Notre Dame Journal of Formal Logic
Volume 23, Number 1, January 1982

Model Structures and Set Algebras for Sugihara Matrices

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1 Introduction Since the work of Lemmon on modal algebras [2], [3], it has been known that there is a close relationship between relational model structures, set algebras, and matrices. The type of result which Lemmon established was to show how to construct a modal set algebra given a modal model structure, or a model structure given an algebra, in such a way that validity in the algebra coincides with validity in the model structure.

The extension of Lemmon's type of result to various cases of relevant algebras and relevant model structures associated with relevant logics has been studied by Brady in [6], by Routley and Meyer in [5] and [6], and by the author in [4] and [6]. The purpose of this paper is to report results connecting model structures and set algebras for the Sugihara matrices, and in particular for two infinite Sugihara matrices, both of which are characteristic for the important logic RM. The Sugihara matrices, or chains, and the logic RM are investigated in Anderson and Belnap's [1]. To date, no semantics for RM has been given which uses only a single relational model structure. This paper provides such a semantics. Earlier results, for example in [6], of such theorems connecting particular relational model structures and particular set algebras have been exclusively for finite cases of such algebras. The present result is new in that it is the first such example of an infinite algebra and model structure.

2 Sugihara matrices, algebras, and model structures Let I be the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and let $I^+ = I \cup \{+\omega, -\omega\}$. The ordering $<_e$ on I^+, called the *extensional ordering*, is defined to be the natural ordering on I together with the proviso that for $x \in I$, $-\omega < x < +\omega$. The *intensional ordering*, $<_i$, on I^+ is defined by $x <_i y$ if either $|x| <_e |y|$ (where $|-\omega| = +\omega$), or $x = -y$ and $x <_e 0$. Let $s_n^0 = \{x : x \in I \,\&\, |x| <_e n + 1\}$, and let $s_n = s_n^0 - \{0\}$. The *Sugihara matrices* are quintuples $\langle \Sigma, \vee, \sim, \rightarrow, \mathcal{D} \rangle$ where: (a) Σ is I^+, $I^+ - \{0\}$, I, $I - \{0\}$, s_n^0, or s_n; (b) $x \vee y = \max \{x, y\}$ relative to $<_e$; (c) $\sim x$ = the numerical

Received May 2, 1980; revised March 30, 1981

negative of x, and $\sim 0 = 0$; (d) $x \rightarrow y$ is defined by: if $x \leqslant_e y$ then $x \rightarrow y = \sim x \vee y$, otherwise $x \rightarrow y = \sim(x \vee \sim y)$; and (e) $\mathcal{D}$ (the set of designated elements) = $\{x: -1 <_e x\}$. If $\Sigma = s_n^0$ or s_n, we describe the matrix as a *finite* or *finite normal* Sugihara matrix respectively, and denote it by S_n^0 or S_n. If $\Sigma = I$ or $I - \{0\}$, we call the matrix the *infinite* or *infinite normal* Sugihara matrix respectively, and denote it by S^0 or S. If $\Sigma = I^+$ or $I^+ - \{0\}$, we call it the *complete infinite* or *complete infinite normal* Sugihara matrix respectively, and denote it by S_ω^0 or S_ω.

We now define our language L to consist of a denumerable number of proposition letters, closed in the usual way under $\vee$, $\sim$, and $\rightarrow$. Capital letters from the front of the alphabet function as metalinguistic variables. $A \& B$, $A \oplus B$, and $A \bigcirc B$ are abbreviations of $\sim(\sim A \vee \sim B)$, $\sim A \rightarrow B$, and $\sim(A \rightarrow \sim B)$, respectively. A function V: $L \rightarrow \Sigma$ is an S_n^0 *valuation* (or S_n, S^0, S, S_ω^0, S_ω valuation, depending on choice of Σ) iff V satisfies $V(A \vee B) = V(A) \vee V(B)$, $V(\sim A) = \sim V(A)$, and $V(A \rightarrow B) = V(A) \rightarrow V(B)$. (Note that the left hand side connective in these equations is the propositional operator. The right hand side connective is the algebraic operator.) A formula A is S_n^0 *valid* (or S_n, etc., valid) (written, as usual $\models_{S_n^0} A$, etc.) iff for all S_n^0 valuations (or S_n, etc., valuations) V, $V(A) \in \mathcal{D}$.

Let $K^0 = \{T, T^*, a_1, a_1^* \ldots\}$ and let $K_n^0 = \{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$. Then a pair $\langle K, R\rangle$, where $K = K^0$ (or K_n^0 for some n) is the *Sugihara model structure* (or a finite Sugihara model structure), denoted by $\underline{S}^0$ (or $\underline{S}_n^0$), iff: (a) the elements of K are distinct, and * is a function $*: K^0 \rightarrow K^0$ satisfying $a^{**} = a$ for all $a \in K^0$; (b) R is a ternary relation on K^0; and (c) if we let $\ldots < a_{i-1}^* < \ldots < a_1^* < T^* < T < a_1 < \ldots < a_{i-1} < \ldots$ and let $a \leqslant b$ iff $a < b$ or $a = b$, then R satisfies $(\forall abc \in K^0)$ ($Rabc$ iff $(\exists d)(a \leqslant d$ and $b \leqslant d^*$ and $d \leqslant c)$ or ($a \leqslant T$ and $b \leqslant c$) or ($a \leqslant T$ and $b \leqslant c$) or ($a \leqslant b = c$) or ($b = c \leqslant a^*$). To obtain the *normal* and *finite normal* Sugihara model structures, $\underline{S}$ and $\underline{S}_n$, set $T = T^*$ in $\underline{S}^0$ and $\underline{S}_n^0$ respectively. A function I: $LxK^0 \rightarrow \{1,0\}$ is an *interpretation* on a Sugihara model structure $\langle K, R\rangle$ iff I satisfies: (a) $I(\sim A, a) = 1$ iff $I(A, a^*) \neq 1$; (b) $I(A \vee B, a) = 1$ iff either $I(A, a) = 1$ or $I(B, a) = 1$; (c) $I(A \rightarrow B, a) = 1$ iff $(\forall bc)$ (not all of $Rabc$ and $I(A, b) = 1$ and $I(B, c) \neq 1$); and (d) if $a \leqslant b$ and $I(A, a) = 1$ then $I(A, b) = 1$. A formula A is *valid* on $\underline{S}^0$ (or $\underline{S}_n^0$, $\underline{S}$, $\underline{S}_n$) iff for all interpretations I on $\underline{S}^0$ (or $\underline{S}_n^0$, etc.), $I(A, T) = 1$.

If $\langle K, R\rangle$ is a Sugihara model structure, we define the *associated Sugihara (set) algebra* to be the quintuple $\langle U, \vee, \sim, \rightarrow, D\rangle$ where: (a) $U \subseteq \mathcal{P}(K)$ (the power set of K) is such that if $a \in x \in U$ and $a \leqslant b$ then $b \in x$; (b) $\vee$ is set theoretic union; (c) $a \in \sim x$ iff $a^* \notin x$; (d) $a \in x \rightarrow y$ iff $(\forall bc)$ (not all of $Rabc$ and $b \in x$ and $c \notin y$); and (e) $x \in D$ iff $T \in x$. For $\langle K, R\rangle = \underline{S}^0$, $\underline{S}_n^0$, $\underline{S}$, $\underline{S}_n$ we call the associated algebra the *Sugihara algebra*, *finite* Sugihara algebra, *normal* and *finite normal* Sugihara algebras, respectively, and denote them by $\underset{\sim}{S}^0$, $\underset{\sim}{S}_n^0$, $\underset{\sim}{S}$, and $\underset{\sim}{S}_n$, respectively. A function $\mathcal{A}$: $L \rightarrow \underset{\sim}{S}^0$ (or $\underset{\sim}{S}_n^0$, etc.) is called an *assignment* on $\underset{\sim}{S}^0$, etc., iff $\mathcal{A}(A \vee B) = \mathcal{A}(A) \vee \mathcal{A}(B)$, $\mathcal{A}(\sim A) = \sim\mathcal{A}(A)$ and $\mathcal{A}(A \rightarrow B) = \mathcal{A}(A) \rightarrow \mathcal{A}(B)$. A formula A is $\underset{\sim}{S}^0$*-valid* or ($\underset{\sim}{S}_n^0$ valid, etc.) iff for all assignments $\mathcal{A}$ on $\underset{\sim}{S}^0$ (etc.), $\mathcal{A}(A) \in D$.

3 ***Results*** The main result reported in this note is that validity in a Sugihara model structure, in its associated set algebra, and in an appropriately chosen Sugihara matrix all coincide. Proof of these facts is quite long, and is given in full in [4]. Here only the main steps are sketched. We begin by noting that part of the result, namely the connection between the model structures and their associated set algebras, is a more-or-less immediate consequence of the general connection between model structures and set algebras established by Routley and Meyer in [5]. Thus:

Theorem 1 $\models_{X} A$ *iff* $\models_{\underset{\sim}{X}} A$, *for* $X = S^0, S_n^0, S$, *or* S_n.

We proceed to connect the set algebras with the matrices. We note an important preliminary fact: that the set algebras in question will all be complete as lattices, since arbitrary set theoretic unions and intersections exist. But the infinite and infinite normal Sugihara matrices originally investigated in [1] are not complete as lattices, since they lack maximal and minimal elements. Naturally, all the finite matrices will be complete. Since we propose to establish isomorphism between the set algebras and suitable matrices, we need to complete the Sugihara matrices, hence the complete Sugihara matrices defined above. It is not difficult to show that validity in the complete matrices coincides with validity in their incomplete counterparts. Hence we have

Theorem 2 $\models_{X} A$ *iff* $\models_{X_\omega} A$, *for* $X = S^0, S$.

We now link the complete matrices with the set algebras. Let $\langle \Sigma, \vee, \sim, \rightarrow, \mathscr{D} \rangle$ be a complete (finite or infinite) Sugihara matrix and $\langle U, \vee, \sim, \rightarrow, D \rangle$ a Sugihara algebra arising from a model structure $\langle K, R \rangle$. K is either $\{T, T^*, a_1, a_1^*, \ldots\}$ or $\{T, T^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$. T and T^* may or may not be distinct, but all other elements are distinct. Let K^+ be the set of unstarred elements of K, including T. Then the *natural correspondence* f between Σ and U is defined to be the function $f: \Sigma \rightarrow U$ satisfying: (a) if $0 \in \Sigma$ then $f(0) = K^+$; (b) for $n > 0$, $f(n) = K^+ \cup \{a_0^*, a_1^*, \ldots a_{n-1}^*\}$ (where $a_0 = T$); and (c) for $n < 0$, $f(n) = K^+ - \{a_0, a_1, \ldots, a_{n-1}\}$.

Theorem 3 *f is a 1-1 correspondence, and $x \in \mathscr{D}$ iff $f(x) \in D$.*

Theorem 4 $f(\sim A) = \sim f(A), f(A \vee B) = f(A) \vee f(B)$.

The proofs of these two theorems are straightforward adaptations of the general results of Routley and Meyer [5].

Theorem 5 $f(A \rightarrow B) = f(A) \rightarrow f(B)$.

In order to establish Theorem 5, we need some lemmas.

Lemma 1 *If $\langle K, R \rangle$ is a Sugihara model structure, $Rabc$ iff Rac^*b^*.*
Lemma 2 *In any Sugihara algebra, $x \rightarrow y = \sim y \rightarrow \sim x$, and $x \;\&\; y = x \cap y$.*
Lemma 3 *In any Sugihara model structure, Raa^*a.*
Lemma 4 *In any Sugihara model structure, $Rabc$ entails ($a \leqslant c$ or $b \leqslant c$) and ($a \leqslant c$ or $a \leqslant b^*$).*
Lemma 5 *In any Sugihara algebra $\langle U, \vee, \sim, \rightarrow, D \rangle$ if $b \notin x \in U$ and $a \leqslant b$ then $a \notin x$.*
Lemma 6 *If $f: \Sigma \rightarrow U$ is the natural correspondence between a Sugihara matrix and algebra, then $x \leqslant y$ (numerically) iff $f(x) \subseteq f(y)$.*

In view of Lemmas 2 and 6 and the definition of $\vee$ and $\rightarrow$ in Sugihara matrices, viz., if $x \leqslant y$ then $x \rightarrow y = {\sim}x \vee y$ and otherwise $x \rightarrow y = {\sim}x \,\&\, y$, it suffices to prove Theorem 5 if we can prove the following for any Sugihara algebra: if $x \subseteq y$ then $x \rightarrow y = {\sim}x \cup y$, and if $y \subset x$ then $x \rightarrow y = {\sim}x \cap y$. We split the proof into two parts and, in turn, divide each of these in two.

1. If $x \subseteq y$ then $x \rightarrow y = {\sim}x \cup y$.

1.1 If $x \subseteq y$ and $a \notin x \rightarrow y$ then $a \notin {\sim}x \cup y$. Assume the antecedent. Now $a \notin {\sim}x \cup y$ iff $a \notin {\sim}x$ and $a \notin y$. So we need to prove that $a \notin {\sim}x$ and $a \notin y$. We prove first that $a \notin y$. Since $a \notin x \rightarrow y$, $(\exists b, c)$ $(Rabc \,\&\, b \in x \,\&\, c \notin y)$. By Lemma 3, $Rabc$ implies $a \leqslant c$ or $b \leqslant c$. If $a \leqslant c$ and $c \notin y$ then by Lemma 5, $a \notin y$. Now $c \notin y$, so by Lemma 5 $b \notin y$. But $x \subseteq y$, so $b \notin x$, contradicting $b \in x$. Hence not $b \leqslant c$, so $a \leqslant c$, so $a \notin y$ as required.

We note now that $x \subseteq y$ implies ${\sim}y \subseteq {\sim}x$. Also by Lemma 2, $x \rightarrow y = {\sim}y \rightarrow {\sim}x$. Thus from the assumption of the antecedent we have ${\sim}y \subseteq {\sim}x$ and $a \notin {\sim}y \rightarrow {\sim}x$. So by a similar argument to the one just given, $a \notin {\sim}x$.

1.2 If $x \subseteq y$ and $a \notin {\sim}x \cup y$ then $a \notin x \rightarrow y$. Suppose the antecedent. As in 1.1, we need to prove that $a \notin {\sim}x$ and $a \notin y$ implies $a \notin x \rightarrow y$. Now $a \notin {\sim}x$ implies $a^* \in x$. By Lemma 3, Raa^*a. So Raa^*a and $a^* \in x$ and $a \notin y$. Hence $a \notin x \rightarrow y$.

2. If $y \subset x$ then $x \rightarrow y = {\sim}x \cap y$.

2.1 If $y \subset x$ and $a \notin x \rightarrow y$ then $a \notin {\sim}x \cap y$. Suppose the antecedent. We need to prove $a \notin {\sim}x \cap y$, i.e., $a \notin {\sim}x$ or $a \notin y$. Since $a \notin x \rightarrow y$, for some b, c we have $Rabc$ and $b \in x$ and $c \notin y$. Now by Lemma 4, $Rabc$ implies $a \leqslant c$ or $a \leqslant b^*$. If $a \leqslant c$ then, since $c \notin y$, by Lemma 5 $a \notin {\sim}y$; so that $a \notin {\sim}x$ or $a \notin y$. Hence, suppose instead $a \leqslant b^*$. Now $b \in x$, so $b^* \notin {\sim}x$. So by Lemma 5, $a \notin {\sim}x$; so that $a \notin {\sim}x$ or $a \notin y$.

2.2 If $y \subset x$ and $a \notin {\sim}x \cap y$ then $a \notin x \rightarrow y$. Now $a \notin {\sim}x \cap y$ iff $a \notin {\sim}x$ or $a \notin y$. So we need to prove both $a \notin {\sim}x$ implies $a \notin x \rightarrow y$ and $a \notin y$ implies $a \notin x \rightarrow y$. Suppose first $a \notin y$. Now $y \subset x$, so not every world is in y. Select the largest world relative to $\leqslant$ not in y. Denote it by b; i.e., $b \notin y$. Now $a \notin y$ and b is the largest world not in y, so $a \leqslant b$. By the third disjunct of the definition of R, viz., $a \leqslant b = c$, we can thus have $Rabb$. But we must also have $b \in x$, because when we construct members of the set algebra we make such sets of worlds progressively larger by adding the largest of the remaining nonmembers. If we add any smaller member than the largest nonmember of y to y then by Lemma 5 we must also add the largest: members of the algebra are closed upwards under $\leqslant$. So since b is the largest nonmember of y, b is in every proper superset of y. But $y \subset x$, so $b \in x$. Thus $Rabb$ and $b \in x$ and $b \notin y$. So $a \notin x \rightarrow y$.

Now let $a \notin {\sim}x$. Since $y \subset x$, ${\sim}x \subset {\sim}y$. Hence, by an identical argument to the one just given, $a \notin {\sim}y \rightarrow {\sim}x$. But ${\sim}y \rightarrow {\sim}x = x \rightarrow y$. Thus, $a \notin x \rightarrow y$.

From the preceding theorems it follows that the natural correspondence f is an isomorphism which also preserves designated elements. Hence validity in the various Sugihara matrices coincides with validity in the various complete Sugihara set algebras. Combining this with Theorems 1 and 2, we have:

Theorem 6 *The following statements are equivalent:*

(1) $\models_{S^0_n} A$. (2) $\models_{\underline{S}^0_n} A$. (3) $\models_{\utilde{S}^0_n} A$.

The following are equivalent:

(4) $\models_{S_n} A$. (5) $\models_{\underline{S}_n} A$. (6) $\models_{\utilde{S}_n} A$.

The following are equivalent:

(7) $\models_{S^0} A$. (8) $\models_{S^0_\omega} A$. (9) $\models_{\underline{S}^0} A$. (10) $\models_{\utilde{S}^0} A$.

The following are equivalent:

(11) $\models_{S} A$. (12) $\models_{S_\omega} A$. (13) $\models_{\underline{S}} A$. (14) $\models_{\utilde{S}} A$.

4 Concluding remarks We conclude with some observations on the connection between model structures and matrices. The connection between syntax and semantics for a logic is in a sense *prima facie* mysterious. Some of the mystery can be removed by showing that syntax and semantics are different sides of the same coin. Several ways of doing this are current in the literature. The Lindenbaum algebra, a linguistic construction, links algebraic semantics with syntax. Similarly, the canonical model links worlds semantics with syntax; and tableau constructions can be semantically or alternatively proof theoretically oriented, the difference in some cases being difficult to discern. In addition to the syntax-semantics link, connections between different semantics for the one logic are worth making, else we might wonder why different kinds of semantics characterise the one logic. Theoretically this might be done via the syntax, but it is always worth making the connection directly, especially if we wish to remain open on the doctrine that semantics can be genuinely explanatory rather than covert syntax. A matrix is a puzzle: how does it arise to characterise a logic? One important aspect of Lemmon's work, developing as it did from Stone's representation theorem for Boolean Algebras was to show that matrices characterising modal logics are algebras and can be viewed as deriving from model structures. If we regard, as many have, the worlds semantics as explanatory of modal logics, Lemmon's results thus provide an explanation of algebraic semantics and matrices of those logics.

The present results can be viewed along these lines. The connection between the worlds semantics and the complete Sugihara matrices via the Sugihara set algebras 'explains' the matrices, by allowing them to be seen as transformations of worlds structures. This raises several questions, however. One is, from whence the incomplete Sugihara matrices S^0 and S, which are not isomorphic with any of the set algebras? The answer is that RM has a certain compactness feature, in that for purposes of validity of a particular formula we only need consider a finite Sugihara algebra, and the union of all such is incomplete as a lattice. It follows that the hope of showing all matrices to be explicable in the direct way indicated above is unwarranted. Nevertheless, large classes of matrices have proved themselves to be amenable to this treatment. If we adopt this view of 'explanation', we can conclude that there is a sense in which the 'real' Sugihara matrices are those explained directly in terms of worlds structures, namely those with maximal and minimal elements $\{+\omega, -\omega\}$. We can also conclude that, in looking for explanations of matrices, none of those which lack maximal and minimal elements, indeed which are incomplete as lattices, are directly so explicable.

Another question is how we manage to deal with algebraic structures such as chains, using as we do an essentially Boolean power-set construction on the set of worlds. The answer is that the so-called Hereditary Condition in the Routley-Meyer worlds semantics for relevant logics forces a collapse of certain elements in the power set algebra on the set of worlds, and thus generates non-Boolean set algebras. On these matters see Mortensen [4] and Routley and Meyer [5].

REFERENCES

[1] Anderson, A. R. and N. D. Belnap, *Entailment*, Princeton University Press, Princeton, New Jersey, 1975.

[2] Lemmon, E. J., "Algebraic semantics for modal logics I," *The Journal of Symbolic Logic* (1966), pp. 45-65.

[3] Lemmon, E. J., "Algebraic semantics for modal logic II," *The Journal of Symbolic Logic* (1966), pp. 191-218.

[4] Mortensen, C. E., *Relevant Algebras and Relevant Model Structures*, Logic Papers No. 8, Logic Group, Research School of Social Sciences, The Australian National University, 1980.

[5] Routley, F. R. and R. K. Meyer, "Algebraic analysis of entailment I," *Logique et Analyse* (1973), pp. 407-428.

[6] Routley, F. R. et al., *Relevant Logics and Their Rivals*, Australian National University, 1981.

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