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ALTERNATIVE FORMS OF PROPOSITIONAL CALCULUS FOR A GIVEN DEDUCTION THEOREM

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In a propositional calculus based on combinatory logic it is necessary to have a restriction on the deduction theorem for implication as otherwise Curry's paradox results (see [5]). In [1] and [2] we restricted the deduction theorem for implication as follows:

DTP $\qquad$ *If* $\Delta, X \vdash Y$, *then* $\Delta, \mathbf{H}X \vdash X \supset Y$,

where Δ is any sequence of obs and $\mathbf{H}X$ stands for "X is a proposition".

Motivation for this deduction theorem was given in [2] using the following three valued tables (that for implication also appears in Kleene [6]).

X	$\mathbf{H}X$
T	T
F	T
N	N

X	ΓX
T	F
F	T
N	N

		Y		
	$X \supset Y$	T	F	N
X	T	T	F	N
	F	T	T	T
	N	T	N	N

where N can stand for "neither T nor F" and Γ (negation) can be defined by $\mathbf{CP}(\Xi\mathbf{HI})$,[1]

A question that arises is: to what extent are the entries in the third column and the third row of the table for implication uniquely determined by DTP, modus ponens and the (fairly obvious) rule:

H $\qquad$ $X \vdash \mathbf{H}X$?

1. Here $\mathbf{P}$ stands for implication. $\Xi\mathbf{HI}$, which can be interpreted as stating that all propositions are provable, is taken as the "standard false" proposition. Given that $\Xi\mathbf{HI}$ is assigned F the table for Γ follows from that for $\supset$.

After part (iii) on page xx, below we assume for $\Xi\mathbf{HI}$:

$$\Xi\mathbf{HI}, \mathbf{H}X \vdash X.$$

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We show below that the three N entries are in fact uniquely determined in the sense that we cannot consistently have a T or an F in that position. Either or both of the T entries, however, may be N as even a slight extension of the above set of rules is insufficient to enforce a T in either place. Thus there are four possible truth tables for $\supset$, each giving rise to a different form of propositional calculus similar to that developed in [2]. The fact that the N in the table for H is unique has already been shown. The simplest version of this appears in [4].[2]

We will say that a particular set of axioms and rules fits a truth table if for every entry T or F we have a corresponding introduction rule such as:

$$X,\ \Gamma Y \vdash \Gamma(X \supset Y)$$
$$Y \vdash X \supset Y$$

and

$$\Gamma X \vdash X \supset Y$$

for any X and Y; for each entry N we require only that there is no introduction rule such as:

$$\Gamma Y \vdash X \supset Y,$$

or

$$\Gamma Y \vdash \Gamma(X \supset Y)$$

for all X and Y (or even for all Y and all X such that not $\vdash X$ and not $\vdash \Gamma X$ (i.e., not $\vdash \mathbf{H}X$)), which would force the assignment of a T or an F where we had an N in the table.

Because of the uniqueness of the Ns in the table for implication it will follow that rules such as:

$$\mathbf{H}X \vdash \mathbf{H}(X \supset Y), \tag{1}$$

which might seem reasonable in view of the deduction theorem, and

$$\mathbf{H}Y \vdash \mathbf{H}(X \supset Y) \tag{2}$$

do not fit any of the possible truth tables and are in fact inconsistent with Modus Ponens, DTP and H.

We now show the uniqueness of each of the Ns.

(i) Let X be any theorem and let

$$Y = \mathbf{Y}(\mathbf{B}(\mathbf{P}X)\Gamma)$$

where $\mathbf{Y}$ is the paradoxical combinator, then

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2. This also shows that the system suggested towards the end of [3] which has $\vdash \mathbf{H}(\mathbf{H}X)$ for all X as well as Modus Ponens, DTP and H inconsistent.

$$Y = X \supset \Gamma Y.$$

Now

$$Y \vdash \Gamma Y$$

so

$$Y \vdash \Xi \mathbf{HI}$$

and by DTP,

$$\mathbf{H}Y \vdash \Gamma Y. \qquad (3)$$

Again by DTP as we must have $\vdash \mathbf{H}X$,

$$\mathbf{H}Y \vdash X \supset \Gamma Y$$

and so

$$\mathbf{H}Y \vdash Y \qquad (4)$$

Thus by (3) and (4) we cannot have $\mathbf{H}Y$ as Y cannot be assigned both T and F hence Y must be assigned N in a truth table. Thus with X being assigned T and ΓY N, $X \supset \Gamma Y$ is assigned N. N must therefore be the entry in the first row and third column of the truth table for implication.

(ii) Let

$$Y = \mathbf{Y}(\mathbf{SP}\Gamma)$$

so that

$$Y = Y \supset \Gamma Y,$$
$$Y \vdash \Gamma Y$$

and so

$$Y \vdash \Xi \mathbf{HI}.$$

Now by DTP,

$$\mathbf{H}Y \vdash \Gamma Y \qquad (5)$$

and again by DTP

$$\mathbf{H}Y \vdash Y \supset \Gamma Y$$

and so

$$\mathbf{H}Y \vdash Y. \qquad (6)$$

Thus by (5) and (6) we again cannot have $\vdash \mathbf{H}Y$, so Y must be assigned N. Also ΓY is assigned N and so is $Y \supset \Gamma Y$. Thus the entry in the third row and third column must be N.

(iii) Let X be any theorem and let

$$Y = \mathbf{Y}(\mathbf{CP}(\Gamma X)),$$

then

$$Y = Y \supset \Gamma X.$$

Thus

$$Y \vdash \Gamma X$$

and as X is a theorem

$$Y \vdash \Xi\mathbf{HI}.$$

Now by DTP applied to both of these steps we have:

$$\mathbf{H}Y \vdash \Gamma Y, \tag{7}$$

and

$$\mathbf{H}Y \vdash Y \supset \Gamma X,$$

which is

$$\mathbf{H}Y \vdash Y. \tag{8}$$

Thus by (7) and (8) we cannot have $\vdash \mathbf{H}Y$, Y must therefore be assigned N, ΓX is assigned F and $Y \supset \Gamma X$ is assigned N. Thus the entry in the third row and second column must be N.

The three N entries in our original truth table for $\supset$ must therefore be N. Now we examine the T entries: We assume:

$$\mathbf{H}X,\ \mathbf{H}Y \vdash \mathbf{H}(X \supset Y) \tag{9}$$

and

$$\vdash \mathbf{H}(\Xi\mathbf{HI}) \tag{10}$$

which certainly fit the truth tables; and we, for the first time make use of the actual definition of our standard false proposition $\Xi\mathbf{HI}$.

(iv) $$X,\ \Gamma X \vdash \Xi\mathbf{HI}$$

so

$$X,\ \Gamma X,\ \mathbf{H}^{k+1} Y \vdash \mathbf{H}^k Y$$

and by repeating this process we obtain:

$$X,\ \Gamma X,\ \mathbf{H}^{k+1} Y \vdash Y.$$

As we have by (9) and (10)

$$\mathbf{H}X \vdash \mathbf{H}(\Gamma X) \tag{11}$$

it follows that

$$X,\ \mathbf{H}^{k+1} Y \vdash \Gamma X \supset Y. \tag{12}$$

If we also have

$$\Gamma X \vdash \mathbf{H}X, \tag{13}$$

which fits the table for Γ, we can prove as well:

$$\Gamma X, \mathsf{H}^{k+1} Y \vdash X \supset Y \tag{14}$$

Thus if we have a Y such that $\vdash \Gamma(\mathsf{H}^k Y)$ for a $k \geqslant 1$, we have a case where Y has the truth value N, X has the truth value F and $X \supset Y$ has the value T.

There may conceivably be obs Y such that we do not have $\vdash \mathsf{H}^k Y$ for any k. In that case no method such as the above provides us with a value for $X \supset Y$ when X is F, so that $X \supset Y$ must be assigned N. This N can be made unique for the second row and third column position if we add as a rule:

$$\mathsf{H}(X \supset Y) \vdash \mathsf{H} Y. \tag{15}$$

Note that (12) or (14) (with appropriate X) and (15) lead to:

$$\mathsf{H}^{k+1} Y \vdash \mathsf{H} Y \tag{16}$$

for all Y, so that it must be impossible to prove $\vdash \Gamma(\mathsf{H}^k Y)$ for all Y, for $k \geqslant 1$.

Alternatively we could have a strong negation satisfying

$$X \vdash \Gamma X \supset Y.^3 \tag{17}$$

This would give us a T in the second row and third column of the truth table for ⊃.

(v) Using DTP it is easy to show:

$$\mathsf{H}^2 X,\ Y \vdash \mathsf{H} X \supset .\ X \supset Y \tag{18}$$

so that if $\vdash \Gamma(\mathsf{H} X)$ (Note: X could be $\mathsf{H}^t Z$ for some Z and t) we have an X with the value N and $\mathsf{H}X$ with the value F. If Y then has the value T, F ⊃. N ⊃ T would have to be T. If we have (17) this is guaranteed irrespective of whether N ⊃ T is N, T, or F. (15) and (16) do not effect the result here so our options are left open completely.

We can assign a unique N to the position in the third row and first column by adding the rule:

$$\mathsf{H}(X \supset Y) \vdash \mathsf{H} X, \tag{19}$$

Alternatively we can assign a T to the position by adding instead of (19) the rule:

$$Y \vdash X \supset Y \tag{20}$$

In [2] we used the following more general rule instead of (9):

PH $\qquad \mathsf{H} X,\ X \supset \mathsf{H} Y \vdash \mathsf{H}(X \supset Y).$

3. This negation is strong in the sense that a false proposition implies every ob instead of just every proposition. In [2] (17) is derived using the axiom $\vdash \mathsf{H}(\mathsf{Q}(\mathsf{KI})(\mathsf{BI}))$ together with certain other rules for Q which represents equality.

This fits the truth table for $\supset$ whether the entry in the third row and first column is T or N. Also it does not clash with (15) or (16), it is in fact derivable from them, Rule H and (9). In addition PH is comptabile with (17). Thus PH is compatible with all possible truth tables.

In [2] the following two "elimination rules" for $\mathbf{H}(X \supset Y)$ were also mentioned:

$$\Gamma Y, \mathbf{H}(X \supset Y) \vdash \mathbf{H} X \tag{21}$$

and

$$X, \mathbf{H}(X \supset Y) \vdash \mathbf{H} Y. \tag{22}$$

In view of (iii) (21) fits all possible truth tables and in view of (i) so does (22).

Other elimination rules for $\supset$ such as:

$$\Gamma(X \supset Y) \vdash X \tag{23}$$
$$\Gamma(X \supset Y) \vdash \Gamma Y \tag{24}$$

and

$$\Gamma Y, X \supset Y \vdash \Gamma X \tag{25}$$

also fit the table, but none of (21)-(25) can be proved in an unrestricted form from Modus Ponens, DTP, H, and (9) or PH.

We should note that all the options we have suggested are left open in work based on [1] and [2], as the general theorems of [2] on which later work is based has all relevant terms restricted to being in H.

We now look at the effect of the various truth tables for $\supset$ on those for $\vee$ and $\wedge$. $\vee$ and $\wedge$ are defined in [2] by:

$$\wedge \equiv [x, y].\mathbf{H}z \supset_z. (x \supset. y \supset z) \supset z$$
$$\vee \equiv [x, y].\mathbf{H}z \supset_z: (x \supset z) \supset. (y \supset z) \supset z$$

We will give $\wedge XY$ the value T if $(X \supset. Y \supset Z) \supset Z$ has the value T whenever $\mathbf{H}Z$ has the value T (i.e., Z has the value T or F). We give $\wedge XY$ the value N if $(X \supset. Y \supset Z) \supset Z$ can have the value N for a Z with $\mathbf{H}Z$ with value T. We give $\wedge XY$ the value F otherwise. $\vee XY$ is assigned values in the same way according to the values of $(X \supset Z) \supset. (Y \supset Z) \supset Z$. We then obtain the following tables:

	$\wedge XY$	Y: T	F	N
X	T	T	F	N
	F	F	F	b
	N	N	a	N

	$\vee XY$	Y: T	F	N
X	T	T	T	c
	F	T	F	N
	N	c	N	N

where if there is a T in the first column and third row of the table for $\supset$, a = F, if there is a T in the third column and second row of the table for $\supset$, b = F and if both these entries are T, c = T. If the appropriate condition does not apply, a, b, or c is N.

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