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A DEDUCTION SYSTEM FOR THE FULL FIRST-ORDER PREDICATE LOGIC

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In [3] H. Hermes and H. Scholz presented an axiomatization which generates exactly the valid formulas of the restricted first-order predicate logic. One of the features of that axiomatization is its symmetry in the underlying deduction rules. In this paper we shall describe an extension and generalization of the axiomatization given by Hermes and Scholz. The axiomatization in [3] is limited to the derivation of valid formulas of the pure first-order predicate logic. Our deduction system $\mathcal{S}$ is formulated for the full first-order predicate logic with identity, including individual constants and functional variables; furthermore, our system $\mathcal{S}$ provides for the deduction of formulas from sets of formulas. The strong completeness and soundness of our system $\mathcal{S}$ guarantees that in $\mathcal{S}$ those and only those deducibility relations are generated which are consequence relations. A deducibility concept, formulated in [1] and in some aspects resembling our formulation, requires a fairly complicated modification in order to obtain soundness (or even the validity of the Deduction Theorem). A preliminary version of our system $\mathcal{S}$ was first presented at the IV'th International Congress for Logic, Methodology, and Philosophy of Science; see [4].

1 The *vocabulary* for the full first-order predicate logic contains (i) a denumerable set of individual variables, (ii) a countable (i.e., finite or denumerable) set of individual constants, (iii) for each integer $n > 0$ a countable set of n-ary functional variables, (iv) for each integer $n \geqslant 0$ a countable set of n-ary predicate variables, (v) the identity symbol $\equiv$, (vi) the propositional connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, (vii) the quantifiers $\forall$ and $\exists$, and (viii) the parentheses (,). The set of *terms* is the smallest set which contains the individual variables and constants and which with any n-ary functional variable f and any n terms $t_1, \ldots, t_n$ also contains $ft_1 \ldots t_n$. Atomic formulas are the 0-ary predicate variables, all expressions of the form $pt_1 \ldots t_n$ where p is any n-ary predicate variable and $t_1, \ldots, t_n$ are any n terms, and all expressions of the form $t_1 \equiv t_2$ where t_1 and t_2 are any terms.

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Formulas are defined inductively by the following conditions:

(1) Each atomic formula is a formula.
(2) If B is a formula then $(\sim B)$ is a formula.
(3) If B and C are formulas then $(B \wedge C)$, $(B \vee C)$, $(B \rightarrow C)$ and $(B \leftrightarrow C)$ are formulas.
(4) If B is a formula and x any individual variable then $(\forall x B)$ and $(\exists x B)$ are formulas.

The notion of *free* occurrence of a term t in a formula A can be described inductively according to the inductive definition of formula as follows:

(1) Any occurrence of a term t in an atomic formula is a free occurrence.
(2) If t occurs free in the formula B then t occurs free in the formula $(\sim B)$.
(3) If t occurs free in the formula B or in the formula C then t occurs free in each of the formulas $(B \wedge C)$, $(B \vee C)$, $(B \rightarrow C)$, $(B \leftrightarrow C)$.
(4) If t occurs free in the formula B and x is an individual variable not occurring in t then t occurs free in the formulas $(\forall x B)$ and $(\exists x B)$.

If A is any formula, x any individual variable and t any term, and there exists a formula B which is the result of replacing in A each free occurrence of x by a free occurrence of t, then B is said to be obtained by a *free substitution* of t for x in A, abbreviated: $\mathsf{Subst}\ A\ x/t\ B$.

2 A *deduction* Δ in the system $\mathcal{S}$ is a finite sequence of ordered pairs $\langle S_k, A_k\rangle$, $1 \leqslant k \leqslant n$ for some positive integer n, where S_k is a (possibly empty) set of formulas upon which the formula A_k depends (according to the regulations set forth below). Each pair $\langle S_k, A_k\rangle$ of Δ must satisfy (at least) one of the following ten conditions:

(a) *Assumption Introduction:* $S_k = \{A_k\}$.
(b) *Assumption Elimination:* There is $i < k$ and a formula B such that $S_k = S_i - \{B\}$ and $A_k = (B \rightarrow A_i)$.
(c) *Tautological Inference:* Either $S_k = \emptyset$ and A_k is tautologous, or there exist $i_1 < k, \ldots, i_m < k$ such that $S_k = S_{i_1} \cup \ldots \cup S_{i_m}$ and $(A_{i_1} \wedge \ldots \wedge A_{i_m}) \rightarrow A_k$ is tautologous.
(d) *Free Substitution:* There is $i < k$, an individual variable x, and a term t such that $S_k = S_i$, x is not free in any of the formulas of S_i, and $\mathsf{Subst}\ A_i\ x/t\ A_k$.
(e) *Antecedent Generalization:* There is $i < k$, formulas B and C, and an individual variable x such that $S_k = S_i$, $A_i = (B \rightarrow C)$ and $A_k = ((\forall x B) \rightarrow C)$.
(f) *Consequent Generalization:* There is $i < k$, formulas B and C, and an individual variable x such that $S_k = S_i$, x is not free in any of the formulas of $S_i \cup \{B\}$, $A_i = (B \rightarrow C)$, and $A_k = (B \rightarrow (\forall x C))$.
(g) *Antecedent Particularization:* There is $i < k$, formulas B and C, and an individual variable x such that $S_k = S_i$, x is not free in any of the formulas of $S_i \cup \{C\}$, $A_i = (B \rightarrow C)$, and $A_k = ((\exists x B) \rightarrow C)$.
(h) *Consequent Particularization:* There is $i < k$, formulas B and C, and an individual variable x such that $S_k = S_i$, $A_i = (B \rightarrow C)$ and $A_k = (B \rightarrow (\exists x C))$.

(i) *Identity Introduction:* There is $i < k$, a formula B, a term t, and an individual variable x not occurring in t such that $S_k = S_i$, $\text{Subst}\, B\; x/t\; A_i$, and $A_k = (\forall x(x \equiv t \rightarrow B))$.
(j) *Identity Elimination:* There is $i < k$, a formula B, a term t, and an individual variable x not occurring in t such that $S_k = S_i$, $A_i = (\forall x(x \equiv t \rightarrow B))$ and $\text{Subst}\, B\; x/t\; A_k$.

A formula A is *deducible* from a set S of formulas, abbreviated: $S \vdash A$, if and only if there is a deduction Δ whose last pair $\langle S_n, A_n \rangle$ is such that $S_n \subseteq S$ and $A_n = A$. For "$\emptyset \vdash A$" where "$\emptyset$" denotes the empty set, we write "$\vdash A$".

3 The notion of deducibility, as formulated here, has the property that $S_k \vdash A_k$ for any pair $\langle S_k, A_k \rangle$ occurring as an element of a deduction. Furthermore, since any deduction is a finite sequence of pairs $\langle S_k, A_k \rangle$ and any formula occurring in S_k must have been introduced originally on account of condition (a) of a deduction, each set S_k must be a finite set. Thus we have at once the finiteness property: $S \vdash A$ if and only if $S' \vdash A$ for some *finite* subset S' of S.

The condition (b) for deductions concerning assumption eliminations becomes redundant if the two conditions (i) and (j) concerning identity introduction and elimination, respectively, are replaced by the following two conditions:

(i′) There is $i < k$, formulas B_1, B_2, and C, a term t and an individual variable x not occurring in t such that $S_k = S_i$, $A_i = (C \rightarrow B_2)$, $A_k = (C \rightarrow (\forall x(x \equiv t \rightarrow B_1)))$ and $\text{Subst}\, B_1\; x/t\; B_2$.
(j′) There is $i < k$, formulas B_1, B_2, and C, a term t and an individual variable x not occurring in t such that $S_k = S_i$, $A_i = (C \rightarrow (\forall x(x \equiv t \rightarrow B_1)))$, $A_k = (C \rightarrow B_2)$ and $\text{Subst}\, B_1\; x/t\; B_2$.

In conjunction with the conditions (a)-(h) for deductions, these conditions (i′) and (j′) yield exactly the same deducibility relations as the conditions (i) and (j). Moreover, any deducibility relation can be established on the basis of a deduction involving only the conditions (a), (c)-(h), (i′), and (j′), without using the condition (b).

The condition (b) for a deduction concerning assumption eliminations leads at once to the so-called Deduction Theorem:

$$\textit{If } S \cup \{A\} \vdash B \textit{ then } S \vdash A \rightarrow B.$$

Indeed, given $S \cup \{A\} \vdash B$, there exists a deduction Δ whose last pair is of the form $\langle S_n, B \rangle$ with $S_n \subseteq S \cup \{A\}$. The deduction Δ may be extended to a deduction Δ' by adding to Δ the pair $\langle S_n - \{A\}, A \rightarrow B \rangle$ on the basis of condition (b). Since $S_n - \{A\} \subseteq S$, Δ' is obviously a deduction for $S \vdash A \rightarrow B$.

The condition (c) of a deduction concerning tautological inferences is semantical in nature, but could be replaced in a variety of ways by purely syntactical conditions. For example, appropriate conditions which provide for the introduction and elimination of propositional connectives as in [2], could serve the purpose of condition (c). However, since the truth-table method provides such a simple means of testing whether or not a formula is tautologous, it appears that for practical purposes our condition (c) is preferable.

4 Our notion of deducibility is defined in such a manner that each of the ten conditions for a deduction leads immediately to a corresponding primary deducibility rule, as listed in the next theorem:

Theorem:

(a) (AI): *If* $A \in S$ *then* $S \vdash A$.
(b) (AE): *If* $S \vdash A$ *then* $S - \{B\} \vdash B \rightarrow A$.
(c) (TA): *If* A *is tautologous then* $\vdash A$;
(TI): *If* $S_1 \vdash A_1, \ldots, S_m \vdash A_m$ *and* $(A_1 \wedge \ldots \wedge A_m) \rightarrow A$ *is tautologous then* $S_1 \cup \ldots \cup S_m \vdash A$.
(d) (FS): *If* $S \vdash A$, Subst $A\ x/t\ B$ *and* x *is not free in any formula of* S *then* $S \vdash B$.
(e) (AG): *If* $S \vdash A \rightarrow B$ *then* $S \vdash \forall xA \rightarrow B$.
(f) (CG): *If* $S \vdash A \rightarrow B$ *and* x *is not free in any formula of* $S \cup \{A\}$ *then* $S \vdash A \rightarrow \forall xB$.
(g) (AP): *If* $S \vdash A \rightarrow B$ *and* x *is not free in any formula of* $S \cup \{B\}$ *then* $S \vdash \exists xA \rightarrow B$.
(h) (CP): *If* $S \vdash A \rightarrow B$ *then* $S \vdash A \rightarrow \exists xB$.
(i) (II): *If* $S \vdash B$, Subst $A\ x/t\ B$ *and* x *does not occur in* t *then* $S \vdash \forall x(x \equiv t \rightarrow A)$.
(j) (IE): *If* $S \vdash \forall x(x \equiv t \rightarrow A)$, Subst $A\ x/t\ B$ *and* x *does not occur in* t *then* $S \vdash B$.

5 In order to illustrate the usefulness of these rules we give several examples.

(I) $\vdash \exists x \forall y A \rightarrow \forall y \exists x A$

Proof:

1.	$\vdash A \rightarrow A$	(TA)
2.	$\vdash \forall yA \rightarrow A$	(AG)
3.	$\vdash \forall yA \rightarrow \exists xA$	(CP)
4.	$\vdash \exists x \forall yA \rightarrow \exists xA$	(AP)
5.	$\vdash \exists x \forall yA \rightarrow \forall y \exists xA$	(CG)

(II) $\vdash \exists xA \leftrightarrow \sim \forall x \sim A$

Proof:

1.	$\vdash \sim A \rightarrow \sim A$	(TA)
2.	$\vdash \forall x \sim A \rightarrow \sim A$	(AG)
3.	$\vdash A \rightarrow \sim \forall x \sim A$	(TI)
4.	$\vdash \exists xA \rightarrow \sim \forall x \sim A$	(AP)
5.	$\vdash A \rightarrow A$	(TA)
6.	$\vdash A \rightarrow \exists xA$	(CP)
7.	$\vdash \sim \exists xA \rightarrow \sim A$	(TI)
8.	$\vdash \sim \exists xA \rightarrow \forall x \sim A$	(CG)
9.	$\vdash \exists xA \leftrightarrow \sim \forall x \sim A$	(TI), 4, 8

(III) $\vdash t \equiv t$

Proof: Let p be a 0-ary predicate variable and let x be an individual variable not occurring in t; then we have:

1. $\vdash (p \to p) \to (x \equiv t \to x \equiv t)$ (TA)
2. $\vdash (p \to p) \to \forall x(x \equiv t \to x \equiv t)$ (CG)
3. $\vdash \forall x(x \equiv t \to x \equiv t)$ (TI)
4. $\vdash t \equiv t$ (IE)

6 For the purpose of demonstrating the completeness of the primary deducibility rules, we establish several additional deducibility rules for the system $\mathcal{S}$.

(AI*): *If* $S \vdash A \to B$ *then* $S \cup \{A\} \vdash B$.

Proof:
1. $S \vdash A \to B$ Hypothesis
2. $\{A\} \vdash A$ (AI)
3. $S \cup \{A\} \vdash B$ (TI), 1, 2

($\forall$E): *If* $S \vdash \forall xA$ *and* Subst $A\ x/t\ B$ *then* $S \vdash B$.

Proof:
1. $S \vdash \forall xA$ Hypothesis
2. $\vdash A \to A$ (TA)
3. $\vdash \forall xA \to A$ (AG)
4. $\vdash \forall xA \to B$ (FS), Subst $A\ x/t\ B$
5. $S \vdash B$ (TI), 1, 4

($\forall$I): *If* $S \vdash A$ *and* x *is not free in any formula of* S *then* $S \vdash \forall xA$.

Proof: Let y be an individual variable which is distinct from x and does not occur in A; then there exists a unique formula B such that Subst $A\ x/y\ B$ and Subst $B\ y/x\ A$; furthermore, x is not free in B. Now:

1. $S \vdash A$ Hypothesis
2. $S \vdash B \to A$ (TI)
3. $S \vdash B \to \forall xA$ (CG); x is not free in $S \cup \{B\}$
4. $S \vdash B$ (FS); x is not free in S
5. $S \vdash \forall xA$ (TI), 3, 4

($\forall$I*): *If* $S \vdash B$, Subst $B\ y/x\ A$, Subst $A\ x/y\ B$ *and* y *is not free in any formula of* S, *then* $S \vdash \forall xA$.

Proof: If x is identical to y then this rule ($\forall$I*) reduces to the rule ($\forall$I). Thus it can be assumed that x is distinct from y. From $S \vdash B$ it follows that there is a finite subset S' of S, say $S' = \{A_1, \ldots, A_r\}$, such that $S' \vdash B$. Let z be an individual variable distinct from y and not occurring in any formula of $S' \cup \{\forall xA\}$. Then there exist unique formulas $B_1, \ldots, B_r$ such that Subst $A_i\ x/z\ B_i$ and Subst $B_i\ z/x\ A_i$ for $i = 1, \ldots, r$. Letting $S'' = \{B_1, \ldots, B_r\}$, we note that y is not free in any formula of S'' since y is not free in any formula of S'; also x is not free in any formula of S'' since x is distinct from z. Now we have:

1. $S' \vdash B$ Hypothesis
2. $\vdash A_1 \to (\ldots \to (A_r \to B) \ldots)$ (AE), r applications

3.	$\vdash B_1 \rightarrow (\ldots \rightarrow (B_r \rightarrow B) \ldots)$	(FS); x/z
4.	$\vdash B_1 \rightarrow (\ldots \rightarrow (B_r \rightarrow A) \ldots)$	(FS); y/x
5.	$\vdash (B_1 \wedge \ldots \wedge B_r) \rightarrow A$	(TI)
6.	$\vdash (B_1 \wedge \ldots \wedge B_r) \rightarrow \forall xA$	(CG)
7.	$\vdash (A_1 \wedge \ldots \wedge A_r) \rightarrow \forall xA$	(FS); z/x
8.	$\vdash A_1 \rightarrow (\ldots \rightarrow (A_r \rightarrow \forall xA) \ldots)$	(TI)
9.	$S' \vdash \forall xA$	(AI*), r applications
10.	$S \vdash \forall xA$	definition of $\vdash$; $S' \subseteq S$

($\exists$I): *If* $S \vdash B$ *and* Subst $A\ x/t\ B$ *then* $S \vdash \exists xA$.

Proof:

1.	$S \vdash B$	Hypothesis
2.	$\vdash A \rightarrow A$	(TA)
3.	$\vdash A \rightarrow \exists xA$	(CP)
4.	$\vdash B \rightarrow \exists xA$	(FS)
5.	$S \vdash \exists xA$	(TI), 1, 4

($\exists$E*): *If* $S \cup \{B\} \vdash C$, Subst $B\ y/x\ A$, Subst $A\ x/y\ B$ *and* y *is not free in any formula of* $S \cup \{C\}$ *then* $S \cup \{\exists xA\} \vdash C$.

Proof:

1.	$S \cup \{B\} \vdash C$	Hypothesis
2.	$S \vdash B \rightarrow C$	(AE)
3.	$S \vdash \sim C \rightarrow \sim B$	(TI)
4.	$S \cup \{\sim C\} \vdash \sim B$	(AI*)
5.	$S \cup \{\sim C\} \vdash \forall x \sim A$	($\forall$I*)
6.	$\vdash \sim A \rightarrow \sim A$	(TA)
7.	$\vdash \forall x \sim A \rightarrow \sim A$	(AG)
8.	$\vdash A \rightarrow \sim \forall x \sim A$	(TI)
9.	$\vdash \exists xA \rightarrow \sim \forall x \sim A$	(AP)
10.	$S \cup \{\sim C\} \vdash \sim \exists xA$	(TI), 5, 9
11.	$S \vdash \sim C \rightarrow \sim \exists xA$	(AE)
12.	$S \vdash \exists xA \rightarrow C$	(TI)
13.	$S \cup \{\exists xA\} \vdash C$	(AI*)

(FS*): *If* $S \vdash A$, Subst $S\ x/t\ S'$ *and* Subst $A\ x/t\ A'$ *then* $S' \vdash A'$.

Note: Subst $S\ x/t\ S'$ means that for each $B \in S$ there is $B' \in S'$ such that Subst $B\ x/t\ B'$, and for each $B' \in S'$ there is $B \in S$ such that Subst $B\ x/t\ B'$.

Proof: Since $S \vdash A$, there is a finite subset $\{A_1, \ldots, A_r\}$ of S such that $\{A_1, \ldots, A_r\} \vdash A$. Let $A_1', \ldots, A_r'$ be such that Subst $A_i\ x/t\ A_i'$ for $i = 1, \ldots, r$ and hence $\{A_1', \ldots, A_r'\} \subseteq S'$. We have now:

1.	$\{A_1, \ldots, A_r\} \vdash A$	Hypothesis
2.	$\vdash A_1 \rightarrow (\ldots \rightarrow (A_r \rightarrow A) \ldots)$	(AE), r applications
3.	$\vdash A_1' \rightarrow (\ldots \rightarrow (A_r' \rightarrow A') \ldots)$	(FS); x/t
4.	$\{A_1', \ldots, A_r'\} \vdash A'$	(AI*), r applications
5.	$S' \vdash A'$	definition of $\vdash$; $\{A_1', \ldots, A_r'\} \subseteq S'$

(II*) *If* $S \vdash A$ *and* Subst $A\ x/t\ B$ *then* $S \cup \{x \equiv t\} \vdash B$.

Proof: Let z be an individual variable distinct from x and not occurring in

A; then there is a unique formula C such that Subst $A\ x/z\ C$ and Subst $C\ z/x\ A$; furthermore, Subst $C\ z/t\ B$ since Subst $A\ x/t\ B$. Now:

1. $S \vdash A$	Hypothesis
2. $\vdash C \rightarrow C$	(TA)
3. $\vdash \forall x(x \equiv z \rightarrow (A \rightarrow C))$	(II)
4. $\vdash x \equiv z \rightarrow (A \rightarrow C)$	(∀E)
5. $\vdash x \equiv t \rightarrow (A \rightarrow B)$	(FS); z/t
6. $\{x \equiv t\} \vdash A \rightarrow B$	(AI*)
7. $S \cup \{x \equiv t\} \vdash B$	(TI), 1, 6

The primary deducibility rules together with the rules established so far for the system $\mathcal{S}$ suffice to guarantee the completeness of our system $\mathcal{S}$. For example, the rules (AI), (AE), (TI), (∀E), (∀I), (FS*), (∃I), (∃E*), and (II*) together with the deducibility relation $\vdash t \equiv t$ are already sufficient to obtain all consequence relations for the full first-order predicate logic; except for the rules (AI), (AE), and (TI), the remaining rules of the above list, dealing with quantification and identity elimination, are generalizations of rules which in [2] are shown to form a complete set of rules for first-order predicate logic with identity.

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