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NOTE ON A STRONG LIBERATED MODAL LOGIC AND ITS RELEVANCE TO POSSIBLE WORLD SKEPTICISM

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We consider a modal language with a set of sentence parameters $P = \{P_1, P_2, \ldots\}$ and the symbols L (necessity), $\supset$, and $\neg$ for sentential connectives. We will consider axiomatic systems based on the following axiom schemes:

A1 All tautologies
A2 $L(E \supset E') \supset (LE \supset LE')$.

As rules of proof, we assume the following:

MP If E and $E \supset E'$ are provable, then so is E'.
Nec If E is provable, then so is LE.

Consider the two following axiom schemes:

A3 $LE \supset E$
A4 $E \supset LE$.

If we add both A3 and A4 to our basic system, we obtain the trivial system in which for any expression E, LE is equivalent to E. The classical systems adopt A3 and exclude A4 to obtain **T** and stronger systems (see [1]). Murungi [6] has given a formal proof of the claim that the system obtained by adding A4 to the basic system, sketched above, is consistent. The system with A4 is called **T'**.

While it is true that **T'** is consistent and does not collapse into the trivial system, **T'** comes *very* close to collapsing into the trivial system, as we shall show by providing a semantics and completeness proof for **T'**. The system **T'** turns out to be one of the so-called liberated modal logics. It is a very strong logic containing all of the liberated systems discussed in [2]-[5].

The semantic range is the usual $\{\mathbf{t},\mathbf{f}\}$. We designate by G an indexed set of functions g mapping the set of sentence parameters P into the semantic range. An interpretation is an ordered triple (g, W, R) where: $g \in G$; $W \subseteq G$; R is a binary relation (accessibility) defined on W; if W is not empty, then

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$g \in W$. The valuation function V is defined on the set of expressions as follows:

$$
\begin{array}{ll}
V(E,g,W,R) & = g(E), \text{ for } E \in P \\
V(\neg E,g,W,R) & = \mathbf{t}, \text{ if } V(E,g,W,R) = \mathbf{f} \\
 & = \mathbf{f}, \text{ otherwise} \\
V(E \supset E',g,W,R) & = \mathbf{t}, \text{ if } V(E,g,W,R) = \mathbf{f} \text{ or } V(E',g,W,R) = \mathbf{t} \text{ (or both)} \\
 & = \mathbf{f}, \text{ otherwise} \\
V(LE,g,W,R) & = \mathbf{t}, \text{ if } V(E,g',W,R) = \mathbf{t} \text{ for all } g' \text{ such that } (g,g') \in R \\
 & = \mathbf{f}, \text{ otherwise.}
\end{array}
$$

For standard modal systems W is required to be non-empty. For the liberated systems, W is allowed to be empty. Characteristic semantic theories for different systems are obtained by imposing various restrictions on R and W. For any system, say **S**, we say that an interpretation is an **S** interpretation just in case the **S** restrictions on W and R are satisfied. We say that an expression is **S** valid just in case it is assigned the value **t** in every **S** interpretation. We write "$\mathbf{S} \Vdash E$" to indicate that E is **S** valid. We write "$\mathbf{S} \vdash E$" to indicate that E is provable in **S**.

Intuitively, a "possible world" is uniquely described by the set of all expressions true under a given interpretation. Once we specify W and R, each function g determines a possible world in one and only one way. So we speak of W as the set of possible worlds.

For the characteristic semantics for $\mathbf{T}'$, we require of W that it either be empty or contain at most one member. We require of R that it be reflexive on W. Of course if W is empty, then R is empty as well.

We can define the conjunction of E with E' as $\neg(E \supset \neg E')$. For any finite set of expressions λ, we mean by $C\lambda$ the conjunction of all members of λ. We say that a finite set of expressions λ is $\mathbf{T}'$ consistent just in case not $\mathbf{T}' \vdash \neg C\lambda$. An infinite set is $\mathbf{T}'$ consistent just in case every finite subset is $\mathbf{T}'$ consistent. An expression E is said to be $\mathbf{T}'$ consistent just in case $\{E\}$ is $\mathbf{T}'$ consistent. A set λ is said to be maximally $\mathbf{T}'$ consistent just in case λ is $\mathbf{T}'$ consistent and for any expression E, either $E \in \lambda$ or $\lambda \cup \{E\}$ is inconsistent. We state the following lemmas without proof; proofs are essentially the same as those found in [1].

Lemma 1 *If λ is any $\mathbf{T}'$ consistent set of expressions, then there is a maximally $\mathbf{T}'$ consistent set λ' such that $\lambda \subseteq \lambda'$.*

Lemma 2 *If λ is any maximally $\mathbf{T}'$ consistent set of expressions, then:*

a. *For any expression E, either $E \in \lambda$ or $\neg E \in \lambda$, but not both.*
b. *For any expression E, if $\mathbf{T}' \vdash E$ then $E \in \lambda$.*
c. *For any expressions E and E', if $E \in \lambda$ and $E \supset E' \in \lambda$, then $E' \in \lambda$.*

Lemma 3 *If $\{\neg LE, LE_1, \ldots, LE_n\}$ is $\mathbf{T}'$ consistent, then so is $\{\neg E, E_1, \ldots, E_n\}$.*

The soundness of $\mathbf{T}'$ with respect to the indicated semantics is easily established in the normal way. One needs to show that all of the axioms are

$\mathbf{T}'$ valid and that the inference rules preserve $\mathbf{T}'$ validity. The details are left to the reader.

For the completeness of $\mathbf{T}'$, we must show that if $\mathbf{T}' \Vdash E$ then $\mathbf{T}' \vdash E$. By the standard Henkin argument, this is equivalent to the following:

Theorem *For any expression E^*, if E^* is $\mathbf{T}'$ consistent, then there is a $\mathbf{T}'$ interpretation (g, W, R) such that $V(E^*, g, W, R) = \mathbf{t}$.*

Proof: Suppose E^* is some $\mathbf{T}'$ consistent expression. By Lemma 1, there is a maximally $\mathbf{T}'$ consistent set λ such that $\{E^*\} \subseteq \lambda$. Let g be the following function:

$$\begin{aligned} g(P_i) &= \mathbf{t}, \text{ iff } P_i \in \lambda \\ &= \mathbf{f}, \text{ otherwise.} \end{aligned}$$

If there is an expression of the form $\neg LE$ in λ, then we put g in W and (g, g) in R. Otherwise W and R are both empty. We now claim that for any expression E, $V(E, g, W, R) = \mathbf{t}$ iff $E \in \lambda$. The proof of this claim is by induction on the complexity of E. The basis step is trivial, since if E is a sentence parameter the result follows immediately from the definition of g. The induction steps for "$\neg$" and "$\supset$" follow the familiar pattern and are left to the reader. The only interesting step is when E is of the form LE'. There are two cases. Case 1: Suppose W is empty. Then there is no expression of the form $\neg LE''$ in λ. So in particular, by Lemma 2, $LE' \in \lambda$. Since W is empty, R must also be empty, so $V(LE', g, W, R) = \mathbf{t}$ follows vacuously. Case 2: Suppose W is not empty. Then $W = \{g\}$, $R = \{(g, g)\}$, and for at least one E'', $\neg LE'' \in \lambda$. Let $N = \{E\colon LE \in \lambda\}$. By axiom scheme A4 and Lemma 2, it follows that $\lambda \subseteq N$. Let E'' be any expression such that $\neg LE'' \in \lambda$. Then using Lemma 3 it is easy to show that $\{\neg E''\} \cup N$ is consistent. We now claim that $\lambda = \{\neg E''\} \cup N$, for suppose not. Then since $\lambda \subseteq N$, there must be some expression $E''' \in \{\neg E''\} \cup N$ such that $E''' \notin \lambda$. By Lemma 2, $\neg E''' \in \lambda$, and so $\neg E''' \in \{\neg E''\} \cup N$ because $\lambda \subseteq \{\neg E''\} \cup N$. It follows that $\{\neg E''\} \cup N$ must be inconsistent because it has the inconsistent subset $\{\neg E''', E'''\}$; but we have already shown that $\{\neg E''\} \cup N$ is consistent. Consequently, $\lambda = \{\neg E''\} \cup N$, where E'' is any expression such that $\neg LE'' \in \lambda$. This result guarantees that: (i) if $LE' \in \lambda$ then $E' \in \lambda$, and (ii) if $\neg LE' \in \lambda$ then $\neg E' \in \lambda$. From (ii) we know by Lemma 2 that if $LE' \notin \lambda$ then $E' \notin \lambda$. So (i) and (ii) guarantee by the induction hypothesis that $V(E', g, W, R) = \mathbf{t}$ iff $LE' \in \lambda$. Since the one and only member of R is (g, g), we conclude that $V(LE', g, W, R) = \mathbf{t}$ iff $LE' \in \lambda$. So we have established our claim that for any expression E, $V(E, g, W, R) = \mathbf{t}$ iff $E \in \lambda$. Since $E^* \in \lambda$, we have the result that $V(E^*, g, W, R) = \mathbf{t}$. QED

Given the characteristic semantics for $\mathbf{T}'$, the sense in which $\mathbf{T}'$ "*almost*" collapses into the trivial system should be clear. Under our definition of interpretation, the semantics for the trivial system would require that W contain exactly one member and that R be reflexive. In other words, the $\mathbf{T}'$ semantics is just like the trivial semantics except that the $\mathbf{T}'$ semantics allows W to be empty. And when W is empty, LE holds for every E.

It is interesting to compare $\mathbf{T}'$ with other liberated modal logics (see [2]-[5]). The strongest liberated modal system discussed thus far is S5*. It is formed by using the inference rules MP and Nec and by adding the following axiom schemes to A1 and A2:

A5 $\neg L \neg E \supset (LE' \supset E')$
A6 $LE \supset LLE$
A7 $\neg L \neg E \supset L \neg L \neg E$.

The semantics for S5* requires that R be reflexive, transitive, and symmetric on W, but no restrictions are placed on W. Note that in the $\mathbf{T}'$ semantics, since W contains at most one member, the fact that R is reflexive on W implies that R is also symmetric and transitive on W. Hence every $\mathbf{T}'$ interpretation is an S5* interpretation, and so $\mathbf{T}'$ contains S5*. But A4 is falsifiable in any S5* interpretation in which W contains two functions which differ in their assignments to at least one sentence parameter; so $\mathbf{T}'$ is not contained in S5* nor in any of the weaker liberated systems. Further, axiom scheme A3 may be falsified in any $\mathbf{T}'$ interpretation with empty W; so $\mathbf{T}'$ does not contain any of the standard modal systems which include A3.

System $\mathbf{T}'$ turns out to be the syntactic counterpart to certain skeptical worries about possible worlds. Many individuals do not wish to banish discourse and argumentation involving "necessity"; they agree with the logical intuitions which give rise to axiom scheme A2 and the rule Nec; and yet they are suspicious of semantic theory based on possible worlds. The skeptical claim that is made is that either the notion of possible worlds is incoherent or the only possible world is the actual world. This claim is equivalent to the position that either there are no possible worlds or the world we know is the only one. But this position is just the characteristic semantics for $\mathbf{T}'$. Thus this study offers a line of defense against such a brand of possible world skepticism. Our completeness result shows that the possible world skeptic must either admit that the sense of "possible worlds" which he questions is not a sense relevant to formal semantics, or he must admit that whatever happens to be the case is logically necessary (i.e., $E \supset LE$). To the extent that fatalism is unacceptable, possible world skepticism is not relevant to formal semantics.

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