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On Extensions of $L_{\omega\omega}(Q_1)$

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Introduction $L_{\omega\omega}(Q_1)$ is the logic that results by adding to first-order logic the quantifier "there are uncountably many", studied by Mostowski, Fuhrken, Vaught, and Keisler. It is countably compact and satisfies a downward Löwenheim-Skolem theorem down to $\aleph_1$ (see [3] and [5]). However, it does not satisfy the interpolation theorem (see [4]). An important unsolved problem about this logic is the existence of countably compact extensions of $L_{\omega\omega}(Q_1)$ satisfying interpolation (see [8] and the discussion in [2], p. 221). There are many interesting countably compact extensions of $L_{\omega\omega}(Q_1)$, some of them satisfying the Löwenheim-Skolem theorem down to $\aleph_1$, for example its closure under Δ-interpolation (cf. [1], [8]) or stationary logic, $L_{\omega\omega}(aa)$, a fragment of second-order logic introduced by Barwise, Kaufmann, and Makkai [2]. But none of the known examples satisfies interpolation.

In this note* we show that the monadic fragment of $L_{\omega\omega}(Q_1)$ satisfies the interpolation theorem and is, in fact, a maximal monadic logic satisfying countable compactness and a form of the downward Löwenheim-Skolem theorem down to $\aleph_1$. This is similar to Lindstrom's theorem for $L_{\omega\omega}$, and it follows from the topological properties of the space of models. We introduce monadic *filter*[1] quantifiers and show that they are essentially the cardinal quantifiers (Section 2). A *back-and-forth* characterization of elementary equivalence is given for those logics obtained by adjoining filter quantifiers to the propositional connectives (Section 3). This is used in Section 4 to show that if two sentences of $L_{\omega\omega}(Q_1)$ have an interpolant in the infinitary logic $L_{\infty\omega}(Q_1)$ allowing conjunctions of arbitrary sets of formulas, then they have an interpolant[2] in $L_{\omega\omega}(Q_1)$. Actually, a stronger result is proved: if L^* and $L^\#$ are

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countably compact extensions of $L_{\omega\omega}(Q_1)$, L^* obtained by adding filter ω_1-*complete* quantifiers, and $L^* \prec L^{\#}$, then two disjoint *PC* classes of $L^{\#}$ that can be separated in the infinitary logic $L^*_{\infty\omega}$, can also be separated in L^*. With this we may show that $L_{\omega\omega}(Q_1)$ is a maximal countably compact sublogic of $L_{\infty\omega}(Q_1)$.

Back-and-forth or game characterizations of elementary equivalence for $L_{\omega\omega}(Q_1)$ have been introduced before by others (see for example [10]) and have been used to prove theorems about preservation of elementary equivalence by various operations. Krawczyk and Krynicki [6] give a general characterization for monotonous quantifiers. Our version differs from those in that in our "game" the "players" do not choose subsets but elements of the universe. By doing so they are choosing equivalence classes with respect to certain equivalence relations. This way, the existence of a back-and-forth relation between two structures becomes a Σ^1_1-definable property in the corresponding logic (see Lemma 3.3).

Structures will be denoted by $\mathfrak{A}, \mathfrak{B}, \ldots$, and their corresponding universes by $A, B, \ldots$; $|X|$ is the cardinal of the set X. If K is a class of structures and c is a cardinal, K^c denotes the class of structures in K of cardinality smaller than or equal to c. An abstract logic is understood as in Barwise [1]. We identify *first-order languages* with similarity types of structures. If L is an (abstract) logic and τ is a language, L_τ denotes the class of sentences of L with symbols in τ. L is said to satisfy $LS_c(d)$ if every consistent set of sentences of L of cardinality smaller than or equal to c has a model of cardinality smaller than or equal to d. If α: $\tau \to \tau'$ is an *interpretation* (see [1]), $\mathfrak{A}$ is a τ'-structure, and $\phi \in L_\tau$, then $\mathfrak{A} \upharpoonright \alpha$ and $\phi^{(\alpha)}$ denote respectively the restriction of $\mathfrak{A}$ and the relativization of ϕ.

1 The monadic fragment of $L_{\omega\omega}(Q_1)$ If τ is a language let E^c_τ denote a set of representatives, with respect to isomorphism, of the class of τ-structures of cardinality smaller than or equal to c. Let L be a logic, then $E^c_\tau(L)$ denotes the topological space that results of giving to E^c_τ the topology generated by the restrictions of the L_τ-elementary classes to E^c_τ. It is not difficult to prove:

Lemma 1.1 *If L is closed under conjunctions and negations, is countably compact, satisfies $LS_{\aleph_0}(c)$, and $|L_\tau| \leqslant \aleph_0$, then $E^c_\tau(L)$ is a compact topological space where the clopen (closed and open) sets form a basis, and coincide with the L_τ-elementary classes restricted to E^c_τ.*

Let $M(Q_1)$ denote the monadic fragment of $L_{\omega\omega}(Q_1)$; it does not have function symbols or n-ary relation symbols for $n > 1$. $M(Q_1)$ is countably compact and satisfies $LS_{\aleph_0}(\aleph_1)$. Moreover, if τ is a finite monadic language then $|M(Q_1)_\tau| \leqslant \aleph_0$ and so the above lemma applies. However, we have a stronger statement:

Lemma 1.2 *Let τ be a finite monadic language, then $E^{\aleph_1}_\tau(M(Q_1))$ is a compact Hausdorff space with the clopen sets as a basis.*

Proof: To prove the Hausdorff separation property it is enough to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are τ-structures with $|A|$, $|B| \leqslant \aleph_1$ then $\mathfrak{A} \equiv_{M(Q_1)} \mathfrak{B}$ implies $\mathfrak{A} \approx \mathfrak{B}$. Let

$\tau = (P_i)_{i<n}$, and for each $f \in 2^n$ form the formula $U_f(x) = \bigwedge_{i<n} P_i^{f(i)}(x)$, where $P^0(x) = P(x)$ and $P^1(x) = \neg P(x)$. For any structure $\mathfrak{A}$ define $U_f^{\mathfrak{A}} = \{a \in A: \mathfrak{A} \models U_f(a)\}$. If $|A| \leqslant \aleph_1$ the isomorphism type of $\mathfrak{A}$ is completely determined by the cardinals $c_f = |U_f^{\mathfrak{A}}|$, where $f \in 2^n$. Since $c_f \leqslant \aleph_1$, these cardinalities are expressible by sets of sentences of $M(Q_1)$ and so they are preserved by elementary equivalence. This proves the claim.

Theorem 1.3 *$M(Q_1)$ satisfies interpolation.*

Proof: Let K_1 and K_2 be disjoint *PC* classes of $M(Q_1)$. Suppose that $K_i = \{\mathfrak{A} \restriction \alpha_i: \mathfrak{A} \models \phi_i\}$ where $\alpha_i: \tau \to \tau_i$ are interpretations (see [1]) and $\phi_i \in M(Q_1)_{\tau_i}$ for $i = 1, 2$. We may assume without loss of generality that τ is finite; then $K_1^{\aleph_1}$ and $K_2^{\aleph_1}$ are disjoint compact subsets of $E_\tau^{\aleph_1}(M(Q_1))$. By well-known separation properties of compact Hausdorff spaces, these classes may be separated by an open set. Since there is a basis of clopen sets, the separation may be realized by a clopen set. But in this topology every clopen set is elementary, thus there is a sentence $\sigma \in M(Q_1)$ such that $K_1^{\aleph_1} \subseteq Mod(\sigma)^{\aleph_1} \subseteq \overline{K}_2^{\aleph_1}$. Suppose $K_1 \not\subseteq Mod(\sigma)$, then there is a structure $\mathfrak{A} \models \phi_1$ such that $\mathfrak{A} \restriction \alpha_1 \not\models \sigma$ and so $\mathfrak{A} \models \neg\sigma^{(\alpha_1)}$. Applying the $LS_{\aleph_0}(\aleph_1)$, the sentences ϕ_1, $\neg\sigma^{(\alpha_1)}$ must have a model of cardinality $\leqslant\aleph_1$, call it $\mathfrak{B}$. Then $\mathfrak{B} \restriction \alpha_1 \in K_1^{\aleph_1}$ and $\mathfrak{B} \notin Mod(\sigma)^{\aleph_1}$, a contradiction. We conclude that $K_1 \subseteq Mod(\sigma)$. Similarly, $Mod(\sigma) \subseteq \overline{K}_2$.

In the next theorem we do *not* assume that M^* is closed under relativization, conjunctions, or negation.

Theorem 1.4 (Lindstrom's theorem for $M(Q_1)$) *Let M^* be a countably compact extension of $M(Q_1)$ satisfying $LS_{\aleph_0}(\aleph_1)$, and let τ be a finite monadic language, then $M_\tau^* \equiv M(Q_1)_\tau$.*

Proof: Let $\phi \in M_\tau^*$, $K_1 = Mod(\phi)$, $K_2 = Mod(\neg\phi)$. Using that M^* is countably compact and satisfies $LS_{\aleph_0}(\aleph_1)$ it is easy to see that $K_1^{\aleph_1}$ and $K_2^{\aleph_1}$ are complementary compact subsets of $E_\tau^{\aleph_1}(M(Q_1))$. As in Theorem 1.3, there exists $\sigma \in M(Q_1)_\tau$ such that $K_1^{\aleph_1} \subseteq Mod(\sigma) \subseteq \overline{K}_2^{\aleph_1}$. By the $LS_{\aleph_0}(\aleph_1)$ in M^* again, we have $K_1 \subseteq Mod(\sigma) \subseteq \overline{K}_2 = K_1$ and so $\phi \equiv \sigma$.

The same method we have used here may be applied to first-order monadic logic, $M_{\omega\omega}$, to give trivial proofs of interpolation, and of "Lindstrom's theorem for $M_{\omega\omega}$", first proved by Tharp [9]. The following is an interesting application of Theorem 1.4.

Corollary 1.5 *If $M(aa)$ is the monadic fragment of stationary logic, then $M(aa) \equiv M(Q_1)$.*

2 Monadic filter quantifiers A *monadic* quantifier is a function C that assigns to each set A a family of its subsets $C(A)$, with the property that if $(A, S) \approx (B, T)$ then $S \in C(A) \Longleftrightarrow T \in C(B)$. The *dual* quantifier $\tilde{C}$ is defined by $\tilde{C}(A) = \{S \subseteq A: A - S \notin C(A)\}$. Obviously $\tilde{\tilde{C}} = C$. Given a family of quantifiers $(C_i)_{i\in I}$, the logic $L(\underline{C}_i)_{i\in I}$ is obtained by adding to the atomic formulas and propositional connectives of first-order logic the quantifier symbols $\underline{C}_i (i \in I)$, allowing formulas of the form $\underline{C}_i x\phi$ in the formation rules, and defining:

$$\mathfrak{A} \models \underline{C}_i x\phi \Longleftrightarrow \{a \in A: \mathfrak{A} \models \phi(a)\} \in C_i(A).$$

Since $\mathfrak{A} \models \tilde{C}_i x\phi \iff \mathfrak{A} \models \neg C_i x \neg\phi$, $L(C_i)$ is equivalent to $L(\tilde{C}_i)$. Abusing the language, we will use the same symbol for the quantifier and the corresponding quantifier symbol.

For example, $\exists(A) = \mathcal{P}(A) - \{\phi\}$, with dual $\tilde{\exists}(A) = \forall(A) = \{A\}$, gives first-order logic $L_{\omega\omega} = L(\exists)$. In general, we define $L_{\omega\omega}(C_i)_{i\in I} = L(\exists, C_i)_{i\in I}$, and $L_{\infty\omega}(C_i)_{i\in I}$ is the infinitary extension that allows conjunctions of arbitrary sets of sentences in its formation rules. Other well-known monadic quantifiers are the (constant) cardinal quantifiers $Q_\alpha(A) = \{S \subseteq A\colon |S| \geqslant \aleph_\alpha\}$, and Chang's quantifier $Ch(A) = \{S \subseteq A\colon |S| = |A|\}$.

Definition 2.1 A monadic quantifier C is a *filter* quantifier if the following are valid schemata of the logic $L_{\omega\omega}(C)$:

(a) $\forall x(\phi \to \psi) \to (Cx\phi \to Cx\psi)$
(b) $Cx(\phi \vee \psi) \to (Cx\phi \vee Cx\psi)$.

Obviously, C is a filter quantifier if and only if:

1. $S \in C(A), S' \supseteq S \Rightarrow S' \in C(A)$.
2. $\bigcup_{i<n} S_i \in C(A) \Rightarrow \exists i < n\colon S_i \in C(A)$. $(n \in \omega)$

This is in turn equivalent to:

$\tilde{C}(A)$ is a filter over A, for all A.

The existential quantifier $\exists$ and the Q_α's are filter quantifiers, and Ch, restricted to infinite structures, is a filter quantifier. These examples can be generalized as follows: let f be a function from cardinals to cardinals and define $Q_f(A) = \{S \subseteq A\colon |S| \geqslant f(|A|)\}$. If $f(c) = 1$ or $f(c) \geqslant \aleph_0$ for every cardinal c, Q_f is a filter quantifier. Surprisingly these exhaust all the possibilities.

Theorem 2.1 *C is a filter quantifier if and only if $C = Q_f$ where f: Cardinals $\to$ $\{1\} \cup$ Infinite Cardinals.*

Proof: Suppose that C is a filter quantifier and define:

$$f(A) = \begin{cases} 2^{|A|} \cdot \aleph_0, & \text{if } C(A) = \phi \\ \text{minimum } |S| \text{ such that } S \in C(A), & \text{if } C(A) \neq \phi. \end{cases}$$

We claim that $S \in C(A) \iff |S| \geqslant f(A)$, for all $S \subseteq A$. If $C(A) = \phi$ this is obvious. If $C(A) \neq \phi$ then $A \in C(A)$ by the monotonicity condition (a) of Definition 2.1. Let μ = minimum $|S|$ such that $S \in C(A)$; it is enough to show that if $T \subseteq A$ and $|T| = \mu$, then $T \in C(A)$; the rest will follow by monotonicity. *Case 1:* there is $S \in C(A)$ with $|S| = \mu$ and $|\bar{S}| = \beta \geqslant \mu$. Choose $T' \subseteq T$ such that $|T'| = \mu$ and $|\bar{T}'| = \beta$, then $(A, T') \approx (A, S)$ and so $T' \in C(A)$ by the isomorphism condition on quantifiers. By monotonicity, $T \in C(A)$. *Case 2:* there is no $S \in C(A)$ with $|S| = \mu$ and $|\bar{S}| \geqslant \mu$. Then $|A| = \mu$. Suppose that $|T| = \mu$ but $T \notin C(A)$. Since $T \cup \bar{T} = A \in C(A)$, then $\bar{T} \in C(A)$ by condition (b) of Definition 2.1; therefore, $|\bar{T}| = \mu$. But this contradicts the assumption because $|\bar{\bar{T}}| = |T| = \mu$. Finally, note that $\mu \neq 2, 3, \ldots$ by condition (b), and $f(A)$ depends only on $|A|$.

3 ***Back-and-forth for filter quantifiers*** Through this section let $\mathcal{C} = (C_\lambda)_{\lambda \in I}$ be a family of filter quantifiers and $L(\mathcal{C})$ the corresponding logic. $\mathfrak{A}$ and $\mathfrak{B}$ are structures of the same similarity type.

Definition 3.1 A *$\mathcal{C}$-back-and-forth from $\mathfrak{A}$ to $\mathfrak{B}$* consists of the following relations, subject to properties (i)-(v) below:

1. A linearly ordered set $\mathcal{P} = (P, <)$ of *parameters.*
2. For each sequence $\sigma \in \bigcup_{n\in\omega} A^n$ a family $(\sim_\sigma^p)_{p\in P}$ of equivalence relations in A with finitely many equivalence classes, such that $p < p'$ implies $\sim_\sigma^{p'} \subseteq \sim_\sigma^p$.
3. For each $\tau \in \bigcup_{n\in\omega} B^n$ a similar family $(\sim_\tau^p)_{p\in P}$ in B.
4. A family $(\sim^p)_{p\in P}$ of relations between sequences of elements of A and sequences of elements of B of the same length, such that $p < p'$ implies $\sim^{p'} \subseteq \sim^p$.

(i) $\phi \sim^p \phi$ (ϕ = the empty sequence)
(ii) $a' \sim_\sigma^p a$, $(\sigma,a) \sim^p (\tau,b)$, $b \sim_\tau^p b' \Rightarrow (\sigma,a') \sim^p (\tau,b')$
(iii-λ) $\sigma \sim^{p'} \tau$, $p < p'$, $a \in A$, and $\{a' : a' \sim_\sigma^p a\} \in C_\lambda(A) \Rightarrow \exists b \in B[(\sigma,a) \sim^p (\tau,b)$ and $\{b' : b' \sim_\tau^p b\} \in C_\lambda(B)]$
(iv-λ) As (iii-λ), alternating the role of A and B
(v) $(a_1, \ldots, a_n) \sim^p (b_1, \ldots, b_n) \Rightarrow f(a_k) = b_k$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Properties (iii-λ) and (iv-λ) must hold for all $\lambda \in I$. The notation $\mathfrak{A} \sim_{\mathcal{C}} \mathfrak{B}$ indicates that there exists a $\mathcal{C}$-back-and-forth from $\mathfrak{A}$ to $\mathfrak{B}$; $\mathfrak{A} \sim_{\mathcal{C}}^{\mathcal{P}} \mathfrak{B}$ indicates which is the set of parameters. We also use $\sim$ to denote the back-and-forth itself.

Definition 3.2 A filter quantifier C is *κ-complete* if $\tilde{C}(A)$ is a κ-complete filter over A. Equivalently, $C = Q_f$, where $cof(f(c)) \geqslant \kappa$ for every cardinal c.

Any filter quantifier is ω-complete; ∃ is κ-complete for any κ; ∃ and Q_1 are the simplest ω_1-complete quantifiers. If $\mathcal{C}$ consists of κ-complete quantifiers ($\kappa > \omega$) we will weaken the finiteness condition, in (2) and (3) of Definition 3.1 of $\mathcal{C}$-back-and-forth, to:

$\sim_\sigma^p$ *has fewer than κ equivalence classes.* (**)

However, in case $\mathcal{C}$ contains non-ω_1-complete quantifiers, a back-and-forth can be taken only in the original sense.

Definition 3.3 $\sim$ is a $\mathcal{C}$-back-and-forth *without parameters* if we drop part 1 of Definition 3.1, postulate nonparameterized relations $\sim_\sigma$, $\sim_\tau$, and $\sim$ in 2, 3, and 4 respectively, and state properties (i)-(v) without the parameter conditions. The existence of such a relation is denoted by $\mathfrak{A} \sim_{\mathcal{C}}^\infty \mathfrak{B}$.

Lemma 3.1 *$\mathfrak{A} \sim_{\mathcal{C}}^\infty \mathfrak{B}$ if and only if $\mathfrak{A} \sim_{\mathcal{C}}^{\mathcal{P}} \mathfrak{B}$ where $\mathcal{P}$ is not well ordered.*

Proof: If $\boldsymbol{\sim}$ is a back and forth without parameters, take any linearly ordered set $\mathcal{P}$ and define: $a \sim_\sigma^p a'$ $(b \sim_\tau^p b') \Longleftrightarrow a \boldsymbol{\sim}_\sigma a'$ $(b \boldsymbol{\sim}_\tau b')$, $\sigma \sim^p \tau \Longleftrightarrow \sigma \boldsymbol{\sim} \tau$. If $\mathfrak{A} \sim_{\mathcal{C}}^{\mathcal{P}} \mathfrak{B}$ and $p_1 > p_2 > \ldots$ is a descending sequence in $\mathcal{P}$, define: $a \boldsymbol{\sim}_\sigma a'$

$(b \approx_\tau b') \iff \exists p_n(a \sim_\sigma^{p_n} a' \; (b \sim_\tau^{p_n} b'))$ and $\sigma \approx \tau \iff \exists p_n(\sigma \sim^{p_n} \tau)$. The equivalence class of $\underline{a}$ with respect to $\approx_\sigma$ is the union of the equivalence classes of $\underline{a}$ with respect to $\sim_\sigma^{p_n}$ ($n = 1, 2, \ldots$). Therefore, the number of equivalence classes of $\approx_\sigma$ is bounded by the number of equivalence classes of any $\sim_\sigma^{p_n}$. If some $\sim_\sigma^{p_n}$ has finitely many equivalence classes, the union may be taken to be finite because there is a coarsest equivalence relation $\sim_\sigma^{p_j}$. To prove properties (iii-λ) and (iv-λ) one uses the fact that in any case $\bigcup_{n \in \omega} E_n \in C(A) \iff \exists n \in \omega(E_n \in C(A))$, where E_n is the equivalence class of $\underline{a}$ with respect to $\sim_\sigma^{p_n}$.

Let $\mathcal{C}$ be a finite family of filter quantifiers. The *quantifier rank* of the sentences of $L(\mathcal{C})$ is defined as usual (see [1]), giving the same weight to every quantifier of $\mathcal{C}$. This way the number of nonequivalent sentences of rank less than or equal to n ($n \in \omega$) becomes finite in case the language τ is finite. $\equiv^n$ denotes elementary equivalence up to sentences of quantifier rank strictly less than n.

Lemma 3.2

(a) *If the language τ is finite then:* $\mathfrak{A} \equiv^n_{L_{\omega\omega}(\mathcal{C})} \mathfrak{B} \Rightarrow \mathfrak{A} \sim^{(n,<)}_{\exists,\mathcal{C}} \mathfrak{B}$.
(b) $\mathfrak{A} \sim^{(n,<)}_{\exists,\mathcal{C}} \mathfrak{B} \Rightarrow \mathfrak{A} \equiv^n_{L_{\omega\omega}(\mathcal{C})} \mathfrak{B}$.
(c) $\mathfrak{A} \sim^{\infty}_{\exists,\mathcal{C}} \mathfrak{B} \Rightarrow \mathfrak{A} \equiv_{L_{\infty\omega}(\mathcal{C})} \mathfrak{B}$.

Proof:

(a) "⇒". For $k < n$, $(\vec{a}, a, a') \in A^{m+2}$, and $(\vec{b}, b, b') \in B^{m+2}$ define:

$$a \sim^k_{\vec{a}} a' \iff (\mathfrak{A}, \vec{a}, a) \equiv^k_{L_{\omega\omega}(\mathcal{C})} (\mathfrak{A}, \vec{a}, a')$$
$$b \sim^k_{\vec{b}} b' \iff (\mathfrak{B}, \vec{b}, b) \equiv^k_{L_{\omega\omega}(\mathcal{C})} (\mathfrak{B}, \vec{b}, b')$$
$$\vec{a} \sim^k \vec{b} \iff (\mathfrak{A}, \vec{a}) \equiv^k_{L_{\omega\omega}(\mathcal{C})} (\mathfrak{B}, \vec{b}).$$

The number of equivalence classes of $\sim^k_{\vec{a}}$ (respectively $\sim^k_{\vec{b}}$) is finite because they are of the form $[a]^k_{\vec{a}} = \left\{a' \colon (\mathfrak{A}, \vec{a}) \models \bigwedge t^k_a(a')\right\}$, where t^k_a is the set of formulas of qr $< k$ satisfied by $\underline{a}$ in $(\mathfrak{A}, \vec{a})$, and there are finitely many of those. To show property (iii-λ) (respectively (iv-λ)), let $(\mathfrak{A}, \vec{a}) \equiv^{k+1}_{L_{\omega\omega}(\mathcal{C})} (\mathfrak{B}, \vec{b})$, $a \in A$, and $[a]^k_{\vec{a}} \in C_\lambda(A)$, then $(\mathfrak{A}, \vec{a}) \models C_\lambda x \bigwedge t^k_a(x)$, $\exists x \bigwedge t^k_a(x)$. Since these sentences have quantifier rank (qr) less than $k + 1$, they are satisfied also by $(\mathfrak{B}, \vec{b})$ and one may choose $b \in B$ such that $(\mathfrak{A}, \vec{a}, a) \equiv^k_{L_{\omega\omega}(\mathcal{C})} (\mathfrak{B}, \vec{b}, b)$, and $[b]^k_{\vec{b}} \in C_\lambda(B)$. The other properties are easy to check.

(b) "⇐". We use induction on the complexity of the formulas to show:

$$\vec{a} \sim^k \vec{b},\ \text{qr}\,(\phi(\vec{y})) < k \Rightarrow ((\mathfrak{A}, \vec{a}) \models \phi(\vec{a}) \iff (\mathfrak{B}, \vec{b}) \models \phi(\vec{b})).$$

Since $\phi \sim^k \phi$ for $k < n$, it follows that $\mathfrak{A} \equiv^n_{L_{\omega\omega}(\mathcal{C})} \mathfrak{B}$. The only interesting step is when $\phi = C\, x \psi(\vec{y}, x)$, $C = C_\lambda$. Suppose $\vec{a} \sim^k \vec{b}$, qr $(\psi(\vec{y}, x)) < k - 1$ and $(\mathfrak{A}, a) \models C\, x \psi(\vec{a}, x)$, then $\{a \colon \mathfrak{A} \models \psi(\vec{a}, a)\} \in C(A)$. By the finiteness of the number of equivalence classes of $\sim^{k-1}_{\vec{a}}$ and property (b) (in Definition 2.1 of filter quantifiers) generalized to finite unions, either the above truth set is empty, in which case both $C(A)$ and $C(B)$ must be trivial (using (iv-∃)), or there is an equivalence class $[a]^{k-1}_{\vec{a}} \in C(A)$ such that $(\mathfrak{A}, \vec{a}) \models \psi(\vec{a}, a)$. In this case using (iii-λ), one finds $b \in B$ such that $[b]^{k-1}_{\vec{b}} \in C(B)$ and $(\vec{a}, a) \sim^{k-1} (\vec{b}, b)$. By induction hypothesis, Definition 3.1 (ii), and monotonicity of C, then $(\mathfrak{A}, \vec{b}) \models C x \psi(b, x)$. The other direction is similar. In case the quantifiers are κ-complete and we have the weaker condition (**), we use that $\bigcup_{\mu<\delta} S_\mu \in C(A) \iff \exists \mu < \delta(S_\mu \in C(A))$ for any $\delta < \kappa$.

(c) Similar to the proof of (b); the converse does not hold even if the language τ is finite.

To show that the converse of Lemma 3.2 (c) fails, assume that $\mathfrak{A} \sim^{\infty}_{\exists, Q_1} \mathfrak{A}$, where the relations $\sim_\sigma$ have at most countably many equivalence classes. Then, as in the proof of Lemma 3.2 (b), $a \sim b$ implies $(\mathfrak{A}, a) \equiv_{L_{\infty\omega}(Q_1)} (\mathfrak{A}, b)$ and so $\mathfrak{A}$ must have countably many $L_{\infty\omega}(Q_1)$-types of elements. Therefore, any structure for a finite language satisfying uncountably many $L_{\infty\omega}(Q_1)$-types will provide a counterexample. Actually, it is easy to give a structure for a binary relation symbol satisfying uncountably many $L_{\omega\omega}$-types; such is the case of $(\omega \cup \mathcal{P}(\omega), \in)$, where we identify $n < m$ with $n \in m$. Every element of ω is defined by a finitary formula and every subset of ω is defined by a countable conjunction of finitary formulas.

In the following lemma it is assumed that the logic $L_{\omega\omega}(Q_\alpha, \mathcal{C})$ is *closed under relativizations* (see [1]). This means essentially that in any formula it must be possible to restrict the meaning of the quantifiers to a subdomain defined by a monadic predicate symbol. Logic with the Chang's quantifier does not enjoy this property. If t: $\tau' \to \tau$ is an interpretation, the universe of the restriction $\mathfrak{A} \upharpoonright t$ is given by the meaning of $t(\forall)$ in $\mathfrak{A}$.

Lemma 3.3 *Let t_i: $\tau \to \tau_i$ $(i = 1, 2)$ be interpretations between finite languages and let $\mathcal{C}$ be a finite family of $\aleph_\alpha$-complete quantifiers. Then there is a sentence $\Delta(\ldots, U_1, U_2, P, E)$ in $L_{\omega\omega}(Q_\alpha, \mathcal{C})$ such that for structures $\mathfrak{A}_i$ of type τ_i $(i = 1, 2)$: $\mathfrak{A}_1 \upharpoonright t_1 \sim^{(\mathbf{P},<)}_{\mathcal{C}} \mathfrak{A}_2 \upharpoonright t_2 \iff \exists \mathfrak{B}$: $(\mathfrak{B}, \mathfrak{A}_1, \mathfrak{A}_2, \mathbf{P}, <) \models \Delta$ when U_i is interpreted by $A_i = |\mathfrak{A}_i|$, and P, E are interpreted by $\mathbf{P}$, $<$.*

Proof: Add to the disjoint union of τ_1 and τ_2 the following new predicates:

$U_1(x)$, $U_2(x)$	denoting the universes A_1 and A_2 respectively.
$R^i(x, x', y)$ $(i = 1, 2)$	denoting a "pairing" relation that permits us to talk about pairs and sequences $\vec{x}$ in the relativized universes $t_i(\forall)$.
$P(x)$, $E(x, x')$	denoting the ordered set of parameters.
$E_i(x, y, z, z')$ $(i = 1, 2)$	denoting the relation $z \sim^x_{\vec{y}} z'$ where $\vec{y}$ is the sequence "coded" by y.
$I(x, z, z')$	denoting $\vec{z} \sim^x \vec{z}'$ where $\vec{z}, \vec{z}'$ are the sequences "coded" by z, z'.
$K(x)$	denoting a set of power less than $\aleph_\alpha$.

All the properties of a $\mathcal{C}$-back-and-forth may be easily translated to this language using the suggested denotations. The quantifiers in $\mathcal{C}$ are used only to express conditions (iii-λ) and (iv-λ). The quantifier Q_α is needed to express the cardinality condition in the equivalence relations:

$$\neg Q_\alpha x K(x) \wedge \forall x, y, z \in t_i(\forall) \exists w[K(w) \wedge E_i(x, y, z, w)].$$

In the translation of (v) one has to be careful to state the isomorphism between the right relations; these are $t_1(R)$ and $t_2(R)$ where R is in τ. It is clear that we need to express the right meaning of C_λ in those subsets of the large structure defined by $t_1(\forall)$ and $t_2(\forall)$. That is the reason that we need relativization in the logic $L_{\omega\omega}(\mathcal{C})$.

If we drop the finiteness condition in the languages and $\mathcal{C}$ in the above lemma, we get a set (or a proper class) of sentences Δ.

4 ***Applications*** Through this section let $\mathcal{C} = \{Q_\alpha, Q_{\alpha_1}, \ldots, Q_{\alpha_n}\}$ where $\mathrm{cof}(\aleph_{\alpha_i}) \geqslant \aleph_\alpha$. Then the quantifiers are all $\aleph_\alpha$-complete, and $L_{\omega\omega}(\mathcal{C})$ is closed under relativization.

Theorem 4.1 *Assume that $L_{\omega\omega}(\mathcal{C})$ is countably compact and $L^\#$ is a countably compact extension of $L_{\omega\omega}(\mathcal{C})$. If K_1 and K_2 are PC classes of $L^\#$, inseparable by elementary classes of $L_{\omega\omega}(\mathcal{C})$, then they are inseparable by elementary classes of $L_{\infty\omega}(\mathcal{C})$.*

Proof: Let $K_i = \{\mathfrak{A} \upharpoonright t_i : \mathfrak{A} \models \phi_i\}$ where $t_i: \tau \to \tau_i$ are interpretations and $\phi_i \in L^\#_{\tau_i}$ $(i = 1, 2)$. Since $\exists$ is κ-complete for any κ and the other quantifiers are $\aleph_\alpha$-complete, we may apply Lemma 3.3 to the family $(\exists, \mathcal{C})$. Let $\Delta(U_1, U_2, P, E, \ldots) \in L_{\omega\omega}(Q_\alpha, \exists, \mathcal{C}) = L_{\omega\omega}(\mathcal{C})$ be the sentence given there. If K_1 and K_2 are inseparable in $L_{\omega\omega}(\mathcal{C})$ there are structures $\mathfrak{A}_n \in K_1$, $\mathfrak{B}_n \in K_2$, such that $\mathfrak{A}_n \equiv^n_{L_{\omega\omega}(\mathcal{C})} \mathfrak{B}_n$ for each $n \in \omega$. Therefore, the sentence $\Delta \wedge \phi_1^{(U_1)} \wedge \phi_2^{(U_2)}$ has models where the cardinality of $\mathbf{P}$ is n, for arbitrary n. Using countable compactness, there is a model $(\ldots, \mathfrak{A}, \mathfrak{B}, \mathcal{P})$ where $\mathcal{P}$ is not well ordered, and so $\mathfrak{A} \upharpoonright t_1 \sim^\infty \mathfrak{B} \upharpoonright t_2$. Hence, $\mathfrak{A} \upharpoonright t_1 \equiv_{L_{\infty\omega}(\mathcal{C})} \mathfrak{B} \upharpoonright t_2$ by Lemma 3.2. But also $\mathfrak{A} \models \phi_1$ and $\mathfrak{B} \models \phi_2$, and so K_1 and K_2 are inseparable in $L_{\infty\omega}(\mathcal{C})$.

Corollary 4.2 *Let $L^*_{\omega\omega}$ be either $L_{\omega\omega}(Q_1)$ or $L_{\omega\omega}(Q_{\alpha_1+1}, \ldots, Q_{\alpha_n+1})$ where $\aleph_{\alpha_i}^{\aleph_0} = \aleph_{\alpha_i}$, and let $L^*_{\infty\omega}$ be the corresponding infinitary logic. Then:*

(a) *Let $\phi, \psi \in L^*_{\omega\omega}$, $\phi \models \psi$. If ϕ and ψ have an interpolant in $L^*_{\infty\omega}$ they have an interpolant in $L^*_{\omega\omega}$.*
(b) *Let $L^\#$ be a countably compact extension of $L^*_{\omega\omega}$, then $L^\# \cap L^*_{\infty\omega} \equiv L^*_{\omega\omega}$. Hence, $L^*_{\omega\omega}$ is a maximal countably compact sublogic of $L^*_{\infty\omega}$.*

Proof: Obviously, Theorem 4.1 applies to $\mathcal{C} = \{Q_1\}$. It also applies to $\mathcal{C} = \{Q_{\alpha_1+1}, \ldots, Q_{\alpha_n+1}\}$ because we may assume $\alpha_1 \leqslant \alpha_i$ and so $\mathrm{cof}(\aleph_{\alpha_i+1}) = \aleph_{\alpha_i+1} \geqslant \aleph_{\alpha_1+1}$. Moreover, $\aleph_{\alpha_i}^{\aleph_0} = \aleph_{\alpha_i}$ implies that $\aleph_0$ is *small for* $\aleph_{\alpha_i+1}$ (see [3], p. 263) and the logic $L_{\omega\omega}(Q_{\alpha_1+1}, \ldots, Q_{\alpha_n+1})$ is countably compact. To prove (a) it is enough to make $L^\# = L^*_{\omega\omega}$ in Theorem 4.1. To prove (b), let $\phi \in L^\# \cap L^*_{\infty\omega}$ and make $K_1 = Mod(\phi)$, $K_2 = Mod(\neg\phi)$ in Theorem 4.1. If $\phi \notin L^*_{\omega\omega}$ these classes are inseparable in $L^*_{\omega\omega}$ and so they are inseparable in $L^*_{\infty\omega}$. This is absurd because $\phi \in L^*_{\infty\omega}$.

The last corollary implies, for example, that if $\Delta(L(aa))$ is the closure of Stationary Logic under Δ-interpolation then $\Delta(L(aa)) \cap L_{\infty\omega}(Q_1) \equiv L_{\omega\omega}(Q_1)$.

Remark: In our doctoral dissertation we show that Corollary 4.2 still holds with $L^*_{\omega\omega}$ equal to stationary logic. Also, we have shown that the results remain valid if we replace $L^*_{\infty\omega}$ with the large infinitary logic obtained by adding to $L^*_{\omega\omega}$ (taken to be $L_{\omega\omega}(Q_1)$ or $L(aa)$) all monadic quantifiers, and we ask the logic $L^{\#}$ in Corollary 4.2 (b) to satisfy a downward Löwenheim-Skolem theorem down to $\aleph_1$. Therefore, we obtain a "Lindstrom's theorem" for $L_{\omega\omega}(Q_1)$ with respect to extensions by monadic quantifiers. Moreover, any extension of this logic satisfying interpolation must include nonmonadic quantifiers.

NOTES

1. In subsequent work related to this paper we have chosen to call these quantifiers, more appropriately, *cofilter* quantifiers.

2. After this paper was written, we learned that this "relative interpolation theorem" and its generalization to stationary logic were proven independently by J. Stavi and J. A. Makowsky. See: J. A. Makowsky, "A note on stationary logic," *Notices of the American Mathematical Society,* vol. 24 (1977), p. A-438.

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