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BEYOND NON-NORMAL POSSIBLE WORLDS

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Formal possible world semantical investigations by Kripke, Hintikka, Lemmon, and others have concerned themselves with the systems C2 and $S2^0$ and their various extensions.[1] We shall refer to these semantics as *Kripke* semantics. The present work introduces a semantical framework for a more comprehensive class of systems, containing Kripke semantics as a special case.[2]

The Kripke semantics for C2 and $S2^0$, upon which the semantics for their various extensions are based, may be stated as follows. A C2-*model structure* is a structure $\mathfrak{M} = \langle w_0, W, N, R\rangle$ such that (1) $w_0 \in W$, (2) $N \subseteq W$, and (3) $R \subseteq W \times W$. The elements of N may be thought of a *normal worlds*, those elements of W outside of N as *non-normal worlds*, and R may be thought of as an *accessibility relation* among worlds. $\mathfrak{M}$ is an $S2^0$-*model structure* if, in addition, w_0 is a normal world. A *valuation function for* $\mathfrak{M}$ is a function v which assigns to each sentential variable at each world a unique truth-value **t** or **f**. The function v is then extended to the full domain of formulas as follows: for all formulas A and B, and all worlds w,

(i) $v(\sim A, w) = \mathbf{t}$ if and only if $v(A, w) = \mathbf{f}$,
(ii) $v(A \wedge B, w) = \mathbf{t}$ if and only if $v(A, w) = \mathbf{t}$ & $v(B, w) = \mathbf{t}$,

and

(iii) $v(\Box A, w) = \mathbf{t}$ if and only if $w \in N$ & $(\forall w' \in W)(wRw' \rightarrow v(A, w') = \mathbf{t})$.

Note here that non-normal worlds are effectively worlds in which every formula $\Box A$ is false. A formula A is *valid in* a model structure $\mathfrak{M}$ =

1. Other contributions here are many. See, for example the works of Prior, Kanger, Montague, Makinson, Routley.

2. Some of the ideas appearing here are to be found in the author's doctoral dissertation, *First-Order Indefinite and Generalized Semantics for Weak Systems of Strict-Implication*, University of Pittsburgh, 1974.

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$\langle w_0, W, N, R\rangle$ if for all valuation functions v for $\mathfrak{M}$, $v(A, w_0) = \mathbf{t}$, and a formula A is C2-(S2^0-) *valid* if A is valid in all C2-(S2^0-) model structures. By imposing certain *model conditions* on model structures, semantics are obtained for the various extensions of C2 and S2^0. The main result established for these systems is their *completeness* and *consistency* with respect to the indicated semantics, namely that a formula A is provable in a system $\mathbf{L}$ if and only if A is $\mathbf{L}$-valid.

We generalize these semantics to include semantics for the systems $\mathbf{N}^0$ and $\mathbf{SN}^0$, which are presented below. As before, the semantics for the various extensions of $\mathbf{N}^0$ and $\mathbf{SN}^0$, among which are C2, S2^0 and their extensions, may be based on those for $\mathbf{N}^0$ and $\mathbf{SN}^0$.

We say that an $\mathbf{N}^0$-*model structure* is a structure $\mathfrak{M} = \langle w_0, W, R, S\rangle$ such that (1) $w_0 \in W$, and (2) $R \subseteq W \times W$ and $S \subseteq W \times W$. A set of *normal* worlds may be defined as $N_{\mathfrak{M}} = \{w: w \in W \text{ and for no } w' \in W, wSw'\}$. Moreover, we say that $\mathfrak{M}$ is an $\mathbf{SN}^0$-*model structure* if $w_0 \in N_{\mathfrak{M}}$. A *valuation function for* $\mathfrak{M}$ is a function v which assigns to each sentential variable, world pair a unique truth-value $\mathbf{t}$ or $\mathbf{f}$. Again, v is extended to the full domain of formulas, but this time as follows: formulas $\sim A$ and $A \wedge B$ are evaluated as before, but formulas $\Box A$ are evaluated in the manner

$$v(\Box A, w) = \mathbf{t} \textit{ if and only if}$$
$$(\forall w' \in W)(wRw' \rightarrow v(A, w') = \mathbf{t})\ \&\ (\forall w' \in W)(wSw' \rightarrow v(A, w') = \mathbf{f}).$$

We note here that worlds normal in Kripke's sense are worlds to which no world is S-related, and worlds non-normal in Kripke's sense are worlds to which some world is *both* R-related and S-related, for only in such worlds is every formula $\Box A$ false. But we note also that in the new semantics there are worlds which are neither normal nor non-normal in the sense of Kripke.

We can now define a C2-*model structure* to be an $\mathbf{N}^0$-model structure in which every world is either normal or non-normal in the sense of Kripke, that is, an $\mathbf{N}^0$-model structure $\mathfrak{M} = \langle w_0, W, R, S\rangle$ such that $S \subseteq R$. If in addition, $w_0 \in N_{\mathfrak{M}}$, then we say that $\mathfrak{M}$ is an S2^0-*model structure.* That such definitions are appropriate follows from the fact that C2-validity in Kripke's sense coincides with C2-validity in the new sense.

Lemma 1 *A formula A is* C2-*valid in Kripke's sense if and only if A is valid in all* $\mathbf{N}^0$-*model structures* $\mathfrak{M} = \langle w_0, W, R, S\rangle$ *such that* $S \subseteq R$.

Proof: The sufficiency clause is seen as follows. Let $\mathfrak{M} = \langle w_0, W, N, R\rangle$ be a C2-model structure, and let v be a valuation for $\mathfrak{M}$ such that $v(A, w_0) = \mathbf{f}$. Let $S_{\mathfrak{M}} = \{\langle w, w\rangle: w \in W\ \&\ w \notin N\}$. Then for all worlds w and all formulas B, $w \in N\ \&\ (\forall w' \in W)(wRw' \rightarrow v(B, w') = \mathbf{t})$ if and only if $(\forall w' \in W)(w(R \cup S_{\mathfrak{M}})w' \rightarrow v(B, w') = \mathbf{t})\ \&\ (\forall w' \in W)(wS_{\mathfrak{M}}w' \rightarrow v(B, w') = \mathbf{f})$. Hence, v is a valuation for the $\mathbf{N}^0$-model structure $\mathfrak{N} = \langle w_0, W, R \cup S_{\mathfrak{M}}, S_{\mathfrak{M}}\rangle$. Also, $S_{\mathfrak{M}} \subseteq R \cup S_{\mathfrak{M}}$. Thus, A is not valid in $\mathfrak{N}$, a C2-model structure in the new sense. Conversely, let $\mathfrak{M} = \langle w_0, W, R, S\rangle$ be an $\mathbf{N}^0$-model structure such that $S \subseteq R$, and let v be a valuation for $\mathfrak{M}$ such that $v(A, w_0) = \mathbf{f}$. Since $S \subseteq R$, we have for all worlds w and all formulas B,

$$(\forall w' \epsilon W)(wRw' \rightarrow v(B, w') = \mathbf{t}) \;\&\; (\forall w' \epsilon W)(wSw' \rightarrow v(B, w') = \mathbf{f})$$
$$\text{if and only if } w \epsilon N_{\mathfrak{M}} \;\&\; (\forall w' \epsilon W)(wRw' \rightarrow v(B, w') = \mathbf{t}).$$

Hence, v is a valuation for the C2-model structure in Kripke's sense $\mathfrak{N} = \langle w_0, W, N_{\mathfrak{M}}, R\rangle$ and A is not valid in $\mathfrak{N}$. A similar reduction holds for S2^0-validity, and thus we see that Kripke semantics are indeed a special case of the semantical framework for $\mathbf{N}^0$ and $\mathbf{SN}^0$.

It remains to characterize the systems $\mathbf{N}^0$ and $\mathbf{SN}^0$ and to show that they are complete and consistent with respect to their semantics. The primitive connectives of $\mathbf{N}^0$ and $\mathbf{SN}^0$ will be '$\sim$', '$\wedge$', and '$\Box$', the other connectives being defined as usual. In particular, '$A \prec B$' is defined as '$\Box(A \supset B)$'. Also, we use 'T' to range over two-valued tautologies. The system $\mathbf{N}^0$ is axiomatized by the axiom-schemes and rules $\{\mathbf{PC}, \mathrm{MP}, \mathrm{R2}\}$ and $\mathbf{SN}^0$ by $\{\Box\mathrm{PC}, \mathrm{MP}, \Box\mathrm{R2}, de\text{-}\Box\}$.

(**PC**). T
($\Box$**PC**). $\Box\mathrm{T}$
(MP). From A and $A \supset B$ infer B
(R2). From $(A \wedge B) \supset C$ and $C \supset (A \vee B)$ infer $(\Box A \wedge \Box B) \supset \Box C$
($\Box$R2). From $(A \wedge B) \prec C$ and $C \prec (A \vee B)$ infer $(\Box A \wedge \Box B) \prec \Box C$
(*de*-$\Box$). From $\Box A$ infer A

It is not difficult to show that $\mathbf{N}^0$ contains Shukla's system T1^0 and is contained in C2, and that $\mathbf{SN}^0$ contains Feys' system S1^0 and is contained in S2^0. It is of interest to note that Group V, used by Lewis and Langford[3] to show the distinctness of S1 and S2, is characteristic for a system that is an extension of $\mathbf{SN}^0$, and is the algebraic equivalent of the $\mathbf{SN}^0$-model structure $\mathfrak{M}(\mathrm{V}) = \langle w_0, W, R, S\rangle$ where $W = \{w_0, w_1\}$ and w_0Rw_0, w_0Rw_1, w_1Rw_1, and w_1Sw_0.

It is a trivial matter to show that $\mathbf{N}^0$ is consistent with respect to its semantics, and we leave it to the reader. Later we demonstrate the consistency of $\mathbf{SN}^0$. Thus we have the following theorem.

Theorem 1 *If* $\mathbf{N}^0 \vdash A$, *then* A *is* $\mathbf{N}^0$*-valid.*

The completeness of $\mathbf{N}^0$ and $\mathbf{SN}^0$ may be shown by using the Henkin method of maximal, consistent sets of formulas employed by Makinson and Routley.[4] For details concerning the lemmas and theorems below the reader is asked to consult these works. We will use the following definitions. Where Δ is a set of formulas, $[\Delta] = \{A: \Box A \epsilon \Delta\}$, $\overline{\Delta} = \{\sim A: A \epsilon \Delta\}$, Δ is *maximal* if for all formulas A, $A \epsilon \Delta$ or $\sim A \epsilon \Delta$, and Δ is $\mathbf{L}$*-consistent* if for no formulas $A_1, \ldots, A_n \epsilon \Delta$, $\mathbf{L} \vdash \sim(A_1 \wedge \ldots \wedge A_n)$.

3. See [1], pp. 493-494.

4. See [2] and [3].

Lemma 2 *If* α *is a maximal,* $\mathbf{N}^0$*-consistent set of formulas such that* $\Box A \notin \alpha$, *then* $[\alpha] \cup \{\sim A\}$ *is* $\mathbf{N}^0$*-consistent or* $\overline{[\alpha]} \cup \{A\}$ *is* $\mathbf{N}^0$*-consistent.* (*Cf.* M., p. 382; R., p. 242).

Lemma 3 *For any system* $\mathbf{L}$ *containing classical two-valued logic, if* Δ *is an* $\mathbf{L}$*-consistent set of formulas, then there is a maximal,* $\mathbf{L}$*-consistent set* α *of formulas such that* $\Delta \subseteq \alpha$. (*Cf.* M., p. 381).

Theorem 2 *If* A *is* $\mathbf{N}^0$*-valid, then* $\mathbf{N}^0 \vdash A$.

Proof: The proof is similar to [2], p. 382, and [3], p. 243. Let A_0 be a non-theorem of $\mathbf{N}^0$. Then by Lemma 3 there is a maximal, $\mathbf{N}^0$-consistent set α_0 such that $\sim A_0 \in \alpha_0$. We define W as the set of all maximal, $\mathbf{N}^0$-consistent sets, R as the set of all pairs (α, β) in W such that $[\alpha] \subseteq \beta$, and S as the set of all pairs (α, β) in W such that $[\alpha] \cap \beta = \emptyset$. Then $\mathfrak{M} = \langle \alpha_0, W, R, S \rangle$ is an $\mathbf{N}^0$-model structure. A valuation function v for $\mathfrak{M}$ is defined by setting $v(P, \alpha) = \mathbf{t}$ if and only if $P \in \alpha$, for all sentential variables P and all $\alpha \in W$. We then extend v to a full valuation function in the usual way. The proof proceeds as in M. and R., using Lemmas 2 and 3 to show that for all formulas A and all $\alpha \in W$, $v(A, \alpha) = \mathbf{t}$ if and only if $A \in \alpha$. Since $\sim A_0 \in \alpha_0$, it follows that $v(A_0, \alpha_0) = \mathbf{f}$.

Lemma 4 (i) *If* A *is* $\mathbf{N}^0$*-valid, then* $\Box A$ *is* $\mathbf{SN}^0$*-valid,*

and

(ii) *if a set* Δ *is* $\mathbf{SN}^0$*-consistent, then* Δ *is* $\mathbf{N}^0$*-consistent.*

Proof: By a simple *reductio* argument (i) may be established. To show (ii) we note that if $\mathbf{N}^0 \vdash A$ then $\mathbf{SN}^0 \vdash \Box A$, as may be established by a simple induction on the length of proof in $\mathbf{N}^0$. Hence, by the rule *de*-$\Box$ we have that if $\mathbf{N}^0 \vdash A$ then $\mathbf{SN}^0 \vdash A$, and (ii) follows.

Theorem 3 *If* A *is* $\mathbf{SN}^0$*-valid, then* $\mathbf{SN}^0 \vdash A$.

Proof: The proof is similar to [3], p. 251. Let A_0 be a non-theorem of $\mathbf{SN}^0$. Then by Lemma 3 there is a maximal, $\mathbf{SN}^0$-consistent set α_0 such that $\sim A_0 \in \alpha_0$. We define W, R, S exactly as in Theorem 2. By Lemma 4 α_0 is a maximal, $\mathbf{N}^0$-consistent set and hence belongs to W. Thus, $\mathfrak{M} = \langle \alpha_0, W, R, S \rangle$ is an $\mathbf{N}^0$-model structure. But, $\Box \top \in \alpha_0$, so that for no $\alpha \in W$ do we have $\alpha_0 S \alpha$, and $\mathfrak{M}$ is an $\mathbf{SN}^0$-model structure. Now, as in Theorem 2 a valuation function v for $\mathfrak{M}$ is defined, and it follows that $v(A_0, \alpha_0) = \mathbf{f}$.

Lemma 5 *If* $\Box A$ *is* $\mathbf{SN}^0$*-valid, then* A *is* $\mathbf{N}^0$*-valid.*

Proof: The proof is similar to one given by Kripke [4] and consists in extending a given $\mathbf{N}^0$-model structure which falsifies a formula A_0 into an $\mathbf{SN}^0$-model structure which falsifies the formula $\Box A_0$.

Theorem 4 *If* $\mathbf{SN}^0 \vdash A$, *then* A *is* $\mathbf{SN}^0$*-valid.*

Proof: The proof follows by a simple induction on the length of proof in $\mathbf{SN}^0$. The cases when a theorem of $\mathbf{SN}^0$ is an axiom or comes by the rule

MP are trivial, and the case when a theorem comes by the rule *de*-$\Box$ follows by Lemma 5. The case when a theorem comes by the rule $\Box$R2 is as follows. By inductive hypothesis the theorems $(A \wedge B) \dashv C$ and $C \dashv (A \vee B)$ are $\mathbf{SN}^0$-valid. By Lemma 5 the formulas $(A \wedge B) \supset C$ and $C \supset (A \vee B)$ are $\mathbf{N}^0$-valid, and by Theorem 2 they are provable in $\mathbf{N}^0$. Thus, $(\Box A \wedge \Box B) \supset \Box C$ is provable in $\mathbf{N}^0$ and by Theorem 1 is $\mathbf{N}^0$-valid. By Lemma 4 it follows that $(\Box A \wedge \Box B) \dashv \Box C$ is $\mathbf{SN}^0$-valid.

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