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# Inexact Geometry

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*1 Introduction* In Tarski [13] elementary geometry is construed as a first-order theory, in the classical predicate calculus, of two predicates: the ternary predicate of betweenness and the quaternary predicate of equidistance.* A simplified version for the one-dimensional case, using only betweenness (and, of course, equality, which can be considered part of the logic), is given in Roberts [11]. A finite structure $(X, b)$ is a model of this theory iff there exists a real-valued function $f$ on $X$ such that for all $x, y, z \in X$

$$b(x, y, z) \text{ iff } |f(x) - f(z)| = |f(x) - f(y)| + |f(y) - f(z)|.$$

With only one very minor modification, but with equality replaced by 'indifference', Roberts shows that his axiom system becomes a theory of what he calls 'tolerance geometry'. A finite structure $(X, i, b)$ is a model of this theory, and of the theory of indifference as described in, e.g., Roberts [10], iff given any positive real number $\varepsilon$ there is a real-valued function $f$ on $X$ such that for all $x, y, z \in X$

$$\begin{array}{ll} i(x, y) & \text{iff } |f(x) - f(y)| < \varepsilon \\ b(x, y, z) & \text{iff } |f(x) - f(y)| + |f(y) - f(z)| - |f(x) - f(z)| < \varepsilon. \end{array}$$

Tolerance geometry is meant to tolerate errors (of measurement, or of perception) smaller than a fixed but arbitrary $\varepsilon$. It is thought to be particularly suitable for the geometry of visual perception. In this context $\varepsilon$ could represent

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differential threshold and $i$ indistinguishability of stimuli. If the 'gap' between two stimuli (e.g., the distance between two points) is smaller than $\varepsilon$, i.e., if they are within the threshold from each other, they are indistinguishable (and see, e.g., Zeeman [14]).

One problem with this approach is that the threshold is not a constant entity. It is defined statistically to be the number $\varepsilon$ such that if the 'distance' between two stimuli is $\varepsilon$ then the stimuli are indistinguishable in exactly half of the trials; and yet it changes from one series of trials to another. This means, in fact, that there are degrees of indistinguishability: a high degree could be attached to a pair of stimuli which are indistinguishable in most cases and a low degree to a pair of stimuli which are indistinguishable in only a few of the cases. We thus suggest that the yes-or-no notion of threshold, based on the 'half-trials' cut-point, should be replaced by a variable notion of indistinguishability. The degrees of this new variable may well be related to magnitudes of the gaps between stimuli, which in turn can sometimes be represented, as we shall soon explain, by distances along a straight line.

This idea applies also to the problem of measurement errors which is among the chief concerns of every scientist, not only the psychologist studying visual perception. In any branch of scientific research errors arising from the dispersive outcomes of repetitive experiments are not fixed and sometimes unbounded. There is a very rich literature dealing with this problem from a statistical-probabilistic point of view, but we cannot refer to it here if we want to keep this paper within manageable proportions. We hope to do so in a subsequent paper, while here we shall only be concerned with the multi-valued logical approach.

That the problem of variable errors can be treated within multi-valued logic was already noted in Goguen's work on the 'logic of inexactness' [5], which he applies to fuzzy sets and to the social sciences, in Giles' work on the 'logic of risk' [4], which he applies to physics, and in Scott's work on the 'logic of error' [12]. All these logics are essentially the Łukasiewicz logic (see Łukasiewicz [8] and Łukasiewicz and Tarski [9]), but the interpretation is new. In a structure $\underline{X}$ of this logic every statement $\varphi$ is a real-valued function on a certain power $X^n$ of the domain $X$ of $\underline{X}$, and the value of the function in a particular point $\bar{x}$ of $X^n$ is the error involved in asserting $\varphi$ in the point $\bar{x}$.

Take, for instance, the case of asserting $b$ (for 'betweenness') at the point $(x, y, z)$. In classical geometry this assertion would be true if $y$ is strictly between $x$ and $z$, and false otherwise. In tolerance geometry it will be true if $y$ is between $x$ and $z$ to within an error not exceeding some fixed $\varepsilon$, and false otherwise. In inexact geometry, which we shall axiomatize in this paper within the logic of inexactness, it will be true up to the deviation of $y$ from being between $x$ and $z$. In other words the statement can be attached to various degrees of (partial) truth; it will be fully true if $y$ is strictly between $x$ and $z$ and partly true otherwise (it will never be fully false, unless we impose an arbitrary maximal error, which is hard to justify in most cases).

Similar remarks apply to the other predicate with which we shall be concerned in this paper, namely 'metric equality' which we denote by $e$. This one should be compared with classical equality and with Roberts' indifference. In the classical case $x = y$ is true if $x$ and $y$ are identical and false otherwise. In the indifference case $i(x, y)$ is true if $x$ and $y$ are indistinguishable (i.e., identical to within $\varepsilon$) and false otherwise. In the multivalued case the (full or partial) degree of truth of $e(x, y)$ is a function of the distance between $x$ and $y$. What happens in the cases of both $e$ and $b$ is that Roberts' fixed error ($\varepsilon$) becomes variable, its values in a structure reflecting truth-values of the logic.

The logic of inexactness will be introduced in Section 2. In Sections 3 and 4 we shall axiomatize (metric) equality and betweenness within this logic so that a real-valued structure $(X, e, b)$ would be a model of our axioms iff there is a real-valued function $f$ on $X$ such that for all $x, y, z \in X$

$$e(x, y) = |f(x) - f(y)|$$

$$b(x, y, z) = \frac{1}{2}(|f(x) - f(y)| + |f(y) - f(z)| - |f(x) - f(z)|).$$

*2 **The logic of inexactness*** Let $L$ be a first-order language with

(1) a countable set $V$ of variables

(2) sets of $m$-place predicate symbols for various $m$'s in $\omega$

(3) the connectives $\wedge$('and'), $\vee$('or'), and $\rightarrow$('implies')

(4) the auxiliary symbols comma and parenthesis.

**Definition 2.1** The formulas of $L$ are defined inductively as follows:

(i) If for some $m \in \omega$, $p$ is an $m$-place predicate of $L$ and $\bar{v} \in V^m$ then $p\bar{v}$ is an atomic formula of $L$

(ii) If $\varphi$ and $\psi$ are (atomic or nonatomic) formulas of $L$ then $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \rightarrow \psi$ are formulas of $L$.

**Definition 2.2** A (real-valued) structure $\underline{X}$ for $L$ consists of

(i) a nonempty set (the domain of $\underline{X}$, to be denoted by $X$)

(ii) for each $m \in \omega$ and each $m$-place predicate $p$ of $L$ a nonnegative real-valued function on $X^m$ (the interpretation of $p$ in $\underline{X}$, to be denoted also by $p$).

We now want to define the (truth-)value of each formula of $L$ in points of the appropriate power of the domain of a given structure for $L$. In the sequel we assume that $\bar{v} \in V^n$ (for some $n \in \omega$) is a list of pairwise distinct variables of $L$, and that if $\bar{u} \in V^m$ (for $m \in \omega$) is the list, in order of appearance, of the not necessarily pair-wise distinct variables of the formula $\varphi$ of $L$, then there is a not necessarily 1-1 function $i:m \rightarrow n$ such that for all $j < m$, $\bar{u}(j) = \bar{v}(i(j))$. Then if $\underline{X}$ is a structure for $L$ and $\bar{x} \in X^n$ we write $\varphi\bar{v}\bar{x}$ for the value of $\varphi$ at the point $(\bar{x}(i(0)), \ldots, \bar{x}(i(m-1)))$ of $X^m$.

**Definition 2.3** The values of formulas of $L$ in points of (powers of) the domain $X$ of a structure $\underline{X}$ for $L$ are defined inductively as follows:

(i) If $p\bar{u}$ is an atomic formula of $L$ then

$$(p\bar{u})\bar{v}\bar{x} = p(\bar{x}(i(0)), \ldots, \bar{x}(i(m-1))),$$

with notations as above and with $p$ on the right standing for the interpretation of $p$ in $\underline{X}$.

(ii) If $\varphi$ and $\psi$ are formulas of $L$ such that $\varphi\bar{v}\bar{x}$ and $\psi\bar{v}\bar{x}$ are already defined, where $\bar{v}$ is the list of all distinct variables of $\varphi$ and $\psi$, then

$$(\varphi \wedge \psi)\bar{v}\bar{x} = \max(\varphi\bar{v}\bar{x},\ \psi\bar{v}\bar{x})$$
$$(\varphi \vee \psi)\bar{v}\bar{x} = \min(\varphi\bar{v}\bar{x},\ \psi\bar{v}\bar{x})$$
$$(\varphi \rightarrow \psi)\bar{v}\bar{x} = \psi\bar{v}\bar{x} \dot{-} \varphi\bar{v}\bar{x}\,.$$

In the last term, and throughout this paper, if $\varepsilon$ and $\delta$ are real numbers then $\varepsilon \dot{-} \delta$ stands for $\max(0, \varepsilon - \delta)$.

Let us now turn to what we call deduction expressions (in the language $L$). These are expressions of the form $\Gamma \vdash \Delta$ (read: '$\Gamma$ yields $\Delta$', or 'from $\Gamma$ deduce $\Delta$') where $\Gamma$ and $\Delta$ are finite sets of formulas of $L$. We adopt the conventions of writing $\varphi$ for $\{\varphi\}$ and $\Gamma, \Gamma'$ for $\Gamma \cup \Gamma'$ in a deduction expression, and of writing $\vdash\Delta$ when $\Gamma$ is empty.

**Definition 2.4** If, for some $n \in \omega$, $\bar{v} \in V^n$ is the list of all distinct variables of the members of $\Gamma \cup \Delta$, then the structure $\underline{X}$ for $L$ is a model of the deduction expression $\Gamma \vdash \Delta$ in $L$ if for all $\bar{x} \in X^n$

$$\max_{\varphi\in\Gamma} \varphi\bar{v}\bar{x} \geqslant \min_{\psi\in\Delta} \psi\bar{v}\bar{x}\,.$$

(Here we shall adopt the convention $\max \emptyset = 0$, where $\emptyset$ is the empty set.)

**Definition 2.5**

(i) A theory $T$ in the language $L$ is a list of deduction expressions in $L$.
(ii) The structure $\underline{X}$ for $L$ is a model of the theory $T$ in $L$ (and we write, somewhat colloquially, $\underline{X} \in T$) if it is a model of each deduction expression of $T$.

In view of our remarks on the logic of inexactness in the Introduction above it should be clear why in the definitions of this section we have reversed the usual semantics of the Łukasiewicz logic. In Definition 2.4 we say, in effect, that $\psi$ is truer than $\varphi$ if the error in asserting $\psi$ is smaller than the error in asserting $\varphi$. In Definition 2.3 (ii) we say that an 'and' assertion maximizes the error, that an 'or' assertion minimizes it, and that the error in asserting $\varphi \rightarrow \psi$ is just the degree of $\varphi$ being truer than $\psi$, as measured by $\psi - \varphi$ (while if $\psi$ is truer than $\varphi$ then there is no error in asserting $\varphi \rightarrow \psi$).

Thus absolute truth attains the value 0 (no error) in our logic. There is no absolute falsehood since there is no maximal potential error, the logic being valued in $[0, \infty)$. This is also the reason why we did not include the negation connective in the language $L$, quite apart from the fact that we do not need negation for what we intend to do in this paper. However, it should be clear that nothing of what we do here would change if we decide to fix an arbitrary maximal admissible error, 1 say, in our logic. Then the range of values would be $[0, 1]$, and negation would be defined in the usual way by

$$\sim\varphi\bar{v}\bar{x} = 1 - \varphi\bar{v}\bar{x}\,,$$

with $\varphi$, $\bar{v}$ and $\bar{x}$ as in Definition 2.3, which in fact is short for

$$\sim\varphi\bar{v}\bar{x} = (\varphi \rightarrow \text{'absolute falsehood'})\,\bar{v}\bar{x}\ .$$

This definition of negation is, of course, purely semantic. So was most of the discussion of the logic of error in this section. A syntactic treatment of this logic, including partial axiomatization (by means of Gentzen-style inference rules), as well as extensions to languages with quantifiers, is to be provided in a subsequent paper (Katz [7]).

*3* ***Metric and linear models*** Let $L$ be a first-order language as before, with a binary predicate $e$ representing equality.

It was noted by Scott [12] that, within the logic of error, equality need not, and should not, be two-valued as it had been in almost all previous treatments of real-valued logic (e.g., in Chang [1], in Fenstand [3], and in Chang and Keisler [2]). Equality in a structure $\underline{X}$ should instead be perceived as a measure of distance between elements of $X$. The error in asserting that $x$ and $y$ are equal increases as the distance between $x$ and $y$ increases. So, the closer are $x$ and $y$ to each other the truer is the assertion that $x$ and $y$ are equal. In this way equality, like any other predicate, may obtain any value in $[0, \infty)$, and we can safely say that "all elements are equal, but some are more equal than others".

**Definition 3.1** The theory $ME$ (in $L$) of metric equality is:

(re) $\vdash e(u, u)$
(sy) $e(u, v) \vdash e(v, u)$
(tr) $e(u, v) \vdash e(v, w) \rightarrow e(w, u)$

where $u, v, w \in V$.
Here (re) stands for reflexivity, (sy) for symmetry, and (tr) for transitivity.

**Definition 3.2** If $\underline{X} \in ME$ then $\underline{X}$ is said to be a metric model, or a model with metric equality.

**Lemma 3.3** *If $\underline{X}$ is a metric model then for all $x, y, z \in X$*

($\underline{\text{re}}$) $e(x, x) = 0$
($\underline{\text{sy}}$) $e(x, y) = e(y,x)$
($\underline{\text{tr}}$) $e(x, y) \geqslant e(z, x) \dot{-} e(y, z)$.

*Proof:* Obvious.

So, if $\underline{X}$ is a metric model then $e$ is a real-valued pseudo-metric on $X$. This seems to be a natural generalization of the two-valued case, since in two-valued logic equality is just the two-valued metric on the domains of the appropriate structures. The requirement that in the real-valued case equality should also be the two-valued metric seems to be less plausible.

In view of these remarks it is now clear that in Definition 3.1 the reason we have transitivity in the form (tr) instead of full-transitivity in the form

(ft) $e(u, v), e(v, w) \vdash e(w, u)$

(note that in real-valued logic $\psi \wedge \theta \to \chi$ is in general stronger than $\psi \to (\theta \to \chi)$), is that for an arbitrary (pseudo-) metric $e$ on a set $X$ it is certainly not true in general that

$$\max(e(x, y), e(y, z)) \geqslant e(z, x)$$

for all $x, y, z \in X$.

Perhaps the first structure that comes to mind when we think about metric models is the structure $\underline{R}$ whose domain is the set $R$ of real numbers and whose equality is measured by the ordinary absolute-value distance. We shall now provide conditions necessary and sufficient for a metric model to be embeddable in $\underline{R}$. Models satisfying these conditions will be called linear models, and the theory appropriate for these models will be called the theory of linear equality.

**Definition 3.4** The theory $LE$ of linear equality is: (sy), (tr),

$$\begin{aligned}
(\mathrm{li})_1 \quad & \vdash (e(u, v) \to e(v, w)) \to e(w, u), \\
& (e(w, u) \to e(u, v)) \to e(v, w), \\
& (e(v, w) \to e(w, u)) \to e(u, v),
\end{aligned}$$

$$\begin{aligned}
(\mathrm{li})_2 \quad & (e(v, w) \to e(w, u)) \to e(u, v), (e(t, w) \to e(w, u)) \to e(u, t) \\
& \vdash (e(v, t) \to e(t, u)) \to e(u, v), (e(t, v) \to e(v, u)) \to e(u, t),
\end{aligned}$$

where $u, v, w, t \in V$.

**Definition 3.5** If $\underline{X} \in LE$ then $\underline{X}$ is said to be a linear model, or a model with linear equality.

**Lemma 3.6** *If $\underline{X}$ is a linear model then for all $x, y, z, p \in X$ we have* $(\underline{\mathrm{sy}})$, $(\underline{\mathrm{tr}})$,

$$\begin{aligned}
(\underline{\mathrm{li}})_1 \quad & \text{either } e(z, x) \dot{-} (e(y, z) \dot{-} e(x, y)) = 0 \\
& \text{or } e(y, z) \dot{-} (e(x, y) \dot{-} e(z, x)) = 0 \\
& \text{or } e(x, y) \dot{-} (e(z, x) \dot{-} e(y, z)) = 0
\end{aligned}$$

$$\begin{aligned}
(\underline{\mathrm{li}})_2 \quad & \max(e(x, y) \dot{-} (e(z, x) \dot{-} e(y, z)), e(x, p) \dot{-} (e(z, x) \dot{-} e(p, z))) \\
& \geqslant \min(e(x, y) \dot{-} (e(p, x) \dot{-} e(y, p)), e(x, p) \dot{-} (e(y, x) \dot{-} e(p, y))).
\end{aligned}$$

*Proof:* Obvious again. Note that $(\underline{\mathrm{re}})$ follows from $(\underline{\mathrm{li}})_1$, so that a linear model is in particular a metric model.

It can be seen from $(\underline{\mathrm{li}})_1$ and $(\underline{\mathrm{li}})_2$ that the intuitive meaning of the first axiom of linearity $(\mathrm{li})_1$ is that for each three "points" one is "between" the two others, and that the intuitive meaning of the second axiom of linearity $(\mathrm{li})_2$ is that if both $v$ and $t$ are "between" $u$ and $w$, then $v$ and $t$ cannot be on "different sides" of $u$. This will be seen more clearly from $(\underline{\mathrm{li}})_1^*$ and $(\underline{\mathrm{li}})_2^*$ below, and still more clearly from $(b)_1$ and $(be)_2$ of Section 4.

**Lemma 3.7** *Let $e$ be a $[0, \infty)$-valued function on $X^2$ for some nonempty set $X$. Then* $(\underline{\mathrm{sy}})$, $(\underline{\mathrm{tr}})$, $(\underline{\mathrm{li}})_1$ *hold for all $x, y, z \in X$ iff the following holds for all $x, y, z \in X$:*

$$\begin{aligned}
(\underline{\mathrm{li}})_1^* \quad & \text{either } e(y, z) = e(z, x) + e(x, y) \\
& \text{or } e(x, y) = e(y, z) + e(z, x) \\
& \text{or } e(z, x) = e(x, y) + e(y, z).
\end{aligned}$$

*Proof:* Note that $(\underline{\text{re}})$ follows from $(\underline{\text{li}})_1^*$. Use this to derive $(\underline{\text{sy}})$ from $(\underline{\text{li}})_1^*$ by substituting $x$ for $z$. Each case of $(\underline{\text{li}})_1$ follows from the corresponding case of $(\underline{\text{li}})_1^*$. If $(\underline{\text{tr}})$ does not hold then none of the cases in $(\underline{\text{li}})_1^*$ can hold, so $(\underline{\text{tr}})$ also follows from $(\underline{\text{li}})_1^*$.

Conversely each case of $(\underline{\text{li}})_1^*$ follows from $(\underline{\text{tr}})$ together with the corresponding case in $(\underline{\text{li}})_1$. Symmetry $(\underline{\text{sy}})$ is needed in order to allow us to interchange $e(y, z)$ with $e(z, x)$ in $(\underline{\text{tr}})$ when $e(z, x) < e(y, z)$.

**Lemma 3.8** *Let e and X be as in the previous lemma, and consider the following*

$$(\underline{\text{li}})_2^* \qquad \begin{aligned} &\max(e(x, y) + e(y, z) - e(z, x), e(x, p) + e(p, z) - e(z, x)) \\ &\geqslant \min(e(x, y) + e(y, p) - e(p, x), e(x, p) + e(p, y) - e(y, x)) \end{aligned}$$

*for all $x, y, z, p \in X$. Then we have*

(i) $(\underline{\text{li}})_2 \rightarrow (\underline{\text{li}})_2^*$

(ii) $(\underline{\text{li}})_1^* + (\underline{\text{li}})_2^* \rightarrow (\underline{\text{li}})_2$.

*Proof:* That (i) is true is obvious since $(\underline{\text{li}})_2^*$ is just the special case of $(\underline{\text{li}})_2$ where all the $\dot{-}$'s are $-$'s.

For (ii) note first that by $(\underline{\text{li}})_1^*$ (in fact, just by $(\underline{\text{tr}})$), the $\dot{-}$'s following $e(x, y)$ and $e(x, p)$ in $(\underline{\text{li}})_2$ are actually $-$'s. If one of the $\dot{-}$'s following $e(z, x)$ in $(\underline{\text{li}})_2$ is not a $-$, then we have $e(x, y)$ (or $e(x, p)$) in the antecedent of $(\underline{\text{li}})_2$ and $e(x, y) - \varepsilon$ (or $e(x, p) - \varepsilon$), for some $\varepsilon \geqslant 0$, in the conclusion of $(\underline{\text{li}})_2$, so that $(\underline{\text{li}})_2$ clearly holds.

Now assume that all the $\dot{-}$'s in the antecedent of $(\underline{\text{li}})_2$ are in fact $-$'s, so that $(\underline{\text{li}})_2$ and $(\underline{\text{li}})_2^*$ have the same antecedent. By $(\underline{\text{li}})_1^*$ there are three possibilities for $x, y, p$:

(1) $e(x, y) = e(y, p) + e(p, x)$
(2) $e(p, x) = e(x, y) + e(y, p)$
(3) $e(y, p) = e(p, x) + e(x, y)$.

Using $(\underline{\text{sy}})$ (which follows from $(\underline{\text{li}})_1^*$ as was mentioned above), under each of (1) and (2) we have 0 in the conclusion of $(\underline{\text{li}})_2$, while under (3) we have

$$\min(e(x, y), e(x, p))$$

in the conclusions of both $(\underline{\text{li}})_2$ and $(\underline{\text{li}})_2^*$.

**Lemma 3.9** *Let e and X be as in the previous lemmas and consider the following:*

$$(\underline{\text{li}})_2^{**} \qquad \begin{aligned} &\textit{if} \quad e(y, p) = e(x, z) = e(x, y) + e(y, z) \\ &\textit{and} \quad e(y, z) = e(x, p) \\ &\textit{and} \quad e(x, y) = e(z, p) \\ &\textit{then}\ e(x, p) = 0 \textit{ or } e(x, y) = 0 \end{aligned}$$

*for all $x, y, z, p \in X$. Then we have*

(i) $(\underline{\text{li}})_2^* \rightarrow (\underline{\text{li}})_2^{**}$

(ii) $(\underline{\text{li}})_1^* + (\underline{\text{li}})_2^{**} \rightarrow (\underline{\text{li}})_2^*$.

*Proof:* Again (i) is trivial since $\underline{(\mathrm{li})}_2^{**}$ is a special case of $\underline{(\mathrm{li})}_2^{*}$, and (ii) is proved by checking various cases.

**Corollary 3.10** *Let $\underline{X}$ be a real-valued model for a language L with a binary predicate e representing equality. The following are equivalent:*

(i) *$\underline{X}$ is a linear model*
(ii) *$\underline{(\mathrm{li})}_1^{*} + \underline{(\mathrm{li})}_2^{*}$ hold for all $x, y, z, p \in X$*
(iii) *$\underline{(\mathrm{li})}_1^{*} + \underline{(\mathrm{li})}_2^{**}$ hold for all $x, y, z, p \in X$.*

*Proof:* By the last three lemmas.

The reasons for introducing all these forms of $(\mathrm{li})_1$ and $(\mathrm{li})_2$ are that

(1) $\underline{(\mathrm{li})}_1$ and $\underline{(\mathrm{li})}_2$ are the direct translations of $(\mathrm{li})_1$ and $(\mathrm{li})_2$

(2) $\underline{(\mathrm{li})}_1^{*}$ and $\underline{(\mathrm{li})}_2^{*}$ are the direct translations of two axioms of betweenness given in the following section

(3) $\underline{(\mathrm{li})}_1^{*}$ and $\underline{(\mathrm{li})}_2^{**}$ are the easiest to use in proofs of some of the following theorems.

We are now in a position to prove the main theorem of this section, the theorem of embeddability (or representation) of linear models. We only sketch the proof here, since a detailed version can be found in an earlier paper [6].

**Theorem 3.11** *The structure $\underline{X}$ (for a language L with a binary predicate e representing equality) is a linear model iff there is a function $f: X \to R$ such that for all $x, y \in X$*

$$e(x, y) = |f(x) - f(y)|.$$

*Proof:* That if such an $f$ exists then $\underline{X}$ satisfies $\underline{(\mathrm{li})}_1^{*}$ and $\underline{(\mathrm{li})}_2^{**}$ is easy to check. For the converse assume $\underline{X}$ satisfies $\underline{(\mathrm{li})}_1^{*}$ and $\underline{(\mathrm{li})}_2^{**}$. Then in particular $\underline{X}$ satisfies $\underline{(\mathrm{re})}$, $\underline{(\mathrm{sy})}$, and $\underline{(\mathrm{tr})}$, so that $e$ is a pseudo-metric on $X$. Now, if the number of equivalence classes of $X$ mod $e = 0$ is 1 the proof is trivial. If not we fix arbitrary $x, y \in X$ such that $e(x, y) > 0$ and for $z \in X$ we define

$$f(z) = \begin{cases} e(x, z) \text{ if } e(y, z) \leqslant \max(e(x, y), e(x, z)) \\ -e(x, z) \text{ if } e(y, z) > \max(e(x, y), e(x, z)). \end{cases}$$

The proof that the function $f$ thus defined on $X$ to $R$ satisfies the claim of the theorem is technical and rather laborious. We have to compute $e(z, p)$ in terms of $f$ for arbitrary $z, p \in X$, checking separately each possible position of $z$ and $p$, relative to our fixed $x$ and $y$, according to $\underline{(\mathrm{li})}_1^{*}$. There are quite a few cases and subcases to check. In most cases we obtain the required result using $\underline{(\mathrm{li})}_1^{*}$ alone. Only in one single case we have to use $\underline{(\mathrm{li})}_2^{**}$ also, so that this condition cannot be considered very 'strong' or 'important'.

In fact it is shown in [6] that only in the case where the number of equivalence classes of $X$ mod $e = 0$ is exactly 4, $\underline{(\mathrm{li})}_2^{**}$ is independent of $\underline{(\mathrm{li})}_1^{*}$. In all other cases $\underline{(\mathrm{li})}_1^{*}$ implies $\underline{(\mathrm{li})}_2^{**}$. Thus the deduction $(\mathrm{li})_2$ can be dropped from the theory $\underline{LE}$ of Definition 3.4 without much altering the appropriate class of models. We call the theory thus obtained the theory of semi-linear equality (to be denoted by SL) and its models semi-linear models. Obviously every linear model is semi-linear. It follows from the remarks in this paragraph that the converse is also necessarily true unless the model has exactly 4 equivalence classes mod $e = 0$.

*4 Betweenness* The notion of betweenness was implicit in our discussion of linear equality, and we now want to make it explicit. We start with a language $L$ containing variables $u, v, w, t, \ldots$, a binary predicate $e$ representing equality and a ternary predicate $b$ representing betweenness.

**Definition 4.1** The theory $LB$ of linear betweenness in $L$ is:

$SL$,
$(\mathrm{b})_1$ $\vdash b(u, v, w), b(v, w, u), b(w, u, v)$
$(\mathrm{b})_2$ $b(t, u, v), b(u, v, w) \vdash e(u, v), b(t, u, w)$
$(\mathrm{b})_3$ $b(u, v, w), b(v, t, w) \vdash b(u, v, t), e(v, w) \wedge e(t, w)$
$(\mathrm{b})_4$ $b(u, v, w) \vdash b(w, v, u)$
$(\mathrm{b})_5$ $b(u, v, u) \vdash e(u, v)$
$(\mathrm{b})_6$ $(e(v, w) \to e(w, u)) \to e(u, v) \vdash b(u, v, w)$.

We remark that this system differs from the one in Roberts [11] in that instead of the theory $SL$ of semi-linear equality Roberts starts with his theory of the indifference predicate, and instead of $(\mathrm{b})_5$ and $(\mathrm{b})_6$ he has

$[\mathrm{b}]_5$ $b(u, v, w), b(v, u, w) \vdash e(u, v), e(v, w) \wedge e(u, w)$
$[\mathrm{b}]_6$ $e(u, v) \vdash b(u, v, w)$.

The reason why in the real-valued case we have to start with something other than the indifference relation is explained in the Introduction to this paper. The reason we replace $[\mathrm{b}]_5$ by $(\mathrm{b})_5$ is that the latter is simpler in its formulation and the two are equivalent in the presence of the remaining axioms. In the two-valued case this equivalence is easy to establish using the fact that the theory of indifference contains reflexicity and symmetry for $e$ (or $I$ in Roberts's notation). In the real-valued case this equivalence is also easy to verify as we shall soon show. Finally, the reason we replace $[\mathrm{b}]_6$ by $(\mathrm{b})_6$ is that whereas in the two-valued case the two are equivalent in the presence of the other axioms (including, we note again, reflexivity and symmetry for $e$), in the real-valued case, as we shall soon see, $(\mathrm{b})_6$ is stronger than $[\mathrm{b}]_6$, and it is the stronger version that we shall need for our purposes.

**Definition 4.2** Let $\underline{X}$ be a real-valued structure for $L$. If $\underline{X} \in LB$ then $\underline{X}$ is said to be a linear betweenness model.

**Lemma 4.3** *If $\underline{X}$ is a linear betweenness model then for all $x, y, z, p \in X$ we have $(\underline{\mathrm{sy}})$, $(\underline{\mathrm{tr}})$, $(\underline{\mathrm{li}})_1$ and*

$(\underline{\mathrm{b}})_1$ $\min(b(x, y, z), b(z, x, y), b(y, z, x)) \leqslant 0$
$(\underline{\mathrm{b}})_2$ $\max(b(p, x, y), b(x, y, z)) \geqslant \min(e(x, y), b(p, x, z))$
$(\underline{\mathrm{b}})_3$ $\max(b(x, y, z), b(y, p, z)) \geqslant \min(b(x, y, p), \max(e(y, z), e(p, z)))$
$(\underline{\mathrm{b}})_4$ $b(x, y, z) \geqslant b(z, y, x)$
$(\underline{\mathrm{b}})_5$ $b(x, y, x) \geqslant e(x, y)$
$(\underline{\mathrm{b}})_6$ $e(x, y) \dot{-} (e(z, x) \dot{-} e(y, z)) \geqslant b(x, y, z)$.

*Proof:* Obvious. It is also obvious that the inequalities in $(\underline{b})_1$ and $(\underline{b})_4$ can be replaced by equalities.

**Lemma 4.4** *If $\underline{X}$ is a linear betweenness model then for all $x, y, z \in X$ we have*

(re) $e(x, x) = 0$
$[\underline{b}]_5$ $\max(b(x, y, z), b(y, x, z)) \geqslant \min(e(x, y), \max(e(y, z), e(x, z)))$
$[\underline{b}]_6$ $e(x, y) \geqslant b(x, y, z)$.

(Note that $[\underline{b}]_5$ and $[\underline{b}]_6$ are the translations in $\underline{X}$ of $[\mathrm{b}]_5$ and $[\mathrm{b}]_6$.)

*Proof:* We have already noted in the previous section that $(\underline{\mathrm{re}})$ follows from $(\underline{\mathrm{li}})_1$. It is obvious that $[\underline{b}]_6$ follows from $(\underline{b})_6$. Finally, in the presence of $(\underline{\mathrm{re}})$, $(\underline{\mathrm{sy}})$, $(\underline{b})_4$ and $[\underline{b}]_6$, $(\underline{b})_5$ implies that $[\underline{b}]_5$ is just a special case of $(\underline{b})_3$.

**Corollary 4.5** *If $\underline{X}$ is a linear betweenness model then for all $x, y \in X$*

$$e(x, y) = b(x, y, x).$$

*Proof:* By $(\underline{b})_5$ and $[\underline{b}]_6$.

Note that if $\underline{X}$ is a real-valued structure satisfying

$(*)$ $SL, (\mathrm{b})_1, (\mathrm{b})_2, (\mathrm{b})_3, (\mathrm{b})_4, [\mathrm{b}]_5, [\mathrm{b}]_6$

then it is easy to check that $\underline{X}$ also satisfies $(\mathrm{b})_5$. However, in that case $\underline{X}$ does not necessarily satisfy $(\mathrm{b})_6$, for let $X = \{x, y, z\}$, fix an $\varepsilon > 0$ and define:

$$\begin{aligned} b(x, y, z) = b(y, z, x) = e(x, y) &= e(y, z) = \varepsilon \\ e(x, z) &= 2\varepsilon \\ b(y, x, z) &= 0. \end{aligned}$$

To obtain a structure $\underline{X}$, determine the remaining values for $e$ and $b$ by means of $(\underline{\mathrm{re}})$, $(\underline{\mathrm{sy}})$, $(\underline{b})_4$, $(\underline{b})_5$, and $[\underline{b}]_6$. It then requires some simple computations to check that $\underline{X}$ satisfies all the axioms listed in $(*)$. Yet $\underline{X}$ does not satisfy $(\mathrm{b})_6$ since

$$e(x, y) \dot{-} (e(x, z) \dot{-} e(y, z)) = 0$$

while

$$b(x, y, z) = \varepsilon > 0.$$

We now turn to one of the central theorems of this section.

**Theorem 4.6** *If $(X, e)$ is a semi-linear model then $(X, e, b)$ is almost a linear betweenness model iff for all $(x, y, z) \in X$*

$$b(x, y, z) = \begin{cases} e(x, y) & \text{if } e(y, z) = e(z, x) + e(x, y) \\ e(y, z) & \text{if } e(x, y) = e(y, z) + e(z, x) \\ 0 & \text{if } e(z, x) = e(x, y) + e(y, z)\,. \end{cases}$$

*Equivalently, iff for all $x, y, z \in X$*

$$b(x, y, z) = \frac{1}{2}\,(e(x, y) + e(y, z) - e(z, x)).$$

(Here by "almost" we mean $(X, e, b)$ is a model of $LB$ except for $(\mathrm{b})_2$.)

*Proof:* That these two conditions are equivalent is trivial. If one of these conditions holds it is a matter of routine computation to check that $(\underline{X}, e, b)$ satisfies all of $(\mathrm{b})_1$, $(\mathrm{b})_3$, $(\mathrm{b})_4$, $(\mathrm{b})_5$, $(\mathrm{b})_6$.

Now assume $(X, e, b)$ is almost a linear betweenness model. Note first that by $(\underline{\mathrm{b}})_6$

if $\quad e(x, y) + e(y, z) - e(z, x) = 0$
then $\quad b(x, y, z) = 0$.

On the other hand assume that

(i) $\quad e(x, y) + e(y, z) - e(z, x) \neq 0$.

Then by $(\underline{\mathrm{li}})_1^*$ we have

either (ii) $\quad e(z, x) + e(x, y) - e(y, z) = 0$
or (iii) $\quad e(y, z) + e(z, x) - e(x, y) = 0$.

Without loss of generality assume (ii). Then by $(\underline{\mathrm{b}})_6$:

$$b(y, x, z) = 0.$$

If we assume that also

$$b(x, y, z) = 0,$$

we obtain from $[\underline{\mathrm{b}}]_5$

$$e(x, y) = 0.$$

Together with (ii), this implies

$$e(x, y) + e(y, z) - e(z, x) = 0,$$

contradicting (i).
We conclude that

$$b(x, y, z) = 0 \text{ iff } e(x, y) + e(y, z) - e(z, x) = 0.$$

Thus we can concentrate on the case where $b(x, y, z) \neq 0$ and either (ii) or (iii) above holds. In that case, by $(\underline{\mathrm{b}})_1$ we have

either $\quad b(y, x, z) = 0$
or $\quad b(x, z, y) = 0$.

Each of these, together with $[\underline{\mathrm{b}}]_5$, $[\underline{\mathrm{b}}]_6$, and $(\underline{\mathrm{re}})$ (and using symmetry of betweenness $(\mathrm{b})_4$ and of equality $\overline{(\mathrm{sy})}$, if needed) implies:

$$b(x, y, z) = \begin{cases} e(x, y) & \text{if (ii)} \\ e(y, z) & \text{if (iii)} \end{cases}$$

and so the proof is complete.

The following immediate corollary to Theorem 4.6 shows that the law of substitution of equals holds for $b$. In this corollary, and in its proof, if $a \in A$ for some set $A$, and $\bar{a} \in A^m$ for some $m \in \omega$, then $\bar{a}[i/a]$, for $i < m$, is the element of $A^m$ obtained from $\bar{a}$ by substituting $a$ for $\bar{a}(i)$.

**Corollary 4.7** *Every linear betweenness model satisfies*

$$e(u, v) \vdash b(u_0, u_1, u_2)[i/u] \to b(u_0, u_1, u_2)[i/v]$$

*where* $u, v, u_0, u_1, u_2 \in V$ *and* $i \leqslant 2$.

*Proof:* $\underline{X}$ satisfies this corollary if and only if for each $i \leqslant 2$ and for all $x, y, x_0, x_1, x_2 \in X$

$$e(x, y) \geqslant b(x_0, x_1, x_2)[i/y] - b(x_0, x_1, x_2)[i/x].$$

To see that this is indeed the case for any linear betweenness model $\underline{X}$, substitute the values for $b$ by the second condition in Theorem 4.6 and then apply $(\underline{\mathrm{tr}})$.

**Lemma 4.8** *In the theory LB of linear betweenness* $(\mathrm{b})_2$ *can be replaced by each of* $(\mathrm{li})_2$ *and*

$(\mathrm{be})_2$ $\quad b(u, v, w), b(u, t, w) \vdash b(u, v, t), b(u, t, v).$

*Proof:* $\underline{X}$ satisfies $(\mathrm{be})_2$ iff for all $x, y, z, p \in X$

$(\underline{\mathrm{be}})_2$ $\quad \max(b(x, y, z), b(x, p, z)) \geqslant \min(b(x, y, p), b(x, p, y)).$

Substituting the values for $b$ by the second equivalent condition in Theorem 4.6, and multiplying both sides of the inequality by 2, we get exactly $(\underline{\mathrm{li}})_2^*$, which is equivalent to $(\underline{\mathrm{li}})_2$ in presence of $(\underline{\mathrm{li}})_1^*$. Thus for a linear betweenness model $\underline{X}$ (where clearly $(\underline{\mathrm{li}})_1^*$ holds) $(\mathrm{be})_2$ and $(\mathrm{li})_2$ are equivalent.

To show that $(\mathrm{li})_2$ can replace $(\mathrm{b})_2$ in $LB$, let $\underline{X}$ be a model of $LB$. Then $(\underline{\mathrm{li}})_2$ holds in $\underline{X}$ iff $(\underline{\mathrm{li}})_2^{**}$ holds in $\underline{X}$ (since $(\underline{\mathrm{li}})_1^*$ holds in $\underline{X}$). But $(\underline{\mathrm{li}})_2^{**}$ is just a special case of $(\underline{\mathrm{b}})_2$. So $(\underline{\mathrm{b}})_2$ implies $(\underline{\mathrm{li}})_2^{**}$, while $(\underline{\mathrm{li}})_2^{**}$ implies one of the cases of $(\underline{\mathrm{b}})_2$. Now it is a matter of simple computation to show that Theorem 4.6 implies the remaining cases of $(\underline{\mathrm{b}})_2$ (and note that $(\underline{\mathrm{b}})_2$ was not used in the proof of Theorem 4.6).

The significance of this lemma is that

(1) it shows that our intuitive understanding of $(\mathrm{li})_2$ in the previous section was correct
(2) it gives two additional formulations of $LB$
(3) it implies immediately the following corollary, which apart from being of interest in itself will be used in the proof of Theorem 4.10 (the embedding theorem).

**Corollary 4.9** *Every linear betweenness model is in particular a linear model.*

*Proof:* $SL + (\mathrm{li})_2$ is exactly the theory $LE$ of linear equality.

We can now easily prove the embedding theorem for linear betweenness models.

**Theorem 4.10** *Let* $\underline{X}$ *be a real-valued structure for a language L with a binary predicate e and a ternary predicate b. Then* $\underline{X}$ *is a linear betweenness model iff there is a map* $f: X \to R$ *such that for all* $x, y, z \in X$

$$b(x, y, z) = \frac{1}{2}\left(|f(x) - f(y)| + |f(y) - f(z)| - |f(x) - f(z)|\right)$$

$$e(x, y) = |f(x) - f(y)|.$$

*Proof:* If such an $f$ exists we require again computations as before to show that $\underline{X}$ is a linear betweenness model.

Conversely, if $\underline{X}$ is a linear betweenness model then by Corollary 4.9 it is in particular a linear model. Then by the embedding theorem for linear models (Theorem 3.11) there is a function $f$ satisfying the above condition for $e$, and by Theorem 4.6 this $f$ also satisfies the required condition for $b$.

Looking through the various results in this section, we see that in a certain sense all that there is to be said about linear betweenness is already being said by the axioms of linear equality. This "certain sense" is made precise in the following theorem and corollary.

**Theorem 4.11** *The theory LB of linear betweenness is* (sy), (tr), $(\mathrm{li})_2$ +

$(\mathrm{be})_1$ $\vdash((e(u, v) \to e(v, w)) \to e(w, u)) \wedge (b(u, v, w) \longleftrightarrow e(u, v)),$
$((e(w, u) \to e(u, v)) \to e(v, w)) \wedge (b(u, v, w) \longleftrightarrow e(v, w)),$
$((e(v, w) \to e(w, u)) \to e(u, v)) \wedge b(u, v, w).$

*Proof:* Axiom $(\mathrm{be})_1$ is just an amalgamation of $(\mathrm{li})_1$ and a formulation in $L$ of the first condition in Theorem 4.6. Note that $\varphi \longleftrightarrow \psi$ is short for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. Obviously its (truth-)value is given by the absolute difference.

**Corollary 4.12** *If $(X, e)$ is a linear model then a ternary real-valued relation $b$ can be defined on $X$ such that $(X, e, b)$ is a linear betweenness model.*

*Proof:* For any $x, y, z \in X$ define $b(x, y, z)$ by means of one of the two equivalent conditions in Theorem 4.6.

On the other hand, we can give a formulation of $LB$ in which the predicate $b$ plays the main role. Here we start with $(X, e)$ being just a metric model (i.e., a model of the theory $ME$ of metric equality).

**Theorem 4.13** *The theory LB of linear betweenness is*

$$ME, (\mathrm{b})_1, (\mathrm{b})_2, (\mathrm{b})_4, (\mathrm{b})_6, (\mathrm{b})_7,$$

*where* $(\mathrm{b})_7$ *is*

$$b(u, v, w) \vdash (e(v, w) \to e(w, u)) \to e(u, v), (e(u, v) \to e(w, u)) \to e(v, w).$$

*Proof:* It is easy to see that a real-valued structure $(X, e, b)$ satisfies the axioms listed in this theorem iff it satisfies the axioms listed in Theorem 4.11.

Note by the way that $(\mathrm{b})_7$ (like $(\mathrm{li})_1$ which appeared in all previous formulations of $LB$) is not necessarily true in the two-valued case.

In a rather artificial way we can obtain a corollary to Theorem 4.13 which corresponds to Corollary 4.12 of Theorem 4.11. We do this by letting $LB^*$ be the theory obtained from that in Theorem 4.13 by replacing, in each of the axioms, every $e$ expression by the appropriate $b$ expression (the appropriate $b$ expression for $e(u, v)$ is $b(u, v, u)$).

**Corollary 4.14** *If $(X, b)$ is a model of $LB^*$ then a binary real-valued relation $e$ can be defined on $X$ such that $(X, e, b)$ is a linear betweenness model.*

*Proof:* For any $x, y \in X$ define $e(x, y)$ by means of the condition in Corollary 4.5.

We summarize this section as follows: we gave various formulations of the theory $LB$ of linear betweenness. We proved an embedding theorem, similar to that in Roberts [11], which fits our intuition about inexact betweenness. Unlike Roberts we did not have to restrict attention to finite models (even the fact that, since the function $f$ of Theorems 3.11 and 4.10 is clearly 1-1 on equivalence classes mod $e = 0$, the number of such classes cannot exceed $2^\omega$, is a result, not a precondition, of our theory), or to impose a fixed maximal error $\varepsilon$ which would have contradicted our understanding of the logic of inexactness. Thus we can consider $LB$ an axiomatization of one-dimensional inexact geometry. Perhaps this deserves further study into the $n$-dimensional case and the inexact equivalent of the second predicate of Tarski [13], namely the equi-distance predicate.

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