemma 3) $L\beta \epsilon \Gamma_j$.

(c) If $L\beta \epsilon \Gamma_j$, then since $CL\beta\beta$ is a thesis of K1, we have $CL\beta\beta \epsilon \Gamma_j$ and thus (by Lemma 3) $\beta \epsilon \Gamma_j$. $CL\beta M\beta$ is also a thesis of K1 and so $CL\beta M\beta \epsilon \Gamma_j$. Consequently (by Lemma 3) $M\beta \epsilon \Gamma_j$. Q.E.D.

**3** Since every known modal system belonging to Family $\mathcal{K}$ can be axiomatized by appending formula *K1* to some of the respective Lewis extensions of S4, it is quite clear that we can construct semantic models for any of the non-Lewis systems in the same way as we have done for K1 (provided of course that we have models for the appropriate Lewis systems which especially trade upon imposing additional requirements on the accessibility relation in an S4-model structure). Now at the time of this writing, in addition to K1, the following enumeration of non-Lewis systems have appeared in the published literature: K2, K3, K3.1, K1.1, K2.1, K3.2, K1.2, and K4. It is well-known that each of these can be axiomatized by appending *K1* to the bases of the following Lewis systems respectively: S4.2, S4.3, S4.3.1, S4.1, S4.2.1, S4.3.2, S4.04, and S4.4.

If we impose the additional requirement that the accessibility relation in an S4-model structure is *convergent* (viz., for any $w_i$, $w_j$, $w_k \epsilon W$ if both $w_i R w_j$ and $w_i R w_k$, then there exists a $w_l \epsilon W$ such that $w_j R w_l$ and $w_k R w_l$) then we obtain a model for system S4.2. If instead we impose the additional requirement of *connectedness* on the accessibility relation in an S4-model, we obtain a model for system S4.3 (*cf.* [3], pp. 288-289 for details). If we now take an S4.3-model and append the additional requirement that the accessibility relation is *discrete* as well, a model for S4.3.1, the Diodoran modal system, results (again *cf.* [3], p. 289 for details). By parity of reasoning, since S4.3.1 is axiomatized by adding

*N1 CLCLCpLppCMLpp*

to the basis of S4.3, it appears that models for S4.1 and S4.2.1 may also be obtained by imposing the additional requirement of discreteness on the accessibility relation in S4- and S4.2-models respectively since both S4.1 and S4.2.1 are also axiomatized by appending *N1* to the respective systems S4 and S4.2 (*cf.* [5], pp. 306 ff, for the axiomatizations of S4.3.1, S4.1, and S4.2.1). In [1], an S4.3.2-model is constructed by requiring that the accessibility relation in an S4-model is also *non-branching*. Again in [1], models for both S4.04 and S4.4 are constructed by imposing the additional requirement that the accessibility relations in S4- and S4.2-models respectively be *remotely symmetrical* (viz., for every $w_i$, $w_j$, $w_k \epsilon W$, if $w_i R w_j$ and $w_j R w_k$, then either $w_k R w_j$ or $w_i = w_j$). Clearly then, by simply introducing abnormal worlds into each of the models mentioned above for the various Lewis extensions of S4, we will have provided ourselves with semantic models for all of the non-Lewis systems. Obviously, the completeness theorems in each case will proceed similarly as for K1. As for the soundness theorems, we shall only prove the ones for K1.2 and K4.

Following the procedure outlined above, $\langle W, R, \mathsf{V}\rangle$ will be a K1.2-model iff (a) it is an S4.04-model, (b) there exists at least one abnormal $w_j \epsilon W$ such that for every normal $w_i \epsilon W$, $w_i R w_j$, and (c) $\mathsf{V}$ is a value-assignment as in a K1-model. The proper axiom of K1.2 is

*H1 CpLCMpp.*

Clearly, in order to establish the soundness of our interpretation for K1.2, we need only demonstrate that *H1* is K1.2-logically true. We proceed in the following way: Assume for the sake of reductio that $\mathsf{V}(CpLCMpp, w_i) = 0$; obviously it follows that

$$(1)\ \mathsf{V}(p, w_i) = 1$$
$$(2)\ \mathsf{V}(LCMpp, w_i) = 0.$$

Thus from (2) we have

$$(3)\ \mathsf{V}(CMpp, w_j) = 0$$

and so

$$(4)\ \mathsf{V}(Mp, w_j) = 1$$
$$(5)\ \mathsf{V}(p, w_j) = 0.$$

Hence from (4) it follows that

$$(6)\ \mathsf{V}(p, w_k) = 1.$$

But $R$ is remotely symmetrical, thus either $w_kRw_j$ or $w_i = w_j$. If $w_kRw_j$, then both $w_j$ and $w_k$ are abnormal since they are accessible from every world in the above model. Thus it follows from (4) that there exists a wff, $p$, such that

$$(7)\ \ \mathsf{V}(p, w_j) = 1.$$

But this contradicts (5). If $w_i = w_j$, then (1) and (5) are inconsistent. Therefore, $\mathsf{V}(CpLCMpp, w_i) = 1$.

A K4-model will also be an ordered triple $\langle W, R, \mathsf{V}\rangle$ iff (a) it is an S4.4-model, (b) there exists at least one abnormal $w_j \in W$ such that for any normal $w_i \in W$, $w_iRw_j$, and (c) $\mathsf{V}$ is a value-assignment as in a K1-model. Remember that $R$ is reflexive, transitive, convergent, and remotely symmetrical. Now we show that

*P1 CMLMpCpLp*

the proper axiom of K4, is K4-logically true. Assume that $\mathsf{V}(CMLMpCpLp, w_i) = 0$, then clearly both

$$(1)\ \ \mathsf{V}(MLMp, w_i) = 1$$
$$(2)\ \ \mathsf{V}(CpLp, w_i) = 0.$$

Thus we have from (2) that

$$(3)\ \ \mathsf{V}(p, w_i) = 1$$
$$(4)\ \ \mathsf{V}(Lp, w_i) = 0.$$

Now we have it from (4) that

$$(5)\ \ \mathsf{V}(p, w_j) = 0.$$

But from (1) we obtain

$$(6)\ \ \mathsf{V}(LMp, w_k) = 1.$$

$R$ is convergent, therefore there exists a $w_l \in W$ such that $w_jRw_l$ and $w_kRw_l$. Thus it follows from (6) that

$$(7)\ \ \mathsf{V}(Mp, w_l) = 1.$$

Since $w_l$ is accessible from every world in this model, it is abnormal. Hence it follows from (7) that there exists a wff, $p$, such that

$$(8)\ \ \mathsf{V}(p, w_l) = 1$$

and so, from (8), there exists a wff, $Lp$, such that

$$(9)\ \ \mathsf{V}(Lp, w_l) = 1.$$

But $R$ is also remotely symmetrical. Thus either $w_lRw_j$ or $w_i = w_j$. If $w_lRw_j$, then it follows from (9) that

$$(10)\ \ \mathsf{V}(p, w_j) = 1$$

which contradicts (5). If $w_i = w_j$, then (3) and (5) are inconsistent with one another. Therefore, $\mathsf{V}(CMLMpCpLp, w_i) = 1$.

**4** In [2], R. I. Goldblatt establishes that each $\mathcal{Z}$ modal system is the intersection of S5 with some system from family $\mathcal{K}$. More specifically, he demonstrates that the connections between the $\mathcal{Z}$ and $\mathcal{K}$ modal systems are as follows:

$$
\begin{aligned}
Z1 &= S5 \cap K1\\
Z2 &= S5 \cap K1.2\\
Z3 &= S5 \cap K1.1\\
Z4 &= S5 \cap K2\\
Z5 &= S5 \cap K2.1\\
Z6 &= S5 \cap K3\\
Z7 &= S5 \cap K3.1\\
Z8 &= S5 \cap K4 = S4.9
\end{aligned}
$$

Given our models for the $\mathcal{K}$ systems, it is clear that the decidability of all the $\mathcal{Z}$ systems are established. All that is required for determining that a given wff is logically true in a given $\mathcal{Z}$-system is showing that it is S5-logically true and that it is also logically true in the appropriate $\mathcal{K}$-system.

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