The Church-Rosser Theorem for the
Typed λ-Calculus with
Surjective Pairing

GARREL POTTINGER

Introduction   The Church-Rosser theorem will be proved for a system obtained from the typed λ-calculus by adding pairing and projection constants. One of the rules for the pairing and projection constants implies that if t has a pair type, then there is a term formed by means of a pairing constant with which t is convertible. Consequently, following the usage of [1], the pairing considered is called 'surjective'.

Klop [2] showed that the Church-Rosser theorem fails for the type-free analog of the calculus considered here and for several simplified versions of that calculus. The result proved here together with the counterexamples of [2] furnish a demonstration that there is a difference in the behavior of typed and type-free combinatory systems with respect to the Church-Rosser property.

The methods of this paper are constructive. According to the referee, an unpublished paper of Girard's contains a proof of the result established here, but Girard's proof is not constructive.[1]

The proof given here is a simple extension of the one in [3]. In contrast with [3], detailed arguments by cases are omitted, and a notation (largely borrowed from [2]) which allows for diagrammatic presentation of arguments about reducibility is employed. Also, terms which are the same up to alphabetic change of bound variables are identified. Identity is expressed by '='. A couple of minor errors in [3] will be noted along the way.

1 The calculus   *Terms* are built up from λ, parentheses, variables (denoted by 'x'), denumerably many pairing constants (denoted by 'P'), denumerably many left projection constants (denoted by 'L'), and denumerably many right projection constants (denoted by 'R'). Propositional formulas built up by means

of $\supset$ and $\&$ are used as type symbols. $A \supset B$ represents a function type, and $A \& B$ represents a pair type. In restoring omitted parentheses, $\supset$ gets a larger scope than $\&$, a dot indicates a left parenthesis with its mate as far to the right as possible, and residual ambiguities are resolved by associating left.

The type symbol assignment T is derived from the given type symbol assignment T_0 according to the rules shown below. T_0 maps the set of variables onto the set of type symbols and assigns denumerably many variables the type symbol A, for all A. T_0 also maps the set of pairing constants 1-1 onto the set of type symbols of the form shown below, and similarly for the projection constants.

$$T(x) = T_0(x)$$
$$T(P) = T_0(P) = A \supset . B \supset A \& B$$
$$T(L) = T_0(L) = A \& B \supset A$$
$$T(R) = T_0(R) = A \& B \supset B$$
$$\frac{T(t) = A \supset B \quad T(u) = A}{T(tu) = B} \qquad \frac{T(t) = B}{T(\lambda xt) = T_0(x) \supset B}$$

Only terms to which T assigns a type symbol are considered from now on.

2 Redexes, contracta, relations, normality $[u/x/t]$ is to be the result of replacing the free occurrences of x in t by occurrences of u. $(\lambda xt)u$ is a β *redex*, and $[u/x/t]$ is its *contractum* if the substitution does not involve variable capture. $\lambda x.tx$ is an η *redex* if x is not free in t, and t is its *contractum*. $L(Ptu)$ is an *LP redex*, and t is its *contractum*. $R(Ptu)$ is an *RP redex*, and u is its *contractum*. $P(Lt)(Rt)$ is a *PLR redex*, and t is its *contractum*. *PLR* contraction makes pairing surjective, in the sense explained in the introduction.

$t \to u$ holds iff u is a result of replacing an occurrence of a redex in t by an occurrence of its contractum. $\twoheadrightarrow$ *(reducibility)* is to be the smallest reflexive, transitive relation containing $\to$, and $\twoheadleftrightarrow$ *(convertibility)* is to be the smallest equivalence relation containing reducibility.

t is *normal* iff no redex occurs in t.

3 Theorems   The following results will be proved.

Theorem 3.1 [normalization]   *There is a normal u such that $t \twoheadrightarrow u$.*

Theorem 3.2 [Church-Rosser]   (1) *If $t \twoheadleftrightarrow u$, then there is a v such that $t \twoheadrightarrow v$ and $u \twoheadrightarrow v$.* (2) *If $t \twoheadrightarrow u$, $t \twoheadrightarrow v$, and u and v are normal, then $u = v$.*

It is easy to see that Church-Rosser (1) and Church-Rosser (2) are equivalent in the presence of the normalization theorem. (1) implies (2) is trivial. To prove (2) implies (1) in the presence of the normalization theorem, use an induction on the number of expansions and contractions involved in converting t to u. Church-Rosser (2) will be proved, after proving a normalization theorem somewhat stronger than Theorem 3.1.

4 Predicative contractions and reductions   The main idea of the proof is to show that in considering normalization and Church-Rosser (2) one can replace reducibility by the relation of predicative reducibility, which is now defined.

$c(A)$ = the number of occurrences of $\supset$ and & in A, and $c(t)$ = the number of occurrences of symbols in t. For the redex t, define $\mathscr{Y}_1(t)$ as follows. If t is an η or PLR redex, then $\mathscr{Y}_1(t) = 0$. If $t = t_1 t_2$ is a β redex, then $\mathscr{Y}_1(t) = c(T(t_1))$. If $t = t_1 t_2$ is an LP or RP redex, then $\mathscr{Y}_1(t) = c(T(t_2))$.

t is a *predicative* redex iff either t is an η, LP, RP, or PLR redex, or $t = (\lambda x t_1) t_2$ and for every redex u occurring in t_2, $\mathscr{Y}_1(u) < \mathscr{Y}_1(t)$. $\underset{P}{\rightarrow}$ is to be the restriction of $\rightarrow$ to cases where a predicative redex is contracted, and $\underset{P}{\twoheadrightarrow}$ is to be the restriction of $\twoheadrightarrow$ to cases where only predicative redexes are contracted.

5 Normalization

Where $1 \leqslant n$, let $\mathscr{R}(n,t)$ be the number of occurrences of redexes u in t such that $\mathscr{Y}_1(u) = n$. Consider a term t, let $0 < n_1 < \ldots < n_m$ be the natural numbers such that $\mathscr{R}(n_1,t), \ldots, \mathscr{R}(n_m,t)$ are not 0, and define:[2]

$$\mathscr{Y}(t) = \begin{cases} \omega^{n_m}\mathscr{R}(n_m,t) + \ldots + \omega^{n_1}\mathscr{R}(n_1,t) + c(t), & \text{if } t \text{ is not normal} \\ 0, & \text{if } t \text{ is normal} \end{cases}$$

Lemma 5.1   *If $t \underset{P}{\rightarrow} u$, then $\mathscr{Y}(u) < \mathscr{Y}(t)$.*

Proof: Check the cases.

Corollary 5.2   *There is a normal u such that $t \underset{P}{\twoheadrightarrow} u$.*

Proof: Immediate from Lemma 5.1.

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Lemma 6.1   *If $t \underset{P}{\rightarrow} u_1$ and $t \underset{P}{\rightarrow} u_2$, then there is a v such that $u_1 \underset{P}{\twoheadrightarrow} v$ and $u_2 \underset{P}{\twoheadrightarrow} v$.*

Proof: Check the cases.[3]

Corollary 6.2   *If $t \underset{P}{\twoheadrightarrow} u_1$, $t \underset{P}{\twoheadrightarrow} u_2$, and u_1 and u_2 are normal, then $u_1 = u_2$.*

Proof: By induction on the maximum number of contractions in a predicative reduction of t to a normal term. If t is normal, there is nothing to prove. Suppose t is not normal. The following picture shows how the induction works.
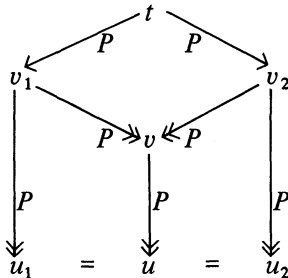
Lemma 6.1 yields the diamond-shaped part of the diagram. Corollary 5.2 provides a predicative reduction of v to a normal u, and the inductive hypothesis yields the identities at the bottom of the diagram.
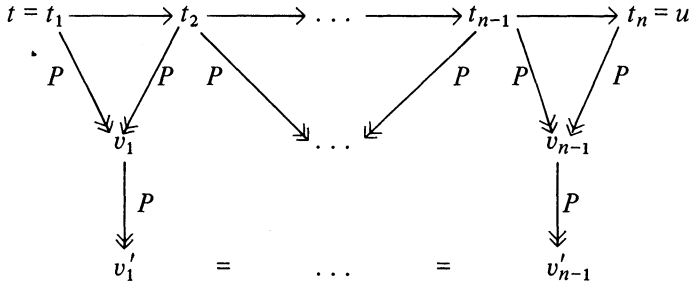
It will now be shown that Church-Rosser (2) can be obtained from Corollary 6.2.

Lemma 6.3   *If $t \to u$, then there is a v such that $t \underset{P}{\twoheadrightarrow} v$ and $u \underset{P}{\twoheadrightarrow} v$.*

Proof: It suffices to consider the case where t is a nonpredicative β redex and u is its contractum. Let $t = (\lambda x t_1)t_2$. Then $u = [t_2/x/t_1]$. Let t_2' be a normal term such that $t_2 \underset{P}{\twoheadrightarrow} t_2'$. Then $(\lambda x t_1)t_2 \underset{P}{\twoheadrightarrow} (\lambda x t_1)t_2' \underset{P}{\twoheadrightarrow} [t_2'/x/t_1]$, and $[t_2/x/t_1] \underset{P}{\twoheadrightarrow} [t_2'/x/t_1]$.

Lemma 6.4   *If $t \twoheadrightarrow u$ and u is normal, then $t \underset{P}{\twoheadrightarrow} u$.*

Proof: The preceding results yield the following picture:

$$t = t_1 \longrightarrow t_2 \longrightarrow \cdots \longrightarrow t_{n-1} \longrightarrow t_n = u$$

with predicative reductions to $v_1, \ldots, v_{n-1}$ and predicative reductions P from each v_i to v_i', with identities $v_1' = \cdots = v_{n-1}'$ shown at the bottom.

Lemma 6.3 yields the predicative reductions of t_i and t_{i+1} to v_i, and Corollary 5.2 provides a predicative reduction of each v_i to a normal v_i'. Corollary 6.2 yields the identities shown at the bottom. Since u is normal, the path leading from t_n to v_{n-1}' *via* v_{n-1} involves no contractions. Hence, the path leading from t_1 to t_n *via* $v_1, v_1', \ldots, v_{n-1}', v_{n-1}$ depicts a predicative reduction of t to u.

Church-Rosser (2) is an immediate corollary of the preceding lemma and Corollary 6.2.

NOTES

1. I am also indebted to the referee for the reference to [1].

2. The corresponding definition of [3], p. 447, is defective and should be replaced by the one given here.

3. This is tedious, but not exactly simple. The following repairs are necessary in the proof of the corresponding lemma given in [3], pp. 448-450. In Case 2.2.1 there may be no redex occurrence of the sort one is advised to contract in order to obtain v, but α-convertibility will yield an appropriate v in such a situation. In Case 2.2.2.2.1 the possibility that $*t_2*$ is the same as $*\lambda x t^1*$ and is an occurrence of an η redex was overlooked. Fortunately, if this happens, then $u_1 = u_2$ and this suffices. In the first line of Case 2.2.2.2.1 the asterisks should be deleted, and on p. 450, line 2, 'x' should be replaced by 'y'.

REFERENCES

[1] Barendreght, H., "Pairing without conventional restraints," *Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik*, vol. 20 (1974), pp. 289-306.

[2] Klop, J. W., *Combinatory Redution Systems*, doctoral dissertation, Rijksuniversiteit te Utrecht, 1980.

[3] Pottinger, G., "Proofs of the normalization and Church-Rosser theorems for the typed λ-calculus," *Notre Dame Journal of Formal Logic*, vol. 19 (1978), pp. 445-451.

Department of History and Philosophy
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213