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Solution to a Problem of Chang and Lee

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In this note we show that input resolution with paramodulation (IP) is strictly weaker than unit resolution with paramodulation (UP).

First we introduce some notation. A is always an atomic sentence and p, q are always statement letters. $\vdash_X$, for X = IP, UP, I (input resolution), or U (unit resolution), means derivability by means of the rules of X.

We work in a fixed first-order language and consider only ground clauses. E is the set of all clauses of the form $\{\neg t_0 = t_1, \neg A(t_i), A(t_{1-i})\}$ together with all those of the form $\{t = t\}$.

A set L of literals is consistent if $\neg \exists l \in L\, \bar{l} \in L$.

If L is a consistent set of literals and C is a clause we say $L \approx C$ if $L \cap C \neq \emptyset$ or $\exists l_1 \in C\, \exists l_2 \in C\, l_1 \neq l_2 \wedge \bar{l}_1 \notin L \wedge \bar{l}_2 \notin L$.

If C_1 and C_2 are clauses define $[C_2/p]C_1 = C_1$ if $p \notin C_1$, $[C_2/p]C_1 = (C_1 - \{p\}) \cup C_2$ if $p \in C_1$. If S is a set of clauses define $[C_2/p]S = \{[C_2/p]C_1 \colon C_1 \in S\}$.

Substitution lemma *Suppose there is a UP derivation of C_1 from S with no clause containing $\neg p$ and with $\{p\}$ at most as its last clause, then for each C_2 there is a $C_3 \subset [C_2/p]C_1$ such that $[C_2/p]S \vdash_{\text{UP}} C_3$.*

The proof of the substitution lemma is routine.

Soundness lemma *If L is a consistent set of literals and S a set of clauses then $L \approx S \cup E \Rightarrow S \cup E \nvdash_{\text{I}} \emptyset$.*

Proof: Prove by induction on the length of an input derivation of C from $S \cup E$ that $\exists l \in C\, (l \in L \vee \bar{l} \notin L)$.

Completeness lemma *If S is a set of clauses then there is a consistent set of literals L such that $S \cup E \nvdash_{\text{I}} \emptyset \Rightarrow L \approx S \cup E$.*

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Proof: Let L be the set of all literals l such that $S \cup E \vdash_{\mathrm{U}} \{l\}$. If $S \cup E \nvdash_{\mathrm{I}} \Phi$ then $S \cup E \nvdash_{\mathrm{U}} \Phi$ so L is consistent. Let $C = \{l_1, \ldots, l_n\} \in S \cup E$. If $n = 1$ then $l_1 \in L$ so $L \mathrel{|\!\approx} C$. If $n > 1$ and $\bar{l}_1, \ldots, \bar{l}_{n-1} \in L$ then $S \cup E \vdash_{\mathrm{U}} \{l_n\}$ so $l_n \in L$ and $L \mathrel{|\!\approx} C$. Thus $L \mathrel{|\!\approx} S \cup E$.

The soundness and completeness lemmas tell us that input resolution with equality axioms is complete for the three-valued semantics represented by consistent sets of literals.

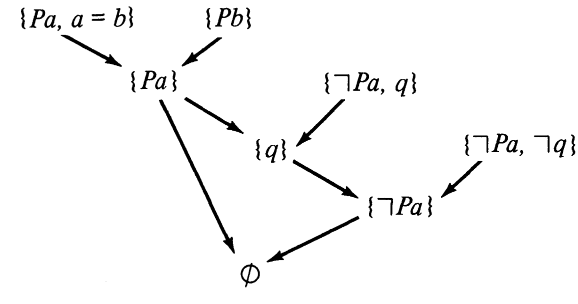
Proposition $\quad S \vdash_{\mathrm{IP}} \Phi \Rightarrow S \cup E \vdash_{\mathrm{I}} \Phi$.

Proof: Prove by induction on the length of an IP derivation of C from S that if L is a consistent set of literals with $L \mathrel{|\!\approx} S \cup E$ then $\exists l \in C\ (l \in L \vee \bar{l} \notin L)$. Thus $S \cup E \nvdash_{\mathrm{I}} \Phi \Rightarrow S \nvdash_{\mathrm{IP}} \Phi$ by the completeness lemma.

Proposition $\quad S \cup E \vdash_{\mathrm{I}} \Phi \Rightarrow S \vdash_{\mathrm{UP}} \Phi$.

Proof: Suppose $S \nvdash_{\mathrm{UP}} \Phi$ and select a new statement letter p. Define $L = \{A: S \vdash_{\mathrm{UP}} \{A\}\} \cup \{\neg A: S \cup \{\{A, p\}\} \vdash_{\mathrm{UP}} \{p\}\}$. By the substitution lemma with $C_2 = \Phi$, L is a consistent set of literals. We shall show that for $C \in S \cup E$ $L \mathrel{|\!\approx} C$.
Case 1. $C = \{A_1, \ldots, A_n, \neg A_{n+1}, \ldots, \neg A_{n+m}, l\} \in S$ and $\neg A_1, \ldots, \neg A_n, A_{n+1}, \ldots, A_{n+m} \in L$. Since $S \nvdash_{\mathrm{UP}} \Phi$, by at most m applications of the substitution lemma $S \vdash_{\mathrm{UP}} \{l\}$ so $l \in L$.
Case 2. $C = \{\neg t_0 = t_1, \neg A(t_i), A(t_{1-i})\} \in E$.
Subcase 1. $t_0 = t_1, A(t_i) \in L$. Since $S \vdash_{\mathrm{UP}} \{t_0 = t_1\}$ and $S \vdash_{\mathrm{UP}} \{A(t_i)\}$, $S \vdash_{\mathrm{UP}} \{A(t_{1-i})\}$ so $L \mathrel{|\!\approx} C$.
Subcase 2. $t_0 = t_1, \neg A(t_{1-i}) \in L$. Since $S \vdash_{\mathrm{UP}} \{t_0 = t_1\}$ and $S \cup \{\{A(t_{1-i}), p\}\} \vdash_{\mathrm{UP}} \{p\}$, we have $S \cup \{\{A(t_i), p\}\} \vdash_{\mathrm{UP}} \{p\}$ so $\neg A(t_i) \in L$ and $L \mathrel{|\!\approx} C$.
Subcase 3. $A(t_i), \neg A(t_{1-i}) \in L$. Since $S \vdash_{\mathrm{UP}} \{A(t_i)\}$ and $S \cup \{\{A(t_{1-i}), p\}\} \vdash_{\mathrm{UP}} \{p\}$, we have $S \cup \{\{t_0 = t_1, p\}\} \vdash_{\mathrm{UP}} \{p\}$ so $\neg t_0 = t_1 \in L$ and $L \mathrel{|\!\approx} C$.
Thus by the soundness lemma $S \cup E \nvdash_{\mathrm{I}} \Phi$.

We now show that $S \vdash_{\mathrm{UP}} \Phi \not\Rightarrow S \cup E \vdash_{\mathrm{I}} \Phi$. We specify the first-order language to contain only the constants a, b, the monadic predicate P, and the statement letter q (together with equality). Let $S = \{\{Pa, a = b\}, \{Pb\}, \{\neg Pa, q\}, \{\neg Pa, \neg q\}\}$ and let $L = \{a = a, b = b, Pb\}$, then L is a consistent set of literals and it is easily verified that $L \mathrel{|\!\approx} S \cup E$. Thus $S \cup E \nvdash_{\mathrm{I}} \Phi$ by the soundness lemma. However, $S \vdash_{\mathrm{UP}} \Phi$ as can be seen from the following UP derivation:



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