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THE ALGEBRA OF RELATIVES

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The term "algebra of relatives" is sometimes regarded as being no more than an outmoded name for the algebra of relations. At best it is associated with the methods and notation developed by C. S. Peirce and E. Schröder. It is true that in Peirce's later work, which was taken up by Schröder, a "relative" is regarded as a set of ordered pairs, hence as a relation. Yet there is evidence that in Peirce's early work he thought of a relative as being something entirely different: a collection of individuals, each of which is related in a given way to some other individual. In the course of its development the algebra of relatives thus metamorphosed into an algebra of relations. The aim of this paper is to see how much sense can be made of an algebra of relatives, where "relative" is taken in its Peircean sense. I first give a sketch of the historical background, then an informal account of relatives and their behaviour, and finally consider how this might be given a sound set-theoretical basis. It turns out that the informal "algebra" of relatives holds this title only by courtesy; nevertheless, by means of a certain artifice, one can give a set-theoretical imitation of it which is an algebra in the strict sense of the word.

1 In Chapter 7 of the *Categories* Aristotle introduces the notion of a relative as follows ([1], 6a36):

> We call *relatives* all such things as are said to be just what they are, *of* or *than* other things, or in some way *in relation to* something else.

The first impression one gets from this passage is that certain individuals are called relatives, namely those that are always referred to in relation to something else, but this impression is countered by the following remark in the translator's Notes:

> He [Aristotle] does not say that 'larger' and 'slave' are relatives, but that the larger and the slave are relatives. However, he does not, of course, mean that, for example, the slave Callias is a relative (he is a substance), but that Callias is a relative in so far as he is *called a slave*; in other words, 'slave' is a relative term.

This passage points out the fundamental difficulty in the conception of a

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relative. When we think of slaves in the abstract, the fact that any slave must be a slave *of* someone inclines us to say that any slave is a relative. But when we come to the individual, to Callias the slave, we find him to be a substance, an independent being. To resolve this difficulty, it seems, Aristotle's translator recommends that relatives are to be neglected in favour of relative terms. But the fact is that a relative term should have some denotation, and it seems natural to say that what it denotes is a relative. Thus, for example, the relative term "slave" is most naturally regarded as denoting the class of slaves; accordingly, the class of slaves must be regarded as a relative. If necessary, an individual member of this or any other class regarded as a relative may then be called an *individual* relative. This, basically, is Peirce's idea in [5], his first paper on the logic of relatives, published in 1870. He classifies logical terms into three classes: absolute terms, (simple) relative terms, and conjugatives; these correspond roughly to monadic, dyadic, and triadic predicates. Concerning relative terms Peirce says:

> The second class embraces terms whose logical form involves the conception of relation, and which require the addition of another term to complete the denotation. These discriminate objects with a distinct consciousness of discrimination. They regard an object as over against another, that is as relative; as father of, lover of, or servant of.

The important point here is that a relative term regards *objects* as relative—precisely the approach cautioned against by Aristotle's translator in the passage quoted above. It is significant that at the time of writing his 1870 paper Peirce was committed to the Aristotelian view that all propositions, including relational ones, are of the subject-copula-predicate form. In pressing relational propositions into this mould he was led to treat relative terms as substantives, rather than verbs. Thus he spoke of lovers instead of loving, of servants instead of serving, and so on (this point is made in [2]). It is then natural to regard relative terms as denoting relatives rather than relations. In [3] it is argued from a consideration of the technicalities in Peirce's 1870 paper that, on the whole, he thought of himself as being concerned with relatives rather than relations, and much the same view is expressed in [4]. But the obscurity of Peirce's exposition of relative terms makes it difficult to say for certain that he thought of relative terms as consistently denoting either relatives or relations, or even that he properly understood the distinction between these two notions. The difficulty is compounded by the fact that he frequently failed to distinguish between individuals and classes of individuals. For this reason the discussion of relatives to which I now turn is independent of Peirce's work.

2 By a *relative* I mean the domain of a binary relation: if R is a relation then its domain

$$\mathbf{D}(R) = \{x \mid (\exists y)[xRy]\}$$

is a relative. The class of lovers is an example of a relative: x is a lover if and only if there is someone whom x loves. A relative is therefore a class of individuals, each of which is related in a given way, which is standard for all the individuals in the class, to some other individual, or to itself. Since relatives are classes of individuals, the usual set-theoretical operations can be applied to them. Thus, taking the class of lovers and the class of servants as examples, one can form their complements, intersection, and union. These *absolute* operations correspond to the following linguistic modifications of the relative terms "lover" and "servant":

Absolute negation: x is *not a lover* if and only if it is not the case that x loves someone—that is, if and only if x does not love anybody.

Absolute conjunction: x is a *lover and a servant* if and only if there is someone whom x loves and there is someone whom x serves—not necessarily the same individual.

Absolute disjunction: x is a *lover or a servant* if and only if: either there is someone whom x loves, or there is someone whom x serves.

To see what further operations can be defined on relatives I consider further modifications of relative terms. Corresponding to the three above are three more:

Relative negation: x is a *non-lover* if and only if there is someone whom x does not love.

Relative conjunction: x is a *lover-and-servant* if and only if there is someone whom x both loves and serves.

Relative disjunction: x is a *lover-or-servant* if and only if there is someone of whom it can be said that x either loves him or serves him.

The set-theoretical operations corresponding to these linguistic modifications I call *relative* complementation, intersection, and union. There are two further operations on relatives; the linguistic modifications corresponding to them are:

Conversion: x is a *loved one* if and only if there is someone who loves x.

Composition: x is a *lover of a servant* if and only if there is someone, y, such that x loves y and y is a servant.

The class of non-lovers is clearly the domain of the absolute complement of the relation of loving, it is therefore a relative. Similarly the class of lovers-and-servants is the domain of the intersection of the relation of loving and the relation of serving, and the class of lovers-or-servants is the domain of their union. The class of loved ones is the domain of the converse $L^{\smile}$ of the relation L of loving, defined by:

$$L^{\smile} = \{\langle y, x\rangle \mid xLy\}.$$

And a little computation shows that the class of lovers of servants is the domain of the relative product $L|S$ of the relation L of loving with the relation S of serving, defined by:

$$L|S = \{\langle x, y\rangle \mid (\exists z)[xLz \ \& \ zSy]\}.$$

A number of points can be made concerning the interplay between operations on relatives. First consider absolute and relative negation. If x is not a lover then he does not love anybody, hence (assuming a nonempty universe) there is someone he does not love and so he is a non-lover. Thus absolute negation implies relative negation. "Lover" and "not a lover" are clearly contradictory terms, but "lover" and "non-lover" are only subcontraries: an individual must be either a lover or a non-lover, and he can be both. The term "not a non-lover" is unambiguous: x is not a non-lover if and only if it is not the case that there is someone whom x does not love—that is, if and only if x loves everybody. This term and "not a lover" are contraries: x cannot both not be a lover and not be a non-lover, and he need not be either. So the four terms under consideration are connected by the standard square of opposition of quantificational logic.

Relative conjunction implies absolute conjunction: if there is someone whom x both loves and serves then there is someone whom he loves and also someone whom he serves. As for relative and absolute disjunction, these must coincide: if there is someone of whom it is known that x either loves him or serves him then it is also known that either there is someone whom x loves or there is someone whom x serves, and conversely. Concerning the converse of a relative little can be said: there is no fixed logical relation between a relative and its converse. The converse of the class of lovers is the class of loved ones, and the converse of this relative is again the class of lovers, *provided* that the relation with respect to which the class of loved ones is regarded as a relative is the relation of being loved by someone—that is, the converse of the relation of loving. If this assumption is not made the converse of the converse of a relative is not necessarily the original relative. Finally, as regards composition of relatives: if x is a lover of a servant then he is still a lover, hence the composition of two relatives, in a given order, is contained in the first relative.

Earlier I pointed to the fundamental difficulty in the conception of a relative: whether an individual (or a class) is to be called a relative depends on whether or not it is viewed in the light of its relation to something else (or the relation of its elements to other individuals). So the first problem in constructing an algebra of relatives is to demarcate the universe of discourse. I propose to admit all sets as relatives, simply because any set is the domain of some relation. And I propose to bring out the behaviour of sets regarded as relatives by giving a technical treatment of their behaviour under *relative operations*. Again there is a problem: any set is the domain of a number of distinct relations, hence any set can be regarded as a relative in many different ways. This is a problem because operations on relatives are to be defined as certain kinds of operations on sets. To clarify the problem, let X and Y be any sets, contained in a universe $\mathbf{U}$, and let R and S be relations such that:

$$X = \mathbf{D}(R) \text{ and } Y = \mathbf{D}(S)$$

Now consider the effect of the operations considered above on X and Y. We get:

(1) Absolute complementation of X yields : $\mathbf{D}(R)'$
Absolute intersection of X and Y yields: $\mathbf{D}(R) \cap \mathbf{D}(S)$
Absolute union of X and Y yields : $\mathbf{D}(R) \cup \mathbf{D}(S)$
Relative complementation of X yields : $\mathbf{D}(R')$
Relative intersection of X and Y yields : $\mathbf{D}(R \cap S)$
Relative union of X and Y yields : $\mathbf{D}(R \cup S)$
Conversion of X yields : $\mathbf{D}(R^{\smile})$
Composition of X and Y yields : $\mathbf{D}(R | S)$.

The problem is that some of these "operations" on relatives are not operations at all, in the sense that they are not single-valued. For example, it is possible that:

$$\mathbf{D}(R) = X = \mathbf{D}(P) \text{ and } \mathbf{D}(S) = Y = \mathbf{D}(Q)$$

where

$$R \cap S \neq \Lambda \text{ and } P \cap Q = \Lambda.$$

In which case we have:

$$\text{Relative intersection of } X \text{ and } Y = \mathbf{D}(R \cap S) \neq \emptyset$$

and also

$$\text{Relative intersection of } X \text{ and } Y = \mathbf{D}(P \cap Q) = \emptyset.$$

The problematical "operations" are relative complementation, relative intersection, conversion, and composition. (Relative union is not a problem because it coincides with absolute union: $\mathbf{D}(R \cup S) = \mathbf{D}(R) \cup \mathbf{D}(S)$.) The fact that these "operations" are not single-valued means that, strictly speaking, the "algebra" of relatives is not an algebra at all, for the standard definition states that an algebra is a class of objects together with a number of single-valued operations on the objects in this class. However, the word "algebra" also has a less rigorous meaning: it indicates that the objects in a certain class exhibit some kind of structure. That there is an "algebra" of relatives in this sense is shown by the informal discussion above.

3 The problem to which I now turn is to find a set-theoretical imitation of this intuitively understood structure which will be an algebra in the narrow sense of the word. The method I propose is this: instead of the four problematical "operations" on relatives, I distinguish four *classes* of operations. For example, the relative complement of a set X depends on which relation X is taken to be the domain of; once this relation is fixed the relative complement is unique, otherwise not. So a distinct operation of relative complementation can be distinguished for every distinct relation R that X is domain of. Similarly for conversion. Also, for any sets X and Y

a distinct operation of relative intersection can be distinguished for every two relations R and S such that X is the domain of R and Y is the domain of S, and similarly for composition. Thus operations on relatives are to be indexed by relations.

A few preliminary results will be useful. The following are known results concerning relations:

R1. $(R \cap S)^{\smile} = R^{\smile} \cap S^{\smile}$ and $R^{\smile\smile} = R$
R2. $\mathbf{D}(R)' \subseteq \mathbf{D}(R')$
R3. $\mathbf{D}(R \cap S) \subseteq \mathbf{D}(R) \cap \mathbf{D}(S)$
R4. $\mathbf{D}(R \cup S) = \mathbf{D}(R) \cup \mathbf{D}(S)$
R5. $\mathbf{D}(R | S) \subseteq \mathbf{D}(R)$.

Given a class X contained in a universe $\mathbf{U}$, the largest relation with domain X is the Cartesian product $X \times \mathbf{U}$. I will frequently use such relations, so I introduce the following abbreviation. I write:

$$X^* \text{ for } X \times \mathbf{U}.$$

The following results are easily proved:

R6. $(X \cap Y)^* = X^* \cap Y^*$ and $(X')^* = (X^*)'$
R7. $\mathbf{D}(X^*) = X$ and $\mathbf{D}((X^*)^{\smile}) = \mathbf{U}$, provided that $X \neq \emptyset$
R8. $R \subseteq (\mathbf{D}(R))^*$
R9. $\mathbf{D}(R \cap X^*) = \mathbf{D}(R) \cap X$
R10. $\mathbf{D}((R \cap X^*) | S) = \mathbf{D}(R | S) \cap X$.

These results furnish the necessary background for the technical treatment of the algebra of relatives. I will consider the set $\mathcal{P}(\mathbf{U})$ consisting of all subsets of a universal set $\mathbf{U}$. Relations over $\mathbf{U}$ are subsets of $\mathbf{V} = \mathbf{U} \times \mathbf{U}$, hence elements of $\mathcal{P}(\mathbf{V})$. Besides the (absolute) operations of complementation, intersection, and union I will define on the elements of $\mathcal{P}(\mathbf{U})$ certain relative operations which are indexed by elements of $\mathcal{P}(\mathbf{V})$. I use the following notations for the relative operations:

Relative complement of X with respect to R:	$\mathsf{C}_R(X)$
Relative intersection of X and Y with respect to R and S:	$\mathsf{I}_{RS}(X, Y)$
Converse of X with respect to R:	$\mathsf{K}_R(X)$
Composition of X and Y with respect to R and S:	$\mathsf{M}_{RS}(X, Y)$.

From (1) we get the following preliminary characterization of the relative operations:

(2) *If* $X = \mathbf{D}(R)$ *and* $Y = \mathbf{D}(S)$ *then:*

$$\begin{aligned} \mathsf{C}_R(X) &= \mathbf{D}(R') \\ \mathsf{I}_{RS}(X, Y) &= \mathbf{D}(R \cap S) \\ \mathsf{K}_R(X) &= \mathbf{D}(R^{\smile}) \\ \mathsf{M}_{RS}(X, Y) &= \mathbf{D}(R | S). \end{aligned}$$

That these are operations in the sense of being single-valued is clear from the fact that $'$, $\cap$, $^{\smile}$, and $|$ are operations on relations, and that the mapping of relations onto their domains is single-valued. However, (2) is not suitable as a definition of the relative operations. If the relative operations are to be indexed, then a relative operation on any set must be defined for any index. Thus, for example, $\mathsf{C}_R(X)$ must be defined even when $X \neq \mathbf{D}(R)$—a requirement not satisfied by (2). Hence (2) must be extended in such a way that any relative operation is defined for any set(s) with respect to any operation(s). Moreover, this must be done in such a way that (2) is not invalidated. Both these requirements are satisfied, as will be shown, by the following definitions:

D1. $\mathsf{C}_R(X) = \mathbf{D}((R \cap X^*)')$
D2. $\mathsf{I}_{RS}(X, Y) = \mathbf{D}((R \cap X^*) \cap (S \cap Y^*))$
D3. $\mathsf{K}_R(X) = \mathbf{D}((R \cap X^*)^{\smile})$
D4. $\mathsf{M}_{RS}(X,Y) = \mathbf{D}((R \cap X^*)\mathbf{|}(S \cap Y^*))$.

For any set $X \subseteq \mathbf{U}$, X^* is a relation contained in $\mathbf{V}$, hence $R \cap X^*$ is defined for any $R \subseteq \mathbf{V}$. The operations $'$, $\cap$, $^{\smile}$, and $\mathbf{|}$ are defined for such relations, the result is again a relation contained in $\mathbf{V}$, and hence its domain is a set contained in $\mathbf{U}$. D1-D4 thus satisfy the first requirement above: any relative operation is defined, and is a single-valued operation, for any set(s) contained in $\mathbf{U}$ and any relation(s) contained in $\mathbf{V}$. Besides the relative operations, grouped into four classes, there are also the standard set-theoretical operations on elements of $\mathbf{U}$. Abstractly seen, the algebra of relatives thus comes out as an algebra of the form:

$$\langle \mathcal{P}(\mathbf{U}), ', \cap, \mathsf{I}_{RS}, \mathsf{C}_R, \mathsf{M}_{RS}, \mathsf{K}_R \rangle_{R,S \in \mathcal{P}(\mathbf{V})}.$$

The main defect of the informal "algebra" of relatives, that it is not an algebra in the strict sense of the word, has thus been eliminated. The operation of set-theoretical union and the constants $\emptyset$ and $\mathbf{U}$ do not appear in the definition of the algebra since they are definable in terms of the other operations. I now proceed to develop this algebra. The first step is to simplify D1-D4.

T1 $\mathsf{C}_R(X) = \mathbf{D}(R') \cup X'$.

Proof: $\mathsf{C}_R(X) = \mathbf{D}((R \cap X^*)') = \mathbf{D}(R' \cup (X^*)') = \mathbf{D}(R' \cup (X')^*)$ by R6
$= \mathbf{D}(R') \cup X'$ by R4 and R7.

T2 $\mathsf{I}_{RS}(X, Y) = \mathbf{D}(R \cap S) \cap X \cap Y$.

Proof: $\mathsf{I}_{RS}(X, Y) = \mathbf{D}((R \cap X^*) \cap (S \cap Y^*)) = \mathbf{D}((R \cap S) \cap (X \cap Y)^*)$ by R6
$= \mathbf{D}(R \cap S) \cap X \cap Y$ by R9.

T3 $\mathsf{K}_R(X) = \mathbf{D}(R^{\smile} \cap (X^*)^{\smile})$.

Proof: From D3 by R1.

T4 $\mathsf{M}_{RS}(X, Y) = \mathbf{D}(R\mathbf{|}(S \cap Y^*)) \cap X$.

Proof: From D4 by R10.

It is now easy to show that D1-D4 also conform to the second requirement mentioned above: they do not invalidate (2). This is established by the next four theorems.

T5 *If* $X = \mathbf{D}(R)$, *then* $\mathsf{C}_R(X) = \mathbf{D}(R')$.

Proof: If $X = \mathbf{D}(R)$, then $X' = \mathbf{D}(R)'$.
Hence $\mathsf{C}_R(X) = \mathbf{D}(R') \cup \mathbf{D}(R)'$ by T1 and hence $\mathsf{C}_R(X) = \mathbf{D}(R')$ by R2.

T6 *If* $X = \mathbf{D}(R)$ *and* $Y = \mathbf{D}(S)$, *then* $\mathsf{I}_{RS}(X, Y) = \mathbf{D}(R \cap S)$.

Proof: If $X = \mathbf{D}(R)$ and $Y = \mathbf{D}(S)$, then
$\mathsf{I}_{RS}(X, Y) = \mathbf{D}(R \cap S) \cap \mathbf{D}(R) \cap \mathbf{D}(S)$ by T2. The conclusion follows by R3.

T7 *If* $X = \mathbf{D}(R)$, *then* $\mathsf{K}_R(X) = \mathbf{D}(R^{\smile})$.

Proof: If $X = \mathbf{D}(R)$ then $R \subseteq X^*$ by R8. Hence $R \cap X^* = R$, hence $(R \cap X^*)^{\smile} = R^{\smile}$ and so $\mathsf{K}_R(X) = \mathbf{D}(R^{\smile})$ by D3.

T8 *If* $X = \mathbf{D}(R)$ *and* $Y = \mathbf{D}(S)$, *then* $\mathsf{M}_{RS}(X, Y) = \mathbf{D}(R \mid S)$.

Proof: If $X = \mathbf{D}(R)$ then $R \subseteq X^*$ by R8; similarly $S \subseteq Y^*$. Hence, from D4, $\mathsf{M}_{RS}(X, Y) = \mathbf{D}(R \mid S)$.

It is now easy to verify all those connections between the operations on relatives observed before. From T1 we get:

T9 $X' \subseteq \mathsf{C}_R(X)$.

T9 shows that X and $\mathsf{C}_R(X)$ are jointly exhaustive but not necessarily mutually exclusive; also that X' and $(\mathsf{C}_R(X))'$ are mutually exclusive but not necessarily jointly exhaustive. This conforms to the earlier observation that absolute and relative negation are connected by the square of opposition for quantificational logic. From T2 it is clear that relative intersections are contained in absolute intersections:

T10 $\mathsf{I}_{RS}(X, Y) \subseteq X \cap Y$.

As for conversion, D3 does not impose any fixed logical relation between a set and its converses: $\mathsf{K}_R(X)$ may be empty without either R or X being empty, it may also be the universal set $\mathbf{U}$, namely when X is nonempty and X^* is contained in R (by R7). The converse of the converse of a set does not in general coincide with that set; as a special case, however, we obtain:

T11 *If* $X = \mathbf{D}(R)$, *then* $\mathsf{K}_{R^{\smile}}(\mathsf{K}_R(X)) = X$.

Proof: If $X = \mathbf{D}(R)$, then $\mathsf{K}_R(X) = \mathbf{D}(R^{\smile})$ by T7. Hence

$$\begin{aligned}\mathsf{K}_{R^{\smile}}(\mathsf{K}_R(X)) &= \mathbf{D}((R^{\smile} \cap \mathbf{D}(R^{\smile})^*)^{\smile}) \text{ by D3}\\ &= \mathbf{D}((R^{\smile})^{\smile}) \text{ by R8}\\ &= \mathbf{D}(R) = X \text{ by R1.}\end{aligned}$$

Finally, from T4 we see that:

T12 $\mathsf{M}_{RS}(X, Y) \subseteq X$.

The set-theoretical algebra constructed here is thus seen to be a reasonable imitation of the informal "algebra" of relatives.

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