Notre Dame Journal of Formal Logic
Volume 23, Number 1, January 1982

# A Result on Propositional Logics Having the Disjunction Property

ROBERT E. KIRK*

It was conjectured in 1952 by Łukasiewicz [5] that the intuitionistic propositional logic ($I$) was the only consistent logic which has all intuitionistically valid formulas as theorems and also has the disjunction property, i.e., the property that $\phi$ or $\psi$ is a theorem whenever $\phi \vee \psi$ is. This conjecture was shown to be false by Kreisel and Putnam [3], who exhibited a logic stronger that $I$ having the disjunction property. Since the disjunction property is of some importance in constructive mathematics, a question arising naturally is whether there is a *maximal* propositional logic having this property. It is the purpose of this article to show that there is not; more precisely, that there is no intermediate logic having the disjunction property which contains as theorems all theorems of such logics.

By a propositional logic we always mean a consistent system formulated in the usual way and closed under substitution and detachment, and by an intermediate logic we mean a propositional logic whose theorems include all intuitionistically valid formulas. Our proof will proceed by exhibiting two intermediate logics with the disjunction property whose union fails to have the property. The intermediate logics used are the system $KP$, used by Kreisel and Putnam to refute Łukasiewicz, which is axiomatized over $I$ by the addition of the formula

$$(\neg p \rightarrow (q \vee r)) \rightarrow ((\neg p \rightarrow q) \vee (\neg p \rightarrow r)),$$

and the logic of finite binary trees $D_1$ of Gabbay and DeJongh [2], which is axiomatized over $I$ by the addition of

*I would like to thank the referee for pointing out a deficiency in an earlier version of this paper.

*Received December 27, 1978; revised April 13, 1981*

$$(((p \to (q \vee r)) \to (q \vee r)) \,\&\, ((q \to (p \vee r)) \to (p \vee r)) \,\&\, ((r \to (p \vee q)) \to (p \vee q))) \to (p \vee q \vee r).$$

By a Kripke frame we shall understand a partially ordered set $\langle A, \leqslant, 0\rangle$ with least element 0, and, as usual, a Kripke model is a Kripke frame together with an assignment of propositional formulas to elements of $A$ satisfying certain conditions (see, e.g., Kripke [4]). We use the notation $[\theta]_a = 1$ to mean that $\theta$ is assigned to $a$ ("$\theta$ is true at $a$"), and $[\theta]_a = 0$ to mean that it is not. A formula $\theta$ is said to be true in a Kripke model $\langle A, \leqslant, 0, [\ ]\rangle$ if $[\theta]_a = 1$ for all $a \in A$. As usual, a Kripke frame is said to be $L$-valid, where $L$ is an intermediate logic, whenever the theorems of $L$ are true in every Kripke model based on the frame.

If $\langle A, \leqslant\rangle$ is a partially ordered set and $B$ a subset of $A$, let $B^+$ and $B^-$ denote, respectively, $\{a|b \leqslant a$ for some $b \in B\}$ and $\{a|a \leqslant b$ for some $b \in B\}$. Finally, let us call $\langle A, \leqslant\rangle$ cohesive if, for every $a \in A$ and $B \subseteq \{a\}^+, \{a\}^+ - B^{+-}$ has at most one minimal element, and let us call $\langle A, \leqslant\rangle$ binary whenever each element of $A$ has at most two immediate successors. Building on a result of Gabbay [1], we prove the following completeness result for the logic $KP + D_1$.

**Theorem 1** *$KP + D_1$ is complete for the class of finite cohesive binary Kripke frames.*

*Proof:* Suppose that $\psi$ is not a theorem of $KP + D_1$. Then, as is well-known, there exists a Kripke model $\langle A, \leqslant, 0, [\ ]\rangle$ in which all the theorems of $KP + D_1$ are true, while $[\psi]_0 = 0$. Following Gabbay, let $\Sigma$ be the closure of the set of subformulas of $\psi$ under the connectives $\to$, &, and $\neg$; and define a relation $R$ on $A$ by setting $aRb$ iff for each $\gamma \in \Sigma$, $[\gamma]_b = 1$ whenever $[\gamma]_a = 1$. Let $S = A/{\sim}$ where $\sim$ is the equivalence relation generated by $R$, and remark, in virtue of results of Diego and McKay (Theorem 16 of [1]), that $S$ is finite (because $\Sigma$ contains only finitely many nonequivalent formulas over $I$). Gabbay shows that the frame $\langle S, R, \tilde{0}\rangle$ is cohesive and that $\psi$ is not true at $\tilde{0}$. We prove now that this frame is binary as well. If not, there are $\tilde{a}, \tilde{b}, \tilde{c}, d \in A$ such that $\tilde{a}, \tilde{b}$, and $\tilde{c}$ are distinct immediate successors of $\tilde{d}$. Without loss of generality, suppose that these are the only immediate successors of $d$, and choose $\alpha, \beta, \gamma \in \Sigma$ such that $[\alpha]_a = 1$, $[\alpha]_b = [\alpha]_c = 0$; $[\beta]_b = 1$, $[\beta]_a = [\beta]_c = 0$; and $[\gamma]_c = 1$, $[\gamma]_a = [\gamma]_b = 0$.

**Claim** $[(\alpha \to (\beta \vee \gamma)) \to (\beta \vee \gamma)]_d = 1$.

If not, there exists $e \in A$, $d \leqslant e$, for which $[\alpha \to (\beta \vee \gamma)]_e = 1$ and $[\beta]_e = [\gamma]_e = 0$. From the second part of this, it follows that $\tilde{a}R\tilde{e}$; and, because $[\alpha]_a = 1$, we have $[\alpha]_e = 1$. Hence, $[\beta \vee \gamma]_e = 1$, a contradiction.

By symmetry, the other conjuncts of the antecedent of the instance of $D_1$ gotten by substituting $\alpha$, $\beta$, and $\gamma$ for $p$, $q$, and $r$ also hold at $d$. But now, since this instance must be true at $d$, we have that $[\alpha \vee \beta \vee \gamma]_d = 1$, a contradiction. Finally, since, as is readily seen, each finite cohesive binary frame is $(KP + D_1)$-valid, the proof of the theorem is complete.

If $\mathfrak{A} = \langle A, \leqslant, 0\rangle$ is a Kripke frame, let us call $a, b \in A$ disjoint if $\{a\}^+$ and $\{b\}^+$ are disjoint sets. Let $\alpha$ be the formula

$$(\neg p \rightarrow \neg(q \,\&\, r)) \vee (\neg q \rightarrow \neg(p \,\&\, r)) \vee (\neg r \rightarrow \neg(p \,\&\, q)).$$

It is easily verified that a necessary (and sufficient) condition for $\mathfrak{A}$ to be $\alpha$-invalid is that $A$ contain three (pairwise) disjoint elements.

**Lemma** *$\alpha$ is a theorem of $KP + D_1$.*

*Proof:* Suppose, in order to reach a contradiction, that $\mathfrak{A}$ is a finite cohesive binary Kripke frame containing three disjoint elements $a_1$, $a_2$, and $a_3$. Let $B_i = \{a \in A | a \not\leqslant a_i\}$ for $i = 1, 2, 3$, and define an assignment in $\mathfrak{A}$ as follows: $[p]_a = 1$ iff $a \in B_3$, $[q]_a = 1$ iff $a \in B_2$, and $[r]_a = 1$ iff $a \in B_1$. (Let [ ] be arbitrary on propositional variables other that $p$, $q$, and $r$.) We next establish:

**Claim** For each $a \in A$, $[q \vee r]_a = 1$ whenever $[p \rightarrow (q \vee r)]_a = 1$.

Suppose that $[p \rightarrow (q \vee r)]_a = 1$. If $a \in B_1 \cup B_2$, $[q \vee r]_a = 1$; and, if $a \in B_3$, $[p]_a = 1$, and our assumption implies that $[q \vee r]_a = 1$. The remaining case to consider is that in which $a \notin B_1 \cup B_2 \cup B_3$. Here we have $a \leqslant a_i$, $i = 1, 2$, and 3. We show next that $A$ must contain an element $c$ such that $a \leqslant c$ and $c \in B_3 - (B_1 \cup B_2)$. Since $\mathfrak{A}$ is cohesive and finite, the set $\{a\}^+ - \{a_3\}^{+-}$ must contain a smallest element $c$, and, since $a_1$, $a_2$ belong to this set, $c$ is of the required sort. But now, because $a \leqslant c$, $[p \rightarrow (q \vee r)]_c = 1$, which implies that $[q \vee r]_c = 1$. This requires that $c \in B_1 \cup B_2$, a contradiction; and the claim is proved.

It is immediate from the claim that $(p \rightarrow (q \vee r)) \rightarrow (q \vee r)$ is true everywhere in $\mathfrak{A}$, and, by symmetry, the antecedent of $D_1$'s characteristic axiom must also be true everywhere in $\mathfrak{A}$. But $[p \vee q \vee r]_0 = 0$, contradicting the $D_1$-validity of $\mathfrak{A}$.

**Theorem 2** *There is no maximal propositional logic having the disjunction property.*

*Proof:* Any such logic $M$ would have to have as theorems all the theorems of $KP + D_1$. But $\alpha$ is such a theorem none of whose disjuncts is classically valid. $M$ would then have a classically invalid formula among its theorems, implying that it is inconsistent.

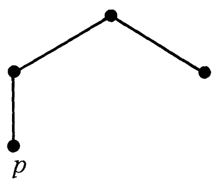
It should be remarked that there are maximal propositional logics having the disjunction property in a weaker sense of 'maximal'. Specifically, if we call a propositional logic weakly maximal (with respect to the disjunction property) when it has this property but no logic properly containing it does, we then have:

**Proposition** *Every propositional logic having the disjunction property has a weakly maximal extension.*

*Proof:* Immediate from Zorn's Lemma, noting that the union of any chain of logics, each having the disjunction property, is itself such a logic.

It would perhaps be of some interest to determine which intermediate logics are weakly maximal. Concerning this question, it may be noted that $KP$ is not weakly maximal since it is properly contained in the logic of finite problems of Medvedev [6], which has the disjunction property. That this inclusion is proper follows from the fact that an axiom due to D. Scott

$((\neg\neg p \rightarrow p) \rightarrow (\neg p \vee \neg\neg p)) \rightarrow (\neg p \vee \neg\neg p)$ is a theorem of Medvedev's logic (see [6]) while not being one of *KP*. The last part of this may be seen by noting that the axiom is refuted in the model




whose frame is *KP*-valid. I do not know whether either $D_1$ or Medvedev's logic is weakly maximal.

## REFERENCES

[1] Gabbay, D., "The decidability of the Kreisel-Putnam system," *The Journal of Symbolic Logic,* vol. 35 (1970), pp. 431-437.

[2] Gabbay, D. and D. H. J. DeJongh, "A sequence of decidable finitely axiomatizable intermediate logics with the disjunction property," *The Journal of Symbolic Logic,* vol. 39 (1974), pp. 67-78.

[3] Kreisel, G. and H. Putnam, "Unableitbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül," *Archiv für mathematische Logik und Grundlagenforschung,* vol. 3 (1957), pp. 74-78.

[4] Kripke, S., "Semantic analysis for intuitionist logic," in *Formal Systems and Recursive Functions,* eds., J. Crossley and M. Dummett, North-Holland, Amsterdam, 1965.

[5] Łukasiewicz, J., "On the intuitionistic theory of deduction," *Indagationes,* vol. 14 (1952), pp. 202-212.

[6] Medvedev, Ju.T., "Finite problems," *Soviet Math. Doklady,* vol. 3 (1962), pp. 227-230.

*Department of Philosophy*
*University of Washington*
*Seattle, Washington 98105*