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A NOTE ON EVALUATION MAPPINGS

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Let $\mathcal{L}$ be a functionally complete sentential language. Let Φ: $\mathcal{L}^n \times \mathcal{A} \to \{0, 1\}$, where $n \geq 1$ and $\mathcal{A}$ is the set of all assignments (i.e., mappings from the set V of all variables to $\{0, 1\}$). Then Φ shall be called an *evaluation mapping on* $\mathcal{L}$ in case for all $\varphi_1, \ldots, \varphi_n \in \mathcal{L}$ and all $\mathfrak{A}, \mathfrak{A}' \in \mathcal{A}$, if $\mathfrak{A}$ and $\mathfrak{A}'$ agree on the variables occurring in $\varphi_1, \ldots, \varphi_n$ then $\Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}) = \Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}')$. The notion of evaluation mapping is a syntactico-semantic generalization of the usual notion of truth-functional connective. For $S \subseteq \mathcal{L}$ and Φ an (n-ary) evaluation mapping:

(1) Φ is *truth-functional on S* in case for all $\varphi_1, \ldots, \varphi_n, \varphi'_1, \ldots, \varphi'_n \in S$ and $\mathfrak{A}, \mathfrak{A}' \in \mathcal{A}$, if $V_{\mathfrak{A}}(\varphi_i) = V_{\mathfrak{A}'}(\varphi'_i)(1 \leq i \leq n)$, then $\Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}) = \Phi(\varphi'_1, \ldots, \varphi'_n, \mathfrak{A}')$.

(2) Φ is *Boolean on S* in case there is $\varphi \in \mathcal{L}$ with n variables such that for all $\varphi_1, \ldots, \varphi_n \in S$ and every $\mathfrak{A} \in \mathcal{A}$, $\Phi(\varphi_1, \ldots, \varphi_n, \mathfrak{A}) = V_{\mathfrak{A}}\left(\varphi \begin{bmatrix} \alpha_1, \ldots, \alpha_n \\ \varphi_1, \ldots, \varphi_n \end{bmatrix}\right)$, where $\alpha_1, \ldots, \alpha_n$ are the variables occurring in φ, $\varphi \begin{bmatrix} \alpha_1, \ldots, \alpha_n \\ \varphi_1, \ldots, \varphi_n \end{bmatrix}$ is the sentence resulting from the simultaneous substitution in φ of φ_i for α_i $(1 \leq i \leq n)$, and $V_{\mathfrak{A}}$ is the sentential valuation induced by $\mathfrak{A}$.

Theorem *For every $S \subseteq \mathcal{L}$ and every evaluation mapping Φ, Φ is Boolean on S if and only if Φ is truth-functional on S.*

Proof: Necessity is obvious. We prove sufficiency. Suppose that Φ: $\mathcal{L}^n \times \mathcal{A} \to \{0, 1\}$ is truth-functional on S. Let f: $\{0, 1\}^n \to \{0, 1\}$ be the Boolean function such that for all $x_1, \ldots, x_n \in \{0, 1\}, f(x_1, \ldots, x_n) = \Phi(p_1, \ldots, p_n, \mathfrak{A})$, where $\mathfrak{A}(p_i) = x_i$ $(1 \leq i \leq n)$, and $p_1, \ldots, p_n$ are the first n variables of V. Then, by the definition of evaluation mapping (the full force of truth-functionality not being needed), f is well-defined, independent of the choice of $\mathfrak{A}$. Let $\varphi(= \varphi(p_1, \ldots, p_n)) \in \mathcal{L}$ express the function f. Then, for every $\mathfrak{A} \in \mathcal{A}$ and for all $\varphi_1, \ldots, \varphi_n \in S$, letting $\mathfrak{A}' \in \mathcal{A}$ such that $\mathfrak{A}'(p_i) = V_{\mathfrak{A}}(\varphi_i)(1 \leq i \leq n)$, we have (since Φ is truth-functional) that

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$$\begin{aligned}\Phi(\varphi_1, \ldots, \varphi_n) &= \Phi(p_1, \ldots, p_n, \mathfrak{A}') \\ &= f(\mathfrak{A}'(p_1), \ldots, \mathfrak{A}'(p_n)) \\ &= f(V_{\mathfrak{A}}(\varphi_1), \ldots, V_{\mathfrak{A}}(\varphi_n)) \\ &= V_{\mathfrak{A}}\left(\varphi\begin{bmatrix}p_1, & \ldots, & p_n \\ \varphi_1, & \ldots, & \varphi_n\end{bmatrix}\right).\end{aligned}$$

Thus, Φ is Boolean on S.

Let $\supset\!\!\!\!\!\cdot$ denote the counterfactual conditional discussed in [1]. The semantics provided therein for $\supset\!\!\!\!\!\cdot$ may be described by saying that for all $\varphi, \psi \in \mathcal{L}$ and every $\mathfrak{A} \in \mathcal{A}$, $V_{\mathfrak{A}}(\varphi \supset\!\!\!\!\!\cdot\; \psi) = 1$ if and only if $V_{\mathfrak{A}'}(\psi) = 1$ for every model $\mathfrak{A}'$ of $S \cup \{\varphi\}$ for any subject S of the collection of sentences true under $\mathfrak{A}$ such that S is maximal with respect to joint consistency with φ. It was shown in [1] that $V_{\mathfrak{A}}(\varphi \supset\!\!\!\!\!\cdot\; \psi) = 1$ if and only if for every disjunctive normal form η in the variables occurring in φ, ψ, if $V_{\mathfrak{A}}(\eta) = 1$ and $\{\eta, \varphi\}$ is consistent, then $\{\eta, \varphi, \psi\}$ is consistent.

Let Φ^*: $\mathcal{L}^2 \times \mathcal{A} \to \{0, 1\}$ be the evaluation mapping such that for all $\varphi, \psi \in \mathcal{L}$ and $\mathfrak{A} \in \mathcal{A}$, $\Phi^*(\varphi, \psi, \mathfrak{A}) = V_{\mathfrak{A}}(\varphi \supset\!\!\!\!\!\cdot\; \psi)$. By a *Boolean domain* for an evaluation mapping Φ, we mean a set S of sentences on which Φ is Boolean but such that Φ is not Boolean on any proper superset of S. Using the "normal form" characterization of the semantics for $\supset\!\!\!\!\!\cdot$, it is easily shown that the evaluation mapping Φ^* is *not* everywhere Boolean. In fact, Φ^* is not even Boolean on the set V of all variables, since, for any two distinct variables α and β, for every $\mathfrak{A} \in \mathcal{A}$, $\Phi^*(\alpha, \beta, \mathfrak{A}) = V_{\mathfrak{A}}(\alpha \wedge \beta)$, but for every $\mathfrak{A} \in \mathcal{A}$, $\Phi^*(\alpha, \alpha, \mathfrak{A}) = 1$, and, hence, for $\mathfrak{A}(\alpha) = 0$, $\Phi^*(\alpha, \alpha, \mathfrak{A}) \neq V_{\mathfrak{A}}(\alpha \wedge \alpha)$.

It is presently an open question as to exactly what are the Boolean domains for Φ^* and how to characterize Φ^* in terms of them. Given two evaluation mappings Φ_1 and Φ_2, we shall say $\Phi_1 \leqslant \Phi_2$ in case Φ_2 is Boolean on every set S on which Φ_1 is Boolean. Then $\leqslant$ is a pre-ordering (i.e., a reflexive, transitive relation). The pre-ordering $\leqslant$ determines an equivalence relation on the set of all evaluation mappings (namely, Φ_1 is *Boolean equivalent* to Φ_2 in case $\Phi_1 \leqslant \Phi_2$ and $\Phi_2 \leqslant \Phi_1$). Note that two evaluation mappings are Boolean equivalent if and only if they have exactly the same Boolean domains. We call the Boolean equivalence classes *Boolean degrees,* since the preordering $\leqslant$ unambiguously determines a partial ordering $\leqslant$ on the set $\mathcal{B}$ of all these equivalence classes. It is hoped that further research will reveal more of the structure of the partially ordered set $\mathcal{B}$ of Boolean degrees. An immediate conjecture to be investigated is whether or not $\langle \mathcal{B}, \leqslant \rangle$ is a lattice.

REFERENCE

[1] Wasserman, Howard C., "An analysis of the counterfactual conditional," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 395-400.

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