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A Predicate Logic Based on Indefinite Description and Two Notions of Identity

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We present a formal[1] system of free predicate logic based on three primitives: indefinite description (using a variant of the Hilbert 'ϵ'-symbol) and two notions of identity, one reflexive and the other nonreflexive. This logic is intended to be of practical use in mathematics. Sample applications are given in Sections 13 and 16.

We assume an axiomatic development of the usual sentence logic. For the predicate logic we list two primitive formulas

(1) $(x \equiv y)$ identity
(2) $(x = y)$ equality

and one primitive term

(3) an x $\underline{\mathrm{p}}'x$ indefinite description.

By (1) we mean identity in the sense of Leibniz, i.e., 'x' and 'y' may be substituted one for the other in any formula without affecting its truth.[2] By (2) we mean identity in the sense of Peirce, i.e., there is some object z such that 'x' names z and 'y' names z.[3] Thus neither (1) nor (2) is a statement about objects x and y but is instead a statement about the variables (or what would in the general case be terms) 'x' and 'y'. More precisely (1) says that 'x' and 'y' are synonymous in the strict sense that if T is a formula obtained from a formula S by replacing 'x' by 'y' then one is entitled to infer T from S. (1) is therefore a syntactical statement which satisfies Carnap's definition of a pseudo-object sentence. Carnap gave, as an example of a pseudo-object sentence, the sentence "Five is not a thing but a number," which he translated as follows: " 'Five' is not a thing-word but a number-word."[4] In most cases (1) holds if and only if (2) holds. The two meanings are in conflict only in the

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case that 'x' and 'y' fail to denote. Suppose that 'x' for the moment does not denote. In this case '$(x \equiv x)$' is true while '$(x = x)$' is false.[5] For a fuller discussion of our interpretations of (1) and (2) see [13].

Terms which do not denote arise in the use of indefinite descriptions. By (3) we mean an object z such that $\underline{p}'z$, if such a z exists. Otherwise (3) is a term with no denotation.

We use the formal inferential language of Morse [11] with the following alterations.

First we maintain the customary distinction between terms and formulas. As a result, only terms may replace schematic expressions such as '$\underline{u}\,x$', '$\underline{u}'xy$', '$\underline{u}''xyz$', '$\underline{v}x$', '$\underline{v}'xy$', etc. Similarly, only formulas may replace schematic expressions such as '$\underline{p}$', '$\underline{q}$', '$\underline{r}$', '$\underline{p}'x$', '$\underline{q}'x$', '$\underline{r}'x$', '$\underline{p}''xy$', etc.

Second, a strengthened form of Morse's rule of schematic substitution is used. It does not require that the schematic expressions being replaced all contain the same string of variables.[6]

Finally, the rule of universalization has been dropped in view of Theorems 6.3 and 6.4.

In 1.1 below, the notion of existence is defined by letting 'ex x' (to be read: x exists) mean $(x = x)$, i.e., there is an object z such that 'x' denotes z. We are then led naturally to Definition 1.2 in relation to which substitution for '$\underline{p}'x$' is much simpler than in Hilbert's[7]

$$(\vee x \underline{p}'x \leftrightarrow \underline{p}' \text{ an } x\underline{p}'x).$$

In 1.4 we define the formula constant '$\mathcal{E}$' to mean that something exists. $\mathcal{E}$ will be a theorem in any mathematical theory which is based on this logic and which axiomatically guarantees the existence of objects. Section 10 deals with the results of assuming $\mathcal{E}$ as a hypothesis. If we were to assume $\mathcal{E}$ as an axiom, the resulting predicate logic would be free (allowing nondenoting terms) but not universal (valid in every domain).

In 1.5-1.7 we define the definite description and two related uniqueness quantifiers.

In Section 17 we introduce forms which are useful in the application of descriptions to the construction of definitions in mathematics.

Sections 1 through 6 constitute the foundations of the formal system.

1 Definitions

1.1 $(\text{ex}\, x \leftrightarrow (x = x))$

1.2 $(\vee x\underline{p}'x \leftrightarrow \text{ex an } x\underline{p}'x)$

1.3 $(\wedge x\underline{p}'x \leftrightarrow \sim\vee x \sim \underline{p}'x)$

1.4 $(\mathcal{E} \leftrightarrow \vee x \text{ ex}\, x)$

1.5 $(\text{the } x\underline{p}'x \equiv \text{an } y \wedge x(\underline{p}'x \leftrightarrow x = y))$

1.6 $(\text{One } x\underline{p}'x \leftrightarrow \text{ex the } x\underline{p}'x)$

1.7 $(\text{Unq } x\underline{p}'x \leftrightarrow (\vee x\underline{p}'x \rightarrow \text{One } x\underline{p}'x))$

2 Axioms for identity and equality

2.1 $((x \equiv y) \rightarrow (\underline{p}'x \rightarrow \underline{p}'y))$

2.2 $((x = y) \rightarrow (x \equiv y))$

3 Axioms for quantification

3.1 $((y = \text{an } x\underline{p}'x) \rightarrow \underline{p}'y)$
3.2 $(\text{ex } y \rightarrow (\underline{p}'y \rightarrow \bigvee x\underline{p}'x))$

4 Completeness axioms

4.1 $(\bigwedge x(\underline{p}'x \leftrightarrow \underline{q}'x) \rightarrow (\text{an } x\underline{p}'x \equiv \text{an } x\underline{q}'x))$
4.2 $(y \equiv \text{an } x\ (x = y))$

5 Theorems of identity and equality

5.1 $(x \equiv y \rightarrow y \equiv y)$
5.2 $(x \equiv y \rightarrow (y \equiv z \rightarrow x \equiv z))$
5.3 $(x \equiv y \rightarrow y \equiv x)$
5.4 $(x \equiv y \rightarrow \underline{p}'x \leftrightarrow \underline{p}'y)$
5.5 $(x = y \rightarrow \underline{p}'x \leftrightarrow \underline{p}'y)$
5.6 $(x = y \rightarrow y = y)$
5.7 $(x = y \rightarrow (y = z \rightarrow x = z))$
5.8 $(x = y \rightarrow y = x)$
5.9 $(x = y \leftrightarrow \text{ex } x \wedge \text{ex } y \wedge x \equiv y)$

Proof:

$$\begin{aligned} &(x = y \rightarrow y = y && [5.6]\\ &\qquad \rightarrow \text{ex } y) && [1.1]\\ &(x = y \rightarrow y = x && [5.8]\\ &\qquad \rightarrow \text{ex } x) && \\ &(x = y \rightarrow x \equiv y). && [2.2] \end{aligned}$$

Thus

$$(x = y \rightarrow \text{ex } x \wedge \text{ex } y \wedge x \equiv y).$$

Conversely,

$$\begin{aligned} &(x \equiv y \wedge \text{ex } x \rightarrow \text{ex } x \wedge (z = x \rightarrow z = y)) && [2.1]\\ &(x \equiv y \wedge \text{ex } x \rightarrow x = x \wedge (x = x \rightarrow x = y) && \\ &\qquad\qquad \rightarrow x = y). && \end{aligned}$$

6 Basic lemmas

6.1 $(\text{ex } y \wedge \bigwedge x\underline{p}'x \rightarrow \underline{p}'y)$

Proof:

$$\begin{aligned} &(\text{ex } y \rightarrow (\sim\underline{p}'y \rightarrow \bigvee x \sim\underline{p}'x) && [3.2]\\ &\qquad \rightarrow (\sim\bigvee x \sim\underline{p}'x \rightarrow \underline{p}'y) && \\ &\qquad \rightarrow (\bigwedge x\underline{p}'x \rightarrow \underline{p}'y)). && [1.3] \end{aligned}$$

6.2 $(\bigvee x\underline{p}'x \leftrightarrow \text{an } x\underline{p}'x = \text{an } x\underline{p}'x)$
6.3 $(\underline{p}'y \rightarrow \sim(y = \text{an } x \sim\underline{p}'x))$[8]

Proof:

$$(y = \text{an } x \sim\underline{p}'x \rightarrow \sim\underline{p}'y). \qquad [3.1]$$

6.4 $(\sim(\text{an}\, x \sim\underline{\text{p}}'x = \text{an}\, x \sim\underline{\text{p}}'x) \to \bigwedge x\underline{\text{p}}'x)$

Proof:

$$(\sim(\text{an}\, x \sim\underline{\text{p}}'x = \text{an}\, x \sim\underline{\text{p}}'x) \to \sim\bigvee x \sim\underline{\text{p}}'x \qquad [6.2]$$
$$\to \bigwedge x\underline{\text{p}}'x). \qquad [1.3]$$

Theorems 6.3 and 6.4 provide the basis for the universalization of theorems. The proof of 6.5 illustrates their use.

6.5 $\bigwedge x(\underline{\text{p}}'x \to \underline{\text{p}}'x)$

Proof:

$$((\underline{\text{p}}'y \to \underline{\text{p}}'y) \to \sim(y = \text{an}\, x \sim(\underline{\text{p}}'x \to \underline{\text{p}}'x))) \qquad [6.3]$$
$$\sim(y = \text{an}\, x \sim(\underline{\text{p}}'x \to \underline{\text{p}}'x))$$
$$\sim(\text{an}\, x \sim(\underline{\text{p}}'x \to \underline{\text{p}}'x) = \text{an}\, x \sim(\underline{\text{p}}'x \to \underline{\text{p}}'x))$$
$$\bigwedge x(\underline{\text{p}}'x \to \underline{\text{p}}'x). \qquad [6.4]$$

Sections 7 through 11 deal with quantification.

7 *Distributive properties*

7.1 $(\bigwedge x(\underline{\text{p}}'x \to \underline{\text{q}}'x) \to (\bigvee x\underline{\text{p}}'x \to \bigvee x\underline{\text{q}}'x))$

Proof:

$$(\bigwedge x(\underline{\text{p}}'x \to \underline{\text{q}}'x) \wedge \bigvee x\underline{\text{p}}'x$$
$$\to \bigwedge x(\underline{\text{p}}'x \to \underline{\text{q}}'x) \wedge \text{an}\, x\underline{\text{p}}'x = \text{an}\, x\underline{\text{p}}'x \qquad [6.2]$$
$$\to \bigwedge x(\underline{\text{p}}'x \to \underline{\text{q}}'x) \wedge \underline{\text{p}}'\,\text{an}\, x\underline{\text{p}}'x \wedge \text{ex an}\, x\underline{\text{p}}'x \qquad [3.1, 1.1]$$
$$\to (\underline{\text{p}}'\,\text{an}\, x\underline{\text{p}}'x \to \underline{\text{q}}'\,\text{an}\, x\underline{\text{p}}'x) \wedge \underline{\text{p}}'\,\text{an}\, x\underline{\text{p}}'x \wedge \text{ex an}\, x\underline{\text{p}}'x \qquad [6.1]$$
$$\to \underline{\text{q}}'\,\text{an}\, x\underline{\text{p}}'x \wedge \text{ex an}\, x\underline{\text{p}}'x$$
$$\to \bigvee x\underline{\text{q}}'x). \qquad [3.2]$$

The remaining proofs, which we omit, are standard.

7.2 $(\bigwedge x(\underline{\text{p}}'x \to \underline{\text{q}}'x) \to (\bigwedge x\underline{\text{p}}'x \to \bigwedge x\underline{\text{q}}'x))$
7.3 $(\bigwedge x(\underline{\text{p}}'x \leftrightarrow \underline{\text{q}}'x) \to \bigvee x\underline{\text{p}}'x \leftrightarrow \bigvee x\underline{\text{q}}'x)$
7.4 $(\bigwedge x(\underline{\text{p}}'x \leftrightarrow \underline{\text{q}}'x) \to \bigwedge x\underline{\text{p}}'x \leftrightarrow \bigwedge x\underline{\text{q}}'x)$
7.5 $(\bigvee x\underline{\text{p}}'x \leftrightarrow \sim\bigwedge x \sim\underline{\text{p}}'x)$
7.6 $(\bigwedge x(\underline{\text{p}}'x \wedge \underline{\text{q}}'x) \leftrightarrow \bigwedge x\underline{\text{p}}'x \wedge \bigwedge x\underline{\text{q}}'x)$
7.7 $(\bigvee x(\underline{\text{p}}'x \vee \underline{\text{q}}'x) \leftrightarrow \bigvee x\underline{\text{p}}'x \vee \bigvee x\underline{\text{q}}'x)$

8 *Behavior with respect to a constant*

8.1 $(\bigvee x\underline{\text{p}} \to \underline{\text{p}})$

Proof:

$$(\bigvee x\underline{\text{p}} \to \text{an}\, x\underline{\text{p}} = \text{an}\, x\underline{\text{p}} \qquad [6.2]$$
$$\to \underline{\text{p}}). \qquad [3.1]$$

The remaining proofs, which we omit, are standard.

8.2 $(\underline{\text{p}} \to \bigwedge x\underline{\text{p}})$
8.3 $(\bigvee x(\underline{\text{p}} \wedge \underline{\text{p}}'x) \leftrightarrow \underline{\text{p}} \wedge \bigvee x\underline{\text{p}}'x)$

8.4 $(\bigwedge x(\underline{p} \lor \underline{p}'x) \leftrightarrow \underline{p} \lor \bigwedge x\underline{p}'x)$
8.5 $(\bigvee x(\underline{p} \lor \underline{p}'x) \rightarrow \underline{p} \lor \bigvee x\underline{p}'x)$
8.6 $(\underline{p} \land \bigwedge x\underline{p}'x) \rightarrow (\bigwedge x(\underline{p} \land \underline{p}'x))$
8.7 $(\bigwedge x(\underline{p} \rightarrow \underline{p}'x) \leftrightarrow (\underline{p} \rightarrow \bigwedge x\underline{p}'x))$
8.8 $(\bigwedge x(\underline{p}'x \rightarrow \underline{p}) \leftrightarrow (\bigvee x\underline{p}'x \rightarrow \underline{p}))$
8.9 $(\bigvee x(\underline{p} \rightarrow \underline{p}'x) \rightarrow (\underline{p} \rightarrow \bigvee x\underline{p}'x))$
8.10 $(\bigvee x(\underline{p}'x \rightarrow \underline{p}) \rightarrow (\bigwedge x\underline{p}'x \rightarrow \underline{p}))$

9 Commutative properties

9.1 $(\bigvee x\bigvee y\underline{p}''xy \leftrightarrow \bigvee y\bigvee x\underline{p}''xy)$

Proof:

$(\bigvee x\bigvee y\underline{p}''xy$
$\rightarrow \text{an } x\bigvee y\underline{p}''xy = \text{an } x\bigvee y\underline{p}''xy$ [6.2]
$\rightarrow \bigvee y\underline{p}''\ \text{an } x\bigvee y\underline{p}''xyy \land \text{ex an } x\bigvee y\underline{p}''xy$ [3.1, 1.1]
$\rightarrow \bigvee y(\underline{p}''\ \text{an } x\bigvee y\underline{p}''xyy \land \text{ex an } x\bigvee y\underline{p}''xy)$ [8.3]
$\rightarrow \bigvee y\bigvee x\underline{p}''xy).$ [3.2, 7.1]

The converse follows by switching variables in the above result.

9.2 $(\bigwedge x\bigwedge y\underline{p}''xy \leftrightarrow \bigwedge y\bigwedge x\underline{p}''xy)$
9.3 $(\bigvee x\bigwedge y\underline{p}''xy \rightarrow \bigwedge y\bigvee x\underline{p}''xy)$

Proof:

$(\bigvee x\bigwedge y\underline{p}''xy$
$\rightarrow \text{an } x\bigwedge y\underline{p}''xy = \text{an } x\bigwedge y\underline{p}''xy$ [6.2]
$\rightarrow \bigwedge y\underline{p}''\ \text{an } x\bigwedge y\underline{p}''xyy \land \text{ex an } x\bigwedge y\underline{p}''xy$ [3.1, 1.1]
$\rightarrow \bigwedge y(\underline{p}''\ \text{an } x\bigwedge y\underline{p}''xyy \land \text{ex an } x\bigwedge y\underline{p}''xy)$ [8.6]
$\rightarrow \bigwedge y\bigvee x\underline{p}''xy).$ [3.2, 7.2]

10 Existence

10.1 $(\mathcal{E} \rightarrow (\bigwedge x\underline{p}'x \rightarrow \bigvee x\underline{p}'x))$

Proof:

$(\text{ex } x \rightarrow (\bigwedge y\underline{p}'y \rightarrow \underline{p}'x))$ [6.1]
$\bigwedge x(\text{ex } x \rightarrow (\bigwedge y\underline{p}'y \rightarrow \underline{p}'x))$ [6.3, 6.4]
$(\bigvee x\ \text{ex } x \rightarrow \bigvee x(\bigwedge y\underline{p}'y \rightarrow \underline{p}'x)$ [7.1]
$\qquad \rightarrow (\bigwedge y\underline{p}'y \rightarrow \bigvee x\underline{p}'x))$ [8.9]
$(\mathcal{E} \rightarrow (\bigwedge x\underline{p}'x \rightarrow \bigvee x\underline{p}'x)).$ [1.4]

Given a nonempty domain, Theorems 10.2 through 10.7 yield the expected converses for Theorems 8.1, 8.2, 8.5, 8.6, 8.9, and 8.10, respectively.

10.2 $(\mathcal{E} \rightarrow \bigvee x\underline{p} \leftrightarrow \underline{p})$
10.3 $(\mathcal{E} \rightarrow \bigwedge x\underline{p} \leftrightarrow \underline{p})$
10.4 $(\mathcal{E} \rightarrow \bigvee x(\underline{p} \lor \underline{p}'x) \leftrightarrow \underline{p} \lor \bigvee x\underline{p}'x)$
10.5 $(\mathcal{E} \rightarrow \bigwedge x(\underline{p} \land \underline{p}'x) \leftrightarrow \underline{p} \land \bigwedge x\underline{p}'x)$
10.6 $(\mathcal{E} \rightarrow \bigvee x(\underline{p} \rightarrow \underline{p}'x) \leftrightarrow (\underline{p} \rightarrow \bigvee x\underline{p}'x))$
10.7 $(\mathcal{E} \rightarrow \bigvee x(\underline{p}'x \rightarrow \underline{p}) \leftrightarrow (\bigwedge x\underline{p}'x \rightarrow \underline{p}))$
10.8 $(\text{ex } x \rightarrow \mathcal{E})$

Proof:

$$(\mathrm{ex}\ x \to \mathrm{ex}\ x \wedge \mathrm{ex}\ x$$
$$\to \bigvee x\ \mathrm{ex}\ x \qquad [3.2]$$
$$\to \mathscr{E}). \qquad [1.4]$$

10.9 $(x = y \to \mathscr{E})$

10.10 $(\bigvee x \underline{\mathrm{p}}'x \to \mathscr{E})$

11 Range of quantification

11.1 $\bigwedge x\ \mathrm{ex}\ x$

Proof:

$$(y = \mathrm{an}\ x \sim\mathrm{ex}\ x \to \mathrm{ex}\ y \wedge \sim\mathrm{ex}\ y) \qquad [5.9, 3.1]$$
$$\sim(y = \mathrm{an}\ x \sim\mathrm{ex}\ x)$$
$$\sim(\mathrm{an}\ x \sim\mathrm{ex}\ x = \mathrm{an}\ x \sim\mathrm{ex}\ x)$$
$$\bigwedge x\ \mathrm{ex}\ x. \qquad [6.4]$$

11.2 $\bigwedge x\ \mathscr{E}$

11.3 $(\bigvee x \underline{\mathrm{p}}'x \leftrightarrow \bigvee x(\mathrm{ex}\ x \wedge \underline{\mathrm{p}}'x) \leftrightarrow \bigvee x(\mathrm{ex}\ x \to \underline{\mathrm{p}}'x))$

11.4 $(\bigvee x \underline{\mathrm{p}}'x \leftrightarrow \bigvee x(\mathscr{E} \wedge \underline{\mathrm{p}}'x) \leftrightarrow \bigvee x(\mathscr{E} \to \underline{\mathrm{p}}'x))$

11.5 $(\bigwedge x \underline{\mathrm{p}}'x \leftrightarrow \bigwedge x(\mathrm{ex}\ x \wedge \underline{\mathrm{p}}'x) \leftrightarrow \bigwedge x(\mathrm{ex}\ x \to \underline{\mathrm{p}}'x))$

11.6 $(\bigwedge x \underline{\mathrm{p}}'x \leftrightarrow \bigwedge x(\mathscr{E} \wedge \underline{\mathrm{p}}'x) \leftrightarrow \bigwedge x(\mathscr{E} \to \underline{\mathrm{p}}'x))$

11.7 $(\mathrm{ex}\ y \leftrightarrow \bigvee x(x = y))$

Proof:

$$(\mathrm{ex}\ y \to \mathrm{ex}\ y \wedge y = y \qquad [1.1]$$
$$\to \bigvee x(x = y)). \qquad [3.2]$$

Conversely,

$$(\bigvee x(x = y) \to \mathrm{an}\ x(x = y) = \mathrm{an}\ x(x = y) \qquad [6.2]$$
$$\to \mathrm{an}\ x(x = y) = y \qquad [3.1]$$
$$\to \mathrm{ex}\ y). \qquad [5.9]$$

11.8 $(\bigvee x(x = y \wedge \underline{\mathrm{p}}'x) \leftrightarrow \mathrm{ex}\ y \wedge \underline{\mathrm{p}}'y)$

11.9 $(\bigwedge x(x = y \to \underline{\mathrm{p}}'x) \leftrightarrow (\mathrm{ex}\ y \to \underline{\mathrm{p}}'y))$

Sections 12 through 14 deal with descriptions and examples. 'One $x\underline{\mathrm{p}}'x$' is Morse's existence and uniqueness quantifier (cf. 13.8 and 13.3).

12 Descriptions

12.1 $(y = \mathrm{the}\ x\underline{\mathrm{p}}'x \leftrightarrow \mathrm{ex}\ y \wedge \bigwedge x(\underline{\mathrm{p}}'x \leftrightarrow x = y))$

12.2 $(y = \mathrm{the}\ x\underline{\mathrm{p}}'x \leftrightarrow \mathrm{ex}\ y \wedge \underline{\mathrm{p}}'y \wedge \bigwedge x(\underline{\mathrm{p}}'x \to x = y))$

12.3 $(y = \mathrm{the}\ x\underline{\mathrm{p}}'x \leftrightarrow \mathrm{ex}\ y \wedge \underline{\mathrm{p}}'y \wedge \mathrm{One}\ x\underline{\mathrm{p}}'x)$

12.4 $(y = \mathrm{the}\ x\underline{\mathrm{p}}'x \leftrightarrow \mathrm{ex}\ y \wedge \underline{\mathrm{p}}'y \wedge \mathrm{Unq}\ x\underline{\mathrm{p}}'x)$

12.5 $(y = \mathrm{the}\ x\underline{\mathrm{p}}'x \to y = \mathrm{an}\ x\underline{\mathrm{p}}'x)$

12.6 $(\mathrm{One}\ x\underline{\mathrm{p}}'x \wedge \bigwedge x(\underline{\mathrm{p}}'x \leftrightarrow \underline{\mathrm{q}}'x) \to \mathrm{the}\ x\underline{\mathrm{p}}'x = \mathrm{the}\ x\underline{\mathrm{q}}'x)$

12.7 $(y = \mathrm{the}\ x(\underline{\mathrm{p}}'x \vee \underline{\mathrm{q}}'x) \to y = \mathrm{the}\ x\underline{\mathrm{p}}'x \vee y = \mathrm{the}\ x\underline{\mathrm{q}}'x)$

12.8 $(z = \mathrm{the}\ x\bigvee y\underline{\mathrm{p}}''xy \to \bigvee y(z = \mathrm{the}\ x\underline{\mathrm{p}}''xy))$

13 Uniqueness

13.1 $(\text{One } x\underline{p}'x \leftrightarrow \bigvee y(y = \text{the } x\underline{p}'x))$

13.2 $(\text{One } x\underline{p}'x \leftrightarrow \bigvee y \bigwedge x(\underline{p}'x \leftrightarrow x = y))$

13.3 $(\text{Unq } x\underline{p}'x \leftrightarrow \bigwedge x \bigwedge y(\underline{p}'x \wedge \underline{p}'y \rightarrow x = y))$

13.4 $(\mathcal{E} \rightarrow \text{Unq } x\underline{p}'x \leftrightarrow \bigvee y \bigwedge x(\underline{p}'x \rightarrow x = y))$

13.5 $(\bigwedge x(\underline{p}'x \leftrightarrow \underline{q}'x) \rightarrow \text{One } x\underline{p}'x \leftrightarrow \text{One } x\underline{q}'x)$

13.6 $(\bigwedge x(\underline{p}'x \leftrightarrow \underline{q}'x) \rightarrow \text{Unq } x\underline{p}'x \leftrightarrow \text{Unq } x\underline{q}'x)$

13.7 $(\text{One } x\underline{p}'x \rightarrow \bigvee x\underline{p}'x)$

13.8 $(\text{One } x\underline{p}'x \leftrightarrow \bigvee x\underline{p}'x \wedge \text{Unq } x\underline{p}'x)$

13.9 $(\underline{p} \rightarrow \text{Unq } x\underline{p}'x \leftrightarrow \text{Unq } x(\underline{p} \wedge \underline{p}'x) \leftrightarrow \text{Unq } x(\underline{p} \rightarrow \underline{p}'x))$

13.10 $(\underline{p} \rightarrow \text{One } x\underline{p}'x \leftrightarrow \text{One } x(\underline{p} \wedge \underline{p}'x) \leftrightarrow \text{One } x(\underline{p} \rightarrow \underline{p}'x))$

14 Examples Formal descriptions are generally not used as a tool in making definitions in mathematics. This has resulted in many circumlocutions in descriptive definitions. We believe the following examples will demonstrate the ease and naturalness with which descriptions can be used. The reader will observe that it is exactly the notion of existence which enables descriptions to function so naturally.

We consider first the definition of limit from calculus. We do not specify any particular formalization of the real number system (e.g., as part of set theory); but, for this example, we use 'real is y' to assert that y is a real number. 14.1 and 14.2 are definitions.[9]

14.1 $((\underline{u}x \rightarrow L \text{ as } x \rightarrow y) \leftrightarrow (\text{real is } y \wedge \bigwedge \epsilon > 0$
$\bigvee \delta > 0 \bigwedge x(0 < |x - y| < \delta \rightarrow |\underline{u}x - L| < \epsilon))).$

14.2 $(\lim x \rightarrow y\underline{u}x \equiv \text{the } L(\underline{u}x \rightarrow L \text{ as } x \rightarrow y))$

In 14.1 we have defined a statement meaning $\underline{u}x$ converges to L as x goes to y. In 14.2 we have defined the limit as x goes to y of $\underline{u}x$. The important point here is that 14.2 in no way prejudices the issue of existence of the limit.[10] We have, for example,

$$(\lim x \rightarrow 2(2 \cdot x + 3) = 7)$$

and

$$\sim\text{ex} \lim x \rightarrow 0 \sin(1/x).$$

The basic theorems concerning limits can be given as follows.

14.3 $\text{Unq } L(\underline{u}x \rightarrow L \text{ as } x \rightarrow y)$ Uniqueness

14.4 $((\underline{u}x \rightarrow L \text{ as } x \rightarrow y) \rightarrow \text{ex } L)$ Existence

14.5 $(L = \lim x \rightarrow y\underline{u}x \leftrightarrow (\underline{u}x \rightarrow L \text{ as } x \rightarrow y))$ Characterization

Theorem 14.3 is often formulated roughly as follows:

$$(\lim x \rightarrow y\underline{u}x = L \wedge \lim x \rightarrow y\underline{u}x = M \rightarrow L = M).$$

Transitivity of equality makes this statement both obvious and trivial, thus obscuring the intended meaning. According to Theorem 13.3, what must be proved in 14.3 is

$$\bigwedge L \bigwedge M((\underline{u}x \rightarrow L \text{ as } x \rightarrow y) \wedge (\underline{u}x \rightarrow M \text{ as } x \rightarrow y) \rightarrow L = M).$$

The simple and obvious Theorem 14.4 cannot be expressed in ordinary logic or in any other logic in which all terms denote.

Theorem 14.5 follows immediately from 14.3 and 14.4 and our Theorem 12.4. An important feature of 14.5 is that both sides of the biconditional assert the existence of L (see 5.9). Usually something very like 14.5 is given as a definition of limit. However, in the context of ordinary logic, 14.5 leads to difficulties. For example, from the usual theorem '$(x = x)$' and the admission of '$\lim x \to y\underline{u}x$' as a term we obtain

$$(\lim x \to 0(1/x) = \lim x \to 0(1/x)).$$

Replacing 'L' by '$\lim x \to 0(1/x)$' in 14.5 we obtain

$$((1/x) \to \lim x \to 0(1/x) \text{ as } x \to 0).$$

Since '$(\underline{u}x \to L \text{ as } x \to y)$' is always understood to imply the existence of L, we reach the absurd conclusion that $\lim x \to 0(1/x)$ exists.

These difficulties are commonly evaded by refusing to admit '$\lim x \to y\underline{u}x$' as a term.[11] As explained in [13], this creates serious problems in the syntax.

As our second example the value of a function at a point[12] is considered. 14.6 and 14.7 are definitions.[13]

14.6 $(\text{function is} f \leftrightarrow (\text{relation is} f \wedge \bigwedge x \text{ Unq } y((x,y) \in f)))$

14.7 $(.fx \equiv \text{the } y \text{ (function is} f \wedge (x,y) \in f))$

We now have the following theorems:

14.8 $(y = .fx \leftrightarrow \text{function is} f \wedge (x,y) \in f)$

Proof:[14]

$$
\begin{aligned}
&(y = .fx \\
&\leftrightarrow y = \text{the } t \text{ (function is} f \wedge (x,t) \in f) \\
&\leftrightarrow \text{ex } y \wedge \text{function is} f \wedge (x,y) \in f \wedge \text{Unq } t \text{ (function is} f \wedge (x,t) \in f) && [12.4] \\
&\leftrightarrow \text{function is} f \wedge (x,y) \in f \wedge \text{Unq } t((x,t) \in f) && [13.9] \\
&\leftrightarrow \text{function is} f \wedge (x,y) \in f \wedge \text{ex } x \wedge \bigwedge s \text{ Unq } t((s,t) \in f) \\
&\qquad\qquad \wedge \text{Unq } t((x,t) \in f) \\
&\leftrightarrow \text{function is} f \wedge (x,y) \in f \wedge \text{ex } x \wedge \bigwedge s \text{ Unq } t((s,t) \in f) && [6.1] \\
&\leftrightarrow \text{function is} f \wedge (x,y) \in f).
\end{aligned}
$$

14.9 $(\text{ex } .fx \leftrightarrow \text{function is} f \wedge x \in \text{dmn} f)$

14.10 $(\bigvee x(y = .fx) \leftrightarrow \text{function is} f \wedge y \in \text{rng} f)$

An application of the results above is given by the following characterization of the continuity of a function at a point on the real line.

14.11 $((f \text{ is continuous at } x) \leftrightarrow .fx = \lim t \to x\ .ft).$

The '$\leftarrow$' direction is obtained from 5.9 and 14.9.

As our final example we mention two consequences in set theory. As in the set theory of Bourbaki, the classifier is definable and the axiom of choice is provable.

15 Consequences of 4.1

15.1 $(\bigwedge x\underline{q}'x \rightarrow \text{an } x\underline{p}'x \equiv \text{an } x(\underline{q}'x \wedge \underline{p}'x) \equiv \text{an } x(\underline{q}'x \rightarrow \underline{p}'x))$

15.2 $(\text{an } x\underline{p}'x \equiv \text{an } x\ (\text{ex } x \wedge \underline{p}'x) \equiv \text{an } x\ (\text{ex } x \rightarrow \underline{p}'x))$

15.3 $(\underline{p} \rightarrow \text{an } x\underline{p}'x \equiv \text{an } x(\underline{p} \wedge \underline{p}'x) \equiv \text{an } x(\underline{p} \rightarrow \underline{p}'x))$

15.4 $(y = \text{an } x(\underline{p} \wedge \underline{p}'x) \leftrightarrow \underline{p} \wedge y = \text{an } x\underline{p}'x)$

15.5 $(\bigwedge x\underline{q}'x \wedge \bigvee x\underline{p}'x \rightarrow \text{an } x\underline{p}'x = \text{an } x(\underline{q}'x \wedge \underline{p}'x) = \text{an } x(\underline{q}'x \rightarrow \underline{p}'x))$

15.6 $(\bigwedge x(\underline{p}'x \leftrightarrow \underline{q}'x) \rightarrow \text{the } x\underline{p}'x \equiv \text{the } x\underline{q}'x)$

16 Consequences of 4.2

16.1 $(x \equiv x)$

16.2 $(x \equiv y \leftrightarrow x = y \vee \sim(\text{ex } x \vee \text{ex } y))$

16.3 $(y \equiv \text{the } x\underline{p}'x \leftrightarrow \bigwedge x(\underline{p}'x \leftrightarrow x = y))$

16.4 $(y \equiv \text{the } x(x = y))$

16.5 $(\text{ex } y \leftrightarrow \sim(y \equiv \text{an } x \sim \text{ex } x))$

16.6 $(x \equiv y \leftrightarrow \bigwedge z(z = x \leftrightarrow z = y))$

17 Terms defined by conditions, by cases, or by representation Formulas 17.1 through 17.5 are definitions.

17.1 $((\underline{p} \rightarrow x) \equiv \text{the } t(\underline{p} \wedge t = x))$

17.2 $((x \Diamond y) \equiv \text{the } t(t = x \vee t = y))$

17.3 $((x \rhd y) \equiv (x \Diamond (\sim\text{ex } x \rightarrow y)))$

17.4 $(\Diamond\ x\underline{u}x \equiv \text{the } t\ \bigvee x(t = \underline{u}x))$

17.5 $(\Diamond\ x{,}y\underline{u}'xy \equiv \text{the } t\ \bigvee x \bigvee y(t = \underline{u}'xy))$

We have the following theorems:

17.6 $(\underline{p} \rightarrow (\underline{p} \rightarrow x) \equiv x)$

17.7 $(\text{ex } (\underline{p} \rightarrow x) \leftrightarrow \underline{p} \wedge \text{ex } x)$

17.8 $(y = (\underline{p} \rightarrow x) \leftrightarrow \underline{p} \wedge y = x)$

17.9 $(x \Diamond y \equiv y \Diamond x)$

17.10 $(\text{ex } (x \Diamond y) \leftrightarrow (\text{ex } x \leftrightarrow \sim\text{ex } y) \vee x = y)$

17.11 $(t = x \Diamond y \rightarrow t = x \vee t = y)$

17.12 $(x = x \Diamond y \leftrightarrow (\text{ex } x \wedge \sim\text{ex } y) \vee x = y)$

17.13 $(\text{ex } (x \rhd y) \leftrightarrow \text{ex } x \vee \text{ex } y)$

17.14 $(z = x \rhd y \rightarrow z = x \vee z = y)$

17.15 $(x = x \rhd y \leftrightarrow \text{ex } x)$

17.16 $(y = x \rhd y \leftrightarrow (\text{ex } y \wedge \sim\text{ex } x) \vee x = y)$

17.17 $(x \rhd (y \rhd z) \equiv (x \rhd y) \rhd z)$

17.18 $(x \rhd y = y \rhd x \leftrightarrow \text{ex}(x \Diamond y))$

17.19 $(\text{ex} \Diamond\ x\underline{u}x \leftrightarrow \text{One } y \bigvee x(y = \underline{u}x))$

17.20 $(y = \Diamond\ x\underline{u}x \leftrightarrow \bigvee x \text{ ex } \underline{u}x \wedge \bigwedge x(\text{ex } \underline{u}x \rightarrow y = \underline{u}x))$

17.21 $(z = \Diamond\ x{,}y\underline{u}'xy \leftrightarrow \bigvee x \bigvee y \text{ ex } \underline{u}'xy \wedge \bigwedge x \bigwedge y(\text{ex } \underline{u}'xy \rightarrow z = \underline{u}'xy))$

Definition 17.1 provides the fundamental form for terms defined by condition. For example, we may define the interior of A with respect to the topology $\mathscr{T}$ so that it has a denotation only if $\mathscr{T}$ is a topology and A is a subset of the underlying space, $\bigcup \mathscr{T}$

$$(\text{int } \mathscr{T}\ A \equiv ((\mathscr{T} \in \text{topology} \wedge A \subset \bigcup \mathscr{T}) \rightarrow \{x\colon \bigvee B \in \mathscr{T}\ (x \in B \subset A)\}))$$[15]

The use of such restrictions is a standard practice in the formulation of mathematical definitions.

Definition by cases is another practice frequently encountered in mathematics. To construct such a definition one may compound terms of the form '$(\underline{p} \to x)$' by means of the '$\Diamond$' connective. For example we may define the absolute value of a real number by

$$(|x| \equiv (((x \geqslant 0) \to x) \Diamond ((x \leqslant 0) \to -x))).$$

Notice that the result is meaningful even if the two cases overlap provided that in such an instance both cases agree.

When the second condition in a definition by cases is "otherwise", the '$\triangleright$' connective is useful. As an example we give the following definition of the characteristic function of a set A.

$$(\chi A \equiv \lambda x(((x \in A) \to 1) \triangleright 0))^{16}$$

It is worthwhile noting that the '$\triangleright$' may be used to compound terms in the same way as '$\Diamond$'. In fact, in the definition of '$|x|$' above, no change in meaning results from replacing '$\Diamond$' by '$\triangleright$'. What distinguishes the two connectives is that, whereas '$\Diamond$' is commutative, '$\triangleright$' assigns the first term priority as explained by 17.15 and 17.16.

'$\Diamond$' is used in definition by representation. An example is the definition of the dimension of a vector space as the cardinality of a basis.

$$(\dim KV \equiv \Diamond B\,((B \text{ is a basis of } V \text{ over } K) \to \text{Card } B)).$$

Note that by Theorem 17.19, '$\dim KV$' is meaningful only if every basis of V has the same cardinality.

An example illustrating the use of the two variable form is the following definition of the sum of two cardinal numbers.

$$((\alpha + \beta) \equiv \Diamond\, x,y((\alpha = \text{Card } x \wedge \beta = \text{Card } y \wedge x \cap y = \phi) \to \text{Card } (x \cup y)))$$

18 Comparison with a related system In order to help place the present system in context we discuss briefly a heirarchy of related systems. These systems are all based on standard sentence logic.

The fact that standard predicate logic makes existential presuppositions has been widely discussed. These presuppositions can be eliminated by taking '$\text{ex } x$' as a primitive predicate. The axioms for a universally free predicate logic can be given as:

18.1 $(\Lambda x(\underline{p}'x \to \underline{q}'x) \to (\Lambda x\underline{p}'x \to \Lambda x\underline{q}'x))$
18.2 $(\underline{p} \to \Lambda x\underline{p})$
18.3 $(\Lambda x\underline{p}'x \to (\text{ex } y \to \underline{p}'y))$
18.4 $\Lambda x \text{ ex } x$

We have in mind for this system the same rules of inference as used in this paper plus universalization. A similar system is formulated by Meyer and Lambert [10].

Hintikka has pointed out the semantic equivalence of '$\text{ex } x$' and '$\vee y(x \equiv y)$'. Thus when considering predicate logic with identity, '$\text{ex } x$' need not be taken as primitive. Instead one may take '$\wedge x\underline{\mathrm{p}}'x$' and '$(x \equiv y)$' as primitive and make the definition

18.5 $(\mathrm{ex}\, x \leftrightarrow \vee y(x \equiv y))$

and assume as axioms 18.1 through 18.4 and

18.6 $((x \equiv y) \rightarrow (\underline{\mathrm{p}}'x \rightarrow \underline{\mathrm{p}}'y))$

and

18.7 $(x \equiv x)$.

We note that in the usual predicate logic identity is merely an adjunct whereas here identity enters into the quantification axioms 18.3 and 18.4 via the definition of 'ex x'.

This logic may be further strengthened by introducing the definite description. In addition to the primitive term 'the $x\underline{\mathrm{p}}'x$', one further axiom will suffice.

18.8 $(z \equiv \text{the } x\underline{\mathrm{p}}'x \leftrightarrow \wedge y(z \equiv y \leftrightarrow \wedge x(\underline{\mathrm{p}}'x \leftrightarrow x \equiv y)))$

Lambert has shown in [2] that 18.7 is a consequence of 18.1 through 18.6 plus 18.8.

This final system is a universally free predicate logic with identity and definite descriptions. There are three primitive expressions and six axioms. By way of comparison the system of this paper has three primitives, six axioms, and does not require universalization as a rule of inference. However, the system of this paper is considerably stronger in that it incorporates indefinite descriptions rather than definite descriptions. Further, the two completeness axioms are not used for the development of quantification theory and are of only marginal use in the theory of descriptions.[17]

NOTES

1. By use of the word "formal" we do not wish to associate ourselves with the view that mathematics consists in the (empty) manipulation of expressions. In the strict observance of linguistic rules we see a means to clarity and precision of expression.

2. "Definition 1. *Same* or *coincident* terms are those which can be substituted for each other anywhere without affecting truth." ([9], Vol. 2, p. 613.) This definition of Leibniz has commonly been confused with Leibniz's principle of the identity of indiscernibles and in the process a use-mention error has been attributed to Leibniz (e.g., [2], p. 300). Ishiguro argues effectively against this confusion in Chapter II of [5]. We believe Leibniz's wording does justice to his meaning. (Accordingly we are not in sympathy with Ishiguro's own account of this definition, cf. [13].)

   The substitutions of 'y' for 'x' or vice versa are performed using Axiom 2.1 and are naturally subject to bound variable restrictions in the substitution procedure (see Note 6).

3. "But identity, though expressed by the line as a dyadic relation, is not a relation between two things, but between two representamens of the same thing" ([3], Vol. IV, p. 372).

4. See [1], Section 74, Pseudo-Object Sentences.

5. Theorem 15.1 expresses the reflexivity of identity, while the nonreflexivity of equality can be seen in the proof of Theorem 6.5. Both reflexive and nonreflexive identities are recognized by Lambert and Scharle [7] when they compare the reflexive identity of a system of theirs with a nonreflexive identity of a system of Lejewski.

6. The restrictions on the substitution procedure are given by Definition 6-4 of [6].

7. Because Hilbert did not consider definitions to be part of the formal system, this formulation of his definition should not be attributed to him.

8. We have not defined '$(x \neq y)$' for use in this context since we believe that in actual practice $(x \neq y)$ implies the existence of both x and y. Thus we prefer the following definition:

$$((x \neq y) \leftrightarrow (\sim(x = y) \wedge \mathrm{ex}\, x \wedge \mathrm{ex}\, y)).$$

9. Both the definiendum of 14.1 and the definiens of 14.2 are used by Morse.

10. We find it interesting to note that according to Heath "There is nothing in connection with definitions which Aristotle takes more pains to emphasize than that a definition asserts nothing as to *existence* or *non-existence* of the thing defined" ([4], Vol. 1, p. 143).

11. Instead of admitting such expressions as terms, they are generally introduced by means of contextual definitions.

12. The problem of evaluating a function at a point outside its domain is treated in a similar manner on pp. 209-210 of [8]. In fact much of the discussion in Chapter 10 is pertinent to the present paper.

13. The forms defined here are taken from Morse.

14. We are assuming that in set theory the following stipulations have been made.

$$((x \in y) \rightarrow \mathrm{ex}\, x \wedge \mathrm{ex}\, y)$$

$$(\mathrm{ex}\, (x,y) \leftrightarrow \mathrm{ex}\, x \wedge \mathrm{ex}\, y)$$

We discuss such stipulations in [13].

15. This sort of conditional definition creates two new ways of handling contextual hypotheses in theorems. On occasion they may be dropped as in

$$(\mathrm{int}\, \mathcal{T}\ \mathrm{int}\, \mathcal{T} A \equiv \mathrm{int}\, \mathcal{T} A).$$

Or they may be included simply by significant usage in the hypothesis, as in

$$(B = \mathrm{int}\, \mathcal{T} A \rightarrow B \subset A).$$

16. '$\lambda x \underline{u} x$' is Morse's bound variable notation for the function which takes each x to $\underline{u} x$.

17. The addition of these two axioms makes the system equivalent to one proved consistent and complete in [12].

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