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On the Number of Nonisomorphic Models of Cardinality λ $L_{\infty\lambda}$-Equivalent to a Fixed Model

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A well-known result of Scott [6] is that if $\mathfrak{M}$ and $\mathfrak{N}$ are countable and $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, then $\mathfrak{M} \cong \mathfrak{N}$. Later, Chang [2] extended this to show that if $cf(\lambda) = \aleph_0$, $\mathfrak{M}$ and $\mathfrak{N}$ have cardinality λ and $\mathfrak{M} \equiv_{\infty\lambda} \mathfrak{N}$, then $\mathfrak{M} \cong \mathfrak{N}$. More recently, Palyutin [5] has shown that if $V = L$, $\mathfrak{M}$ has cardinality $\aleph_1$, and $K = \{\mathfrak{N}: \mathfrak{N} \equiv_{\infty\omega_1} \mathfrak{M}$ and $\mathfrak{N} = \aleph_1\}$, then, up to isomorphism, K contains either one member or $2^{\aleph_1}$ members. It has long been known that the first case was not exclusive (cf. [4]).

For $\lambda = \aleph_1$ Palyutin needed the fact that $V = L$ implies $\diamondsuit_S$ for every stationary $S \subseteq \omega_1$. In the Theorem below, we extend Palyutin's result to most other uncountable regular cardinals. Our proof, however, requires a stronger combinatorial principle of Beller and Litman [1] which does not hold in the case of λ weakly compact, and so the restriction in the Theorem.

By Shelah [6] the *GCH* is not enough to guarantee the conclusion even for $\lambda = \aleph_1$, because the "theorem" would imply the following. For λ regular and G a λ-free group of cardinality λ, up to isomorphism $Ext(G, Z)$ has either 1 or 2^λ members. However, by [6], "$ZFC + GCH + Ext(G, Z) = Q$ for some G, $\overline{\overline{G}} = \aleph_1$" is consistent.

We now proceed to the theorem and its proof. The result was announced in [8].

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Theorem ($V = L$) *Let λ be regular and not weakly compact.*[1] *Let $\mathfrak{M}$ be a model of cardinality λ and $K = \{\mathfrak{N}: \mathfrak{N} \equiv_{\infty\lambda} \mathfrak{M}$ and $\overline{\overline{N}} = \lambda\}$. Then, up to isomorphism, K contains either 1 or 2^λ members.*

Proof: We may assume without loss of generality that $\mathfrak{M}$ has universe λ itself. For $\alpha < \lambda$ we use α^* to denote the sequence of length α whose β^{th} entry is β. We use $\bar{x}_\alpha$ to denote the sequence of variables of length α whose β^{th} entry is x_β. It is well-known (cf. [2]) that for any sequence m^* of length less than λ there is a formula φ_{m^*} of $L_{(2^\lambda)^+\lambda}$ such that for any model $\mathfrak{N}$ and sequence n^* of the same length as m^*, $\mathfrak{N} \models \varphi_{m^*}(n^*)$ iff $(\mathfrak{M}, m^*) \equiv_{\infty\lambda} (\mathfrak{N}, n^*)$. In other words, φ_{m^*} describes the $\infty\lambda$-type of m^* in $\mathfrak{M}$, $tp_{\infty\lambda}(m^*, \mathfrak{M})$.

We now define a set S of ordinals less than λ that will be used for the rest of the proof. Let

$$S = \left\{\alpha < \lambda\colon\ \mathfrak{M} \models \forall \bar{x}_\alpha\left(\bigwedge_{\beta<\alpha} \varphi_{\beta^*}(\bar{x}_\beta) \rightarrow \varphi_{\alpha^*}(\bar{x}_\alpha)\right)\right\}.$$

The proof divides into two cases, depending on whether or not S is stationary. At first, the definition of S may look a bit puzzling since the situations for limit and successor ordinals seem different. However, because we only care whether S is stationary, we are essentially only interested in the limit ordinals anyway. We consider first the case in which S is not stationary. The proof does not differ from that in [5] in any material way, but we include it here to make our paper self-contained.

Claim *If S is not stationary, then all members of K are isomorphic.*

In this case there is, by definition, a closed set C unbounded in λ and disjoint from S. Since λ is regular we may write $C = \{\delta_\alpha\colon \alpha < \lambda\}$ where δ_α is increasing and continuous in α.

Let $\mathfrak{N} \in K$. Again we may assume $\mathfrak{N}$ has universe λ. For each $\alpha < \lambda$ we will define a partial isomorphism f_α from $\mathfrak{M}$ to $\mathfrak{N}$. The domain and range of f_α will each include α. In addition, if $\beta < \alpha$, f_α will be an extension of f_β. Thus $f = \bigcup\{f_\alpha\colon \alpha < \lambda\}$ will be an isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$. It will also be arranged so that for $\alpha > 0$, f_α has domain δ_β for some $\beta \geqslant \alpha$, and so that $(\mathfrak{M}, \delta_\beta^*) \equiv_{\infty\lambda} (\mathfrak{N}, f_\alpha(\delta_\beta^*))$, where $f_\alpha(\delta_\beta^*)$ is the sequence of length δ_β whose ξ^{th} element is $f_\alpha(\xi)$.

First, for $\alpha = 0$, we let f_0 be the empty function.

Next, suppose $\alpha = \beta + 1$ and f_β has been defined with domain δ_γ so that $(\mathfrak{M}, \delta_\gamma^*) \equiv_{\infty\lambda} (\mathfrak{N}, f_\beta(\delta_\gamma^*))$. First, by the back-and-forth property, there is some $\xi < \lambda$ such that $(\mathfrak{M}, \delta_\gamma^*, \xi) \equiv_{\infty\lambda} (\mathfrak{N}, f_\beta(\delta_\gamma^*), \beta)$. Now, choose $\rho > \gamma$ such that $\xi < \delta_\rho$. Next, choose a sequence $\langle a_\nu \rangle_{\nu<\delta_\rho}$ such that $(\mathfrak{M}, \delta_\gamma^*, \xi, \delta_\rho^*) \equiv_{\infty\lambda} (\mathfrak{M}, f_\beta(\delta_\gamma^*), \beta, \langle a_\nu \rangle_{\nu<\delta_\rho})$. Now, define f_α so that $f_\alpha(\nu) = a_\nu$, for $\nu < \delta_\rho$, and f_α will extend f_β.

Finally, suppose α is a limit ordinal. This is the more interesting situation. Let us suppose that for each $\beta < \alpha$ we have defined f_β as required with domain $\gamma_\beta \in C$. Let $\mu = \bigcup_{\beta<\alpha} \gamma_\beta$. Then $\mu \in C$ since C is closed. We will let $\delta_\alpha = \mu$ and define $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. We must show that this choice will satisfy our requirements. First, the requirements on the domain and range are satisfied by induction. Since $\mu \in C$, $\mu \notin S$. Thus, $\mathfrak{M} \models \forall \bar{x}_\mu\left(\bigwedge_{\beta<\mu} \varphi_{\beta^*}(\bar{x}_\beta) \rightarrow \varphi_{\mu^*}(\bar{x}_\mu)\right)$. Since $\mathfrak{N} \equiv_{\infty\lambda} \mathfrak{M}$, $\mathfrak{N} \models$

$\forall \bar{x}_\mu\left(\bigwedge_{\beta<\mu} \varphi_{\beta^*}(\bar{x}_\beta) \to \varphi_{\mu^*}(\bar{x}_\mu)\right)$. Furthermore, since for each $\beta < \alpha$ $(\mathfrak{M},\gamma_{\beta^*}) \equiv_{\infty\lambda} (\mathfrak{N},f_\beta(\gamma_\beta^*))$, we also have $\mathfrak{N} \models \varphi_{\beta^*}(f(\gamma_{\beta^*})|\beta)$, and hence $\mathfrak{N} \models \bigwedge_{\beta<\mu} \varphi_{\beta^*}(f(\gamma_{\beta^*})|\beta)$. Consequently, we must also have $\mathfrak{N} \models \varphi_{\mu^*}(f(\mu^*))$. Then, of course, $(\mathfrak{M},\mu^*) \equiv_{\infty\lambda} (\mathfrak{N},f(\mu^*))$, and this finishes the proof in the first case.

We now must consider the more difficult case in which S is stationary. Our object is to prove the following:

Claim *If S is stationary, then K has 2^λ nonisomorphic models.*

For each $\sigma \in 2^\lambda$ we will construct a model $\mathfrak{M}_\sigma \in K$ with universe λ such that if $\sigma \neq \sigma' \in 2^\lambda$, then $\mathfrak{M}_\sigma \not\cong \mathfrak{M}_{\sigma'}$. This will, of course, prove the claim.

In order to carry out the above, for each $\alpha < \lambda$ and $\eta \in 2^\alpha$ we will define ordinals $\delta_\eta, \rho_\eta < \lambda$ and a function $f_\eta\colon \delta_\eta \xrightarrow[\text{onto}]{1-1} \rho_\eta$. We regard f_η as defining a model $\mathfrak{M}_\eta$ with universe δ_η. The isomorphism type of $\mathfrak{M}_\eta$ is obtained by letting $tp(\delta_\eta^*, \mathfrak{M}_\eta) = tp(f_\eta(\delta_\eta^*), \mathfrak{M})$, for quantifier-free formulas. We also need to control $\tau_\eta = tp_{\infty\lambda}(f_\eta(\delta_\eta^*), \mathfrak{M})$. The idea is to view $\mathfrak{M}_\eta$ as an approximation to $\mathfrak{M}_\sigma$ with universe λ where $\eta = \sigma|\alpha$. In order for this to make sense, we must arrange things so that if $\alpha < \beta$, then $\tau_{\sigma|\alpha} \subseteq \tau_{\sigma|\beta}$, though not necessarily so that $f_{\sigma|\alpha} \subseteq f_{\sigma|\beta}$. In fact, this last requirement would create serious problems when we had to "split" so as to obtain nonisomorphic models at the end. On the other hand, a certain amount of this sort of extension is necessary in order to have $\mathfrak{M}_\sigma \equiv_{\infty\lambda} \mathfrak{M}$.

In the construction we will use two combinatorial principles which hold in L. The first is that $\diamondsuit_X$ holds for each stationary $X \subseteq \lambda$. We state $\diamondsuit_X$ in the following form:

> For each $\alpha < \lambda$ there are $\eta_\alpha \neq \nu_\alpha$ and $g_\alpha\colon \alpha \to \alpha$ such that for any $\sigma \neq \sigma' \in 2^\lambda$ and $g\colon \lambda \to \lambda$,
>
> $$\{\alpha \in X\colon \sigma|\alpha = \eta_\alpha, \sigma'|\alpha = \nu_\alpha \text{ and } g|\alpha = g_\alpha\}$$
>
> is stationary in λ.

The second is due to Beller and Litman [1]:

> Let X be stationary in λ. Then there is a set $X_0 \subseteq X$, and for each limit $\alpha < \lambda$ a set C_α such that:
>
> (i) X_0 is stationary in λ
> (ii) for all $\alpha < \lambda$, $X_0 \cap \alpha$ is not stationary in α
> (iii) C_α is closed unbounded in α
> (iv) $C_\alpha \cap X_0 = 0$
> (v) if γ is a limit point of C_α, then $C_\gamma = C_\alpha \cap \gamma$.

Now, since S is stationary we may apply the Beller-Litman principle and obtain sets S_0, C_α, $\alpha < \lambda$ as described. Now we begin the details of the argument. Leaving the construction for the end, let us assume that for each $\alpha < \lambda$ and $\eta, \nu \in 2^\alpha$ we have defined $\delta_\eta, \rho_\eta < \lambda$ and f_η such that

(1) $f_\eta\colon \delta_\eta \xrightarrow[\text{onto}]{1-1} \rho_\eta$
(2) if $\beta < \alpha$, then $\delta_{\eta|\beta} \leqslant \delta_\eta$, $\rho_{\eta|\beta} \leqslant \rho_\eta$, and $\tau_{\eta|\beta} \subseteq \tau_\eta$
(3) if $\alpha \notin S_0$, α a limit ordinal, and δ is a limit point of C_α, then $f_{\eta|\delta} \subseteq f_\eta$
(4) $\alpha \subseteq \delta_\eta$, $\alpha \subseteq \rho_\eta$

(5) if $\alpha \in S_0$, $\delta_{\eta_\alpha|\beta}, \rho_{\eta_\alpha|\beta}, \delta_{\nu_\alpha|\beta}, \rho_{\nu_\alpha|\beta} < \alpha$ for all $\beta < \alpha$ and $g_\alpha\colon \alpha \xrightarrow[\text{onto}]{1-1} \alpha$, then $\langle f_{\eta_\alpha}(g(\alpha^*))\rangle$ and $\langle f_{\nu_\alpha}(\alpha^*)\rangle$ realize contradictory $L_{\infty\lambda}$-types.

We may now form models $\mathfrak{M}_\sigma$ for each $\sigma \in 2^\lambda$ as described earlier. We show now that these models behave as claimed.

A. If $\sigma \neq \sigma' \in 2^\lambda$, then $\mathfrak{M}_\sigma \not\cong \mathfrak{M}_{\sigma'}$.

Suppose to the contrary that g is an isomorphism from $\mathfrak{M}_\sigma$ onto $\mathfrak{M}'_\sigma$. In particular then, $g\colon \lambda \xrightarrow[\text{onto}]{1-1} \lambda$. It is then easy to see that the set $A = \{\alpha < \lambda\colon g\colon \alpha \xrightarrow[\text{onto}]{1-1} \alpha\}$ is closed unbounded in λ. By assumption, the set $B = \{\alpha \in S_0\colon \sigma|\alpha = \eta_\alpha, \sigma'|\alpha = \nu_\alpha$ and $g|\alpha = g_\alpha\}$ is stationary in λ. Furthermore, by condition (2), the set $C = \{\alpha < \lambda\colon \delta_{\eta_\alpha|\beta}, \rho_{\eta_\alpha|\beta}, \delta_{\nu_\alpha|\beta}, \rho_{\nu_\alpha|\beta} < \alpha$ for all $\beta < \alpha\}$ is also closed unbounded in λ. Thus $A \cap B \cap C$ is not empty. Now, for $\alpha \in A \cap B \cap C$, by condition (5) $\langle f_{\eta_\alpha}(g(\alpha^*))\rangle$ and $\langle f_{\eta_\alpha}(\alpha^*)\rangle$ realize contradictory types. Since these are respectively just the types of $g^{-1}(\alpha^*)$ in $\mathfrak{M}_\sigma$ and α^* in $\mathfrak{M}_{\sigma'}$, g is not an isomorphism, contrary to our assumption.

B. If $\sigma \in 2^\lambda$, then $\mathfrak{M}_\sigma \equiv_{\infty\lambda} \mathfrak{M}$.

Consider the set

$$F = \{f_{\sigma|\delta}\colon \delta \text{ is a limit of } C_\alpha \text{ for some } \delta < \alpha < \lambda, \alpha \notin S_0\}.$$

By definition of $\mathfrak{M}_\sigma$, F is a set of partial isomorphisms from $\mathfrak{M}_\sigma$ to $\mathfrak{M}$. By conditions (3) and (4) and the properties of the Beller-Litman family, F is seen to have the Karp back-and-forth property corresponding to $L_{\infty\lambda}$, since λ is regular.

C. The construction can be carried out.

The proof is by induction on $\alpha < \lambda$. For $\alpha = 0$ and η the empty sequence we may take $\delta_\eta = \rho_\eta = f_\eta = 0$. For $\alpha = \beta + 1$ and $\eta \in 2^\alpha$, we may disregard condition (3) and by a previous observation, since without loss of generality we may assume S_0 contains only limit ordinals, we may also disregard (5). It is quite easy to satisfy conditions (1), (2), and (4). Simply let $\delta_\eta = \delta_{\eta|\beta} + 1$, $\rho_\eta = \rho_{\eta|\beta} + 1$ and $f_\eta = f_{\eta|\beta} \cup \{\langle \delta_{\eta|\beta}, \rho_{\eta|\beta}\rangle\}$. The above conditions will then hold by induction.

For α a limit ordinal we will consider two subcases separately, determined by whether or not C_α contains a last limit point. Before doing this we make the following subclaim.

Subclaim *Without loss of generality we may assume that for every $\alpha \in S_0$, C_α has no last limit point.*

Proof: First, we consider the case in which $\lambda = \aleph_1$. It is easy to see that without loss of generality we could have assumed that S_0 contained only ordinals γ such that $\gamma > 0$ and if $\xi < \gamma$, then $\xi + \omega < \gamma$. Now, if $\alpha \in S_0$, choose $\alpha_n < \alpha$, $n < \omega$ such that $\alpha_n + \omega < \alpha_{n+1}$, α_n is a limit ordinal, and $\alpha = \bigcup_{n<\omega} \alpha_n$. Now let $C_\alpha = \{\alpha_n + k\colon n \in \omega, 1 \leqslant k \leqslant \omega\}$. Then $C_\alpha \cap S_0 = 0$ and C_α has no last limit point.

Now we consider the case in which $\lambda > \aleph_1$. First we consider the easy case in which S_0 contains only ordinals of cofinality $\geqslant \aleph_1$. Now if γ were the last limit point of C_α, then since C_α is closed unbounded in α we would have $\gamma = \gamma_0 < \gamma_1 <, \ldots,$ an ω-sequence of elements of C_α, viz., the successors of γ in

C_α increasing to α, and contradicting the assumption that α has cofinality $\geqslant \aleph_1$.

Next we consider the more general case.

Stage A: We define by induction on $n \in \omega$, for every increasing sequence v of length n of ordinals $<\lambda$, a closed, bounded subset C_v of λ with last element a limit such that

1. $C_{v|l}$ is an initial segment of C_v, for $l < n$
2. the last element of C_v is bigger than the last element of v (for $n > 0$)
3. the set $S_v = \{\delta: \delta < \lambda, C_v$ is an initial segment of C_δ, $cf(\delta) = \aleph_1\}$ is a stationary subset of λ.

The C_v's may be defined as follows. First let $C_{\langle\rangle} = \phi$. Assume C_v is defined. For each $\alpha < \lambda$ and for every $\delta \in S_v$, except for $<\lambda$ many, there is an initial segment $C_{v,\alpha,\delta}$ of C_δ with last element $\gamma(v,\alpha,\delta)$ a limit $>\alpha$, and bigger than the last element of C_v. By Fodor's Lemma there is some γ such that $\{\delta \in S_v: \gamma(v,\alpha,\delta) = \gamma\}$ is stationary. Now define $C_{v^\frown\langle\alpha\rangle} = C_\gamma$.

Stage B: Now we redefine the C_δ's to satisfy the requirements. We let

$$C^* = \{\delta: \text{if } v \in \delta^{<\omega}, \text{ then } C_v \subseteq \delta\}$$
$$S_0^* = S_0 \cap C^*$$
$$C_\delta^* = \begin{cases} C_\delta & \text{if } \delta \notin S_0^* \text{ or } cf(\delta) \geqslant \aleph_1 \\ \bigcup_{n<\omega} C_{v|n} & \text{if } \delta \text{ not as above, where} \\ & v \in \delta, \text{ increasing and unbounded.} \end{cases}$$

It is easy to check that the sets S_0^*, C_δ^*, satisfy the requirements for S, C_δ.

We now may return to the limit case of the construction. We consider first the case in which C_α has a last limit point β. We may write $C-\beta$ as $\{\beta = \beta_0, \beta_1, \beta_2, \ldots\}$ with $\beta_n < \beta_{n+1}$, $n \in \omega$. By the subclaim we may assume $\alpha \notin S_0$ and so we need not be concerned with the "splitting" condition (5). Let $\delta = \bigcup_{n\in\omega} \delta_{\eta|\beta_n}$, $\rho = \bigcup_{n\in\omega} \rho_{\eta|\beta_n}$ and $\tau = \bigcup_{n\in\omega} \tau_{\eta|\beta_n}$. Then τ is an $L_{\infty\lambda}$-type realized in $\mathfrak{M}$. To see this we use the fact that for $n < m$, $\tau_{\beta_n} \to \exists \vec{x} \tau_{\beta_m}(\vec{x})$, where $\vec{x}$ is the sequence of variables $\langle x_\xi \rangle_{\delta_{\eta|\beta_n} \leqslant \xi < \delta_{\eta|\beta_m}}$, to construct a realizing sequence (recall that τ is equivalent to a formula in $L_{\mu\lambda}$ for some fixed μ sufficiently large). Then $\tau_{\eta|\beta} \models \exists \vec{x}\, \tau[f_{\eta|\beta}(\delta_{\eta|\beta}^*), \vec{x}]$, where $\vec{x}$ is the sequence of variables $\langle x_\xi \rangle_{\delta_{\eta|\beta} \leqslant \xi < \delta}$. Now, select a sequence $a^* = \langle a_\xi \rangle_{\delta_{\eta|\beta} \leqslant \xi < \delta}$ of elements of $\mathfrak{M}$ such that $\mathfrak{M} \models \tau[f_{\eta|\beta}(\delta_{\eta|\beta}^*), a^*]$. Now, define f'_η so that f'_η extends $f_{\eta|\beta}$ and so that $f'_\eta(\xi) = a_\xi$ for $\delta_{\eta|\beta} \leqslant \xi < \delta$. Now, let ρ_η be large enough to contain range $f'_\eta \cup \rho$. Next, choose δ_η so that $\overline{\overline{\delta_\eta - \delta}} = \overline{\overline{\rho_\eta - (\text{range } f'_\eta \cup \rho)}}$. Finally extend f'_η to f_η taking the elements of $\delta_\eta - \delta$ 1 − 1 onto the elements of $\rho_\eta - (\text{range } f'_\eta \cup \rho)$ in any way whatsoever. This will satisfy conditions (1)-(4), by induction.

We now consider the case in which C_α has no last limit point. First, in view of condition (3) we may define a function f by $f = \cup \{f_{\eta|\beta}: \beta$ is a limit point of $C_\alpha\}$. Now, if $\alpha \notin S_0$ we may define $f_\eta = f$ and δ_η and ρ_η in the obvious way.

Finally let us suppose $\alpha \in S_0$ and condition (5) does apply, with all notation as in the condition. Let us also assume that $\langle f(g(\alpha^*))\rangle$ and $\langle f_{\nu_\alpha}(\alpha^*)\rangle$ do realize the same type, where f_ν is defined (without having any difficulties) as

$\bigcup\{f_{\nu_\alpha|\beta}\colon \beta$ is a limit point of $C_\alpha\}$. If they realize different types then we can just let $f_{\eta_\alpha} = f$, etc. Now we must "split". This is no problem since $\alpha \in S_0$. Simply choose a sequence $\langle a_\xi \rangle_{\xi<\alpha}$ of elements of $\mathfrak{M}$ such that for each $\beta < \alpha$, $\langle a_\xi \rangle_{\xi<\beta}$ realizes $\tau_{\eta|\beta}$, but $\langle a_\xi \rangle_{\xi<\alpha}$ and $\langle f(\alpha^*) \rangle$ realize contradictory types. Now, define $f'\colon \alpha \to M$ by $f'(\xi) = a_{g^{-1}(\xi)}$. Finally, let f_{η_α} extend f' and be from some ordinal 1–1 onto another ordinal containing $\bigcup_{\beta<\alpha} \rho_{\eta_\alpha|\beta}$ in its range. Then f_{η_α}, along with the obvious choices for δ_{η_α} and ρ_{η_α} will satisfy all conditions since $\alpha \in S_0$ and so condition (3) is vacuous.

NOTE

1. Remark added in proof: Now we know that the theorem is false for λ weakly compact; it is possible to get any number of models between 1 and λ^+. See [9].

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