Notre Dame Journal of Formal Logic
Volume XX, Number 4, October 1979
NDJFAM

VARIABLE BINDING TERM OPERATORS IN λ-CALCULUS

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A variable binding term operator (vbto) is any operator which binds one or more variables in a term or well formed formula. We will represent an arbitrary vbto by v and the result of applying it to a term or wf $F(x)$ by $\mathrm{v}xF(x)$. Examples are proper integrals ($\mathrm{v}xF(x) = {}_a\int^b F(x)dx$), all integral transforms, λ abstraction ($\mathrm{v}xF(x) = \lambda xF(x)$), the quantifiers ($\mathrm{v}xF(x) = \forall xF(x)$ or $\exists xF(x)$), the class forming operator ($\mathrm{v}xF(x) = \{x: F(x)\}$), and the description operators ι and ε ($\mathrm{v}xF(x) = \iota xF(x)$ or $\varepsilon xF(x)$).

Most references such as [4] and [6] give two characteristic axioms for vbtos:

A1 $\mathrm{v}xF(x) = \mathrm{v}yF(y)$

where y is free in $F(y)$ exactly where x was free in $F(x)$ and vice versa,

and

A2 $\forall x(F(x) \equiv G(x)) \supset \mathrm{v}xF(x) = \mathrm{v}xG(x)$.

Clearly, however, about half of our examples, as they concern terms rather than well formed formulas, do not fit A2, so we use as an alternative the weaker:

A2′ $\forall x(F(x) = G(x)) \supset \mathrm{v}xF(x) = \mathrm{v}xG(x)$.

This version of A2 is given for a particular vbto in [4] and was mentioned in [7].

Clearly λ-abstraction is a vbto which satisfies A1 and A2′, but conversely any vbto satisfying A1 and A2 can be represented using λ-abstraction and another operator that is not variable binding. If we replace $\mathrm{v}xF(x)$ by $\mathbf{V}(\lambda xF(x))$ in Axioms A1 and A2′, these become theorems of the λβ-calculus, which includes the axioms:

(α) *If y does not occur free in X*, $\lambda y[y/x]X = \lambda xX$
(β) $(\lambda xX)Y = [Y/x]X$

Received April 24, 1978

(μ) $X = Y \Rightarrow ZX = ZY$
(ξ) $X = Y \Rightarrow \lambda xX = \lambda xY$.[1]

The stronger version A2 can be obtained in an illative combinatory logic such as that of [1] in which first order predicates and sets of classes are identified[2] and which has extensionality.

If $\mathbf{FAH}X$ and $\mathbf{FAH}Y$ (i.e., if X and Y are sets (or classes) and also first order predicates) we have by extensionality:

$$Au \supset_u (Xu \equiv Yu) \;.\!\supset\!.\; X = Y^3$$

If we add to our pure λ-calculus the axiom:

(η) $\lambda x(Mx) = M$, where x is not free in M,

we obtain by (ξ) and (μ):

$$Au \supset_u (Xu \equiv Yu) \;.\!\supset\!.\; \mathbf{V}(\lambda u(Xu)) = \mathbf{V}(\lambda u(Yu)),$$

which is exactly A2.

We now show that for two particular vbtos which satisfy A2 the V can also be defined in terms of illative combinatory logic.

$\{x: F(x)\}$, the class of all x such that $F(x)$, may be characterized simply as $\lambda xF(x)$, or as F, thus $\mathsf{V} = \mathbf{I}$.

The Hilbert symbol ι is a description operator, $\iota xF(x)$ is the unique x such that $F(x)$ if there is one. For the special case of a system where all individuals are sets (i.e., there are no "ur-elementen"), ι becomes simply the sum set operator defined in [1] by:

$$\{\mathbf{Un}\} = \lambda y\, \lambda x\ .\, \Sigma\mathbf{A}(\lambda u(ux \wedge yu))^4$$

Clearly, if the u such that yu is unique, $\{\mathbf{Un}\}y = u$.

NOTES

1. See [5] for these and the other axioms of λ-calculus.

2. The identification of sets (and classes) and first order predicates was made in [1] and is discussed in [2] and [3].

3. $\mathbf{A}$ is the class of individuals over which quantification ranges. "$\mathbf{A}u \supset_u \ldots$" thus means "for all u in $\mathbf{A} \ldots$". [1] uses "$\sim$" instead of "$\equiv$" for "if and only if".

4. "$\Sigma\,\mathbf{A}(\lambda u \ldots)$" is interpreted as "$\exists u \ldots$".

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