Notre Dame Journal of Formal Logic
Volume 22, Number 3, July 1981

Equivalents of a Weak Axiom of Choice

MELVEN KROM

Some mathematical theorems that have informal proofs involving choices can be proved with at most only a very weak version of the axiom of choice (e.g., The Ball-Game Theorem [12]). In many of these cases the axiom of choice for countable families of finite sets will suffice. This weak axiom of choice is denoted here by AC^f_ω. As an aid in deciding for questionable cases whether any axiom of choice is needed, it may be helpful to note different equivalents of AC^f_ω. Two important combinatorial theorems, Ramsey's theorem and König's infinity lemma, are known to be equivalent to AC^f_ω ([8]; [1], p. 203; [7], p. 298; and [6], pp. 105-106). Here a topological theorem is established as another equivalent of AC^f_ω. Also it is shown that the stronger axiom of choice for arbitrary families of finite sets (to be denoted here by AC^f) is not equivalent to the corresponding stronger topological statement.

For these observations it is assumed that the Zermelo-Fraenkel set theory, ZF, is consistent. Recall that the axiom of choice for finite collections of sets is a theorem of ZF ([7], p. 160) while the slightly stronger AC^f_ω is independent of ZF ([7], p. 167; and [2]). The Tychonoff theorem, which says that the product of a family of compact topological spaces is compact in the product topology, is equivalent to the full axiom of choice, while the Tychonoff theorem limited to Hausdorff spaces is equivalent to the weaker prime ideal theorem ([5]; [10], fn 5; [4], pp. 26-27; and [9]). Let TYC^f_ω denote the following still more limited version of the Tychonoff theorem: "For any countable family of finite sets each regarded as a topological space with the discrete topology, the product of the family is compact in the product topology". Let TYC^f be the statement obtained from TYC^f_ω by omitting the condition that the family is countable.

Received September 3, 1980

Theorem TYC^f_ω is equivalent to AC^f_ω, TYC^f implies AC^f, and TYC^f is independent of AC^f.

Proof: The implications $TYC^f_\omega \Rightarrow AC^f_\omega$ and $TYC^f \Rightarrow AC^f$ follow immediately by specializing a known proof that the general Tychonoff theorem implies the full axiom of choice ([5]; [10], fn 5; and [4], p. 26, problem 8). For each member of a given family of nonempty sets for which a choice function is to be established this proof considers a topological space formed by adjoining one new element and letting all finite subsets of the resulting set be closed. Thus for any family of nonempty finite sets a choice function is obtained using the Tychonoff theorem for a product of finite discrete topological spaces and it is a countable product when the given family of finite sets is countable.

Known facts that show TYC^f is independent of AC^f are: TYC^f implies the prime ideal theorem ([4], p. 27; [9]; and [11]) the prime ideal theorem implies the ordering principle ([4], pp. 16 and 19) and the ordering principle is independent of AC^f ([4], p. 104).

König's infinity lemma, which says that any infinite finitary tree has an infinite branch, is equivalent to AC^f_ω ([1], p. 203; [7], p. 298; and [6], pp. 105-106). So the proof, sketched below (cf. the proof that ${}^\omega 2$ is compact: [3], p. 18), of TYC^f_ω based on König's lemma shows that AC^f_ω implies TYC^f_ω and completes a proof of the above theorem. Let $\{X_i | i \in \omega\}$ be a countable family of finite topological spaces each with the discrete topology and let $Y = \prod_{i \in \omega} X_i$ be their topological product. For $n \in \omega$ let $Z_n = \prod_{i=0}^{n} X_i$ and for $\sigma \in Z_n$ let $\mathscr{B}(\sigma) = \{f \in Y | f(i) = \sigma(i) \text{ for } 0 \leqslant i \leqslant n\}$. Then, for any $n \in \omega$, $\{\mathscr{B}(\sigma) | \sigma \in Z_n\}$ is a finite open cover of Y. König's lemma will be used to show that for any open cover $\mathscr{C}$ of Y there is $n \in \omega$ such that $\{\mathscr{B}(\sigma) | \sigma \in Z_n\}$ is a finite refinement of $\mathscr{C}$. Thus any open cover has a finite subcover. Assume that $\mathscr{C}$ is an open cover of Y and suppose for any $n \in \omega$ there is some $\sigma \in Z_n$ such that $\mathscr{B}(\sigma)$ is not a subset of any member of $\mathscr{C}$. Then $\{\sigma | \sigma \in Z_n$ for some $n \in \omega$ and $\mathscr{B}(\sigma) \nsubseteq C$ for any $C \in \mathscr{C}\}$ forms an infinite finitary tree consisting of finite sequences with order determined by sequence extension. The infinite branch of this tree ensured by König's lemma shows that there is an element of Y not covered by $\mathscr{C}$.

REFERENCES

[1] Drake, F. R., *Set Theory, An Introduction to Large Cardinals*, North-Holland Publishing Co., Amsterdam, 1974.

[2] Feferman, S., "Some applications of the notions of forcing and generic sets," *Fundamenta Mathematica*, vol. 56 (1965), pp. 325-345.

[3] Hinman, P. G., *Recursion-Theoretic Hierarchies*, Springer-Verlag, Berlin, 1978.

[4] Jech, T. J., *The Axiom of Choice*, North-Holland Publishing Co., Amsterdam, 1973.

[5] Kelley, J. L., "The Tychonoff product theorem implies the axiom of choice," *Fundamenta Mathematica*, vol. 37 (1950), pp. 75-76.

[6] Kuratowski, K. and A. Mostowski, *Set Theory*, North-Holland Publishing Co., Amsterdam, 1968.

[7] Levy, A., *Basic Set Theory*, Springer-Verlag, Berlin, 1979.

[8] Lolli, G., "On Ramsey's theorem and the axiom of choice," *Notre Dame Journal of Formal Logic*, vol. 18 (1977), pp. 599-601.

[9] Loś, J. and C. Ryll-Nardzewski, "Effectiveness of the representation theory for Boolean algebras," *Fundamenta Mathematica*, vol. 41 (1955), pp. 49-56.

[10] Loś, J. and C. Ryll-Nardzewski, "On the application of Tychonoff's theorem in mathematical proofs," *Fundamenta Mathematica*, vol. 38 (1951), pp. 233-237.

[11] Mycielski, J., "Two remarks on Tychonoff's product theorem," *Bulletin de L'Academie Polonaise des Sciences*, Series des sciences mathematique, astronomiques et physiques, vol. 12 (1964), pp. 439-441.

[12] Smullyan, R. M., "Trees and ball games," *Annals of the New York Academy of Sciences*, vol. 321 (1979), pp. 86-90.

University of California
Davis, California