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A SECOND DEDUCTION THEOREM FOR REJECTION THESES IN ŁUKASIEWICZ'S SYSTEM OF MODAL LOGIC

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In an earlier paper[1] I proved the following deduction theorem for rejection theses of Jan Łukasiewicz's system[2] of modal logic:

$$\dashv \mathbf{A}_1, \ldots, \dashv \mathbf{A}_{m-1}, \dashv \mathbf{A}_m \vdash_{Ł} \dashv \mathbf{B} \Rightarrow \dashv \mathbf{A}_1, \ldots, \dashv \mathbf{A}_{m-1} \vdash_{Ł} \ulcorner \dashv NC\mathbf{B}\mathbf{A}_m \urcorner.$$

In this paper I shall prove a second deduction theorem for such theses, viz.

$$\vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_{m-1}, \vdash \mathbf{A}_m \vdash_{Ł} \dashv \mathbf{B} \Rightarrow \vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_{m-1} \vdash_{Ł} \ulcorner \dashv K\mathbf{A}_m\mathbf{B} \urcorner.$$

As mentioned in my earlier paper, by "rejection thesis" I mean the last line of any valid deduction in Łukasiewicz's system of the form $\dashv \Delta$. There are several such theses already proved by Łukasiewicz, e.g., theses 115-122,[3] but obviously there are many more which are able to be proved. This second deduction theorem is, like the first, designed to facilitate the deduction of these latter theses.

The strategy to be used in proving this theorem will be to first assume that the antecedent of the theorem is true, i.e., that there is a finite sequence of wffs which constitutes a demonstration of $\dashv \mathbf{B}$ from the premises $\vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_m$, and then to indicate how one may construct, from this initial sequence, a second sequence which will constitute a demonstration of $\ulcorner \dashv K\mathbf{A}_m\mathbf{B} \urcorner$ from the premises $\vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_{m-1}$.

Before giving the details of my proof, I shall first introduce some conventions:

R_1 = Rule of substitution for assertions: from $\vdash \mathbf{A}$ one may infer the result of substituting a wff $\mathbf{B}$ for a sentential variable c throughout $\mathbf{A}$.
R_2 = Rule of detachment for assertions: from $\vdash \mathbf{A}$ and $\ulcorner \vdash C\mathbf{A}\mathbf{B} \urcorner$ one may infer $\vdash \mathbf{B}$.
R_3 = Rule of substitution for rejections: from $\dashv \mathbf{A}$, where $\mathbf{A}$ is a substitution instance of $\mathbf{B}$, one may infer $\dashv \mathbf{B}$.[4]
R_4 = Rule of detachment for rejections: from $\ulcorner \vdash C\mathbf{A}\mathbf{B} \urcorner$ and $\dashv \mathbf{B}$ one may infer $\dashv \mathbf{A}$.[5]

Theorem $\vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_{m-1}, \vdash \mathbf{A}_m \vdash_{Ł} \dashv \mathbf{B} \Rightarrow \vdash \mathbf{A}_1, \ldots, \vdash \mathbf{A}_{m-1} \vdash_{Ł} \ulcorner \dashv K\mathbf{A}_m\mathbf{B} \urcorner$.

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Proof: Assume $\vdash\mathbf{A}_1, \ldots, \vdash\mathbf{A}_m \vdash_{\mathrm{L}} \dashv\mathbf{B}$ and let $\Sigma_1, \ldots, \Sigma_i, \ldots, \Sigma_k$ denote its demonstration. Construct the sequence which for every Σ_i of the form $\vdash\Gamma$ has a Σ_j^* of the form $\ulcorner\vdash C\mathbf{A}_m\Gamma\urcorner$ and which for every Σ_i of the form $\dashv\Delta$ has a Σ_j^* of the form $\ulcorner\dashv K\mathbf{A}_m\Delta\urcorner$. We now show how to use this latter sequence to construct a demonstration of $\vdash\mathbf{A}_1, \ldots, \vdash\mathbf{A}_{m-1} \vdash_{\mathrm{L}} \ulcorner\dashv K\mathbf{A}_m\mathbf{B}\urcorner$. (In order to avoid problems regarding substitution in hypotheses, allow $p_1, \ldots, p_n$ in the following to be distinct variables which do not occur in any hypotheses or anywhere in the original sequence.) Consider the following cases:

Case 1: $\vdash\Sigma_i$ is an assertion axiom, an axiom variant or $\vdash\mathbf{A}_j (1 \leqslant j \leqslant m - 1)$. Use the following proof to demonstrate $\vdash C\mathbf{A}_m\Sigma_i$:

1.	$\vdash Cp_1Cp_2p_1$	PC[6]
2.	$\vdash C\Sigma_iC\mathbf{A}_m\Sigma_i$	1; R_1, p_1/Σ_i, $p_2/\mathbf{A}_m$
3.	$\vdash\Sigma_i$	
4.	$\vdash C\mathbf{A}_m\Sigma_i$	2; 3; R_2

Case 2: $\vdash\Sigma_i$ is $\vdash\mathbf{A}_m$. Use the following proof to demonstrate $\ulcorner\vdash C\mathbf{A}_m\Sigma_i\urcorner$, i.e., $\ulcorner\vdash C\mathbf{A}_m\mathbf{A}_m\urcorner$:

1.	$\vdash CCp_1Cp_2p_3CCp_1p_2Cp_1p_3$	PC
2.	$\vdash CCp_1Cp_2p_1CCp_1p_2Cp_1p_1$	1; R_1, p_3/p_1
3.	$\vdash Cp_1Cp_2p_1$	PC
4.	$\vdash CCp_1p_2Cp_1p_1$	2; 3; R_2
5.	$\vdash CCp_1Cp_2p_1Cp_1p_1$	4; R_1, p_2/Cp_2p_1
6.	$\vdash Cp_1p_1$	3; 5; R_2
7.	$\vdash C\mathbf{A}_m\mathbf{A}_m$	6; R_1, $p_1/\mathbf{A}_m$

Case 3: $\vdash\Sigma_i$ is generated by the use of R_2 and two earlier steps in the sequence, $\vdash\Sigma_a$ and $\ulcorner\vdash C\Sigma_a\Sigma_i\urcorner$. Use the following proof to demonstrate $\ulcorner\vdash C\mathbf{A}_m\Sigma_i\urcorner$:

1.	$\vdash C\mathbf{A}_m\Sigma_a$	
2.	$\vdash C\mathbf{A}_mC\Sigma_a\Sigma_i$	
3.	$\vdash CCp_1Cp_2p_3CCp_1p_2Cp_1p_3$	PC
4.	$\vdash CC\mathbf{A}_mC\Sigma_a\Sigma_iCC\mathbf{A}_m\Sigma_aC\mathbf{A}_m\Sigma_i$	3; R_1, $p_1/\mathbf{A}_m$, p_2/Σ_a, p_3/Σ_i
5.	$\vdash CC\mathbf{A}_m\Sigma_aC\mathbf{A}_m\Sigma_i$	2; 4; R_2
6.	$\vdash C\mathbf{A}_m\Sigma_i$	1; 5; R_2

Case 4: $\dashv\Sigma_i$ is a rejection axiom. Use the following proof to demonstrate $\ulcorner\dashv K\mathbf{A}_m\Sigma_i\urcorner$:

1.	$\vdash CKp_2p_1p_1$	PC
2.	$\vdash CK\mathbf{A}_m\Sigma_i\Sigma_i$	1; R_1, p_1/Σ_i, $p_2/\mathbf{A}_m$
3.	$\dashv\Sigma_i$	
4.	$\dashv K\mathbf{A}_m\Sigma_i$	2; 3; R_4

Case 5: Σ_a is a substitution instance of Σ_i, and $\dashv\Sigma_a$ is a rejection axiom. Use the following proof to demonstrate $\ulcorner\dashv K\mathbf{A}_m\Sigma_i\urcorner$:

1.	$\vdash CKp_2p_1p_1$	PC
2.	$\dashv\Sigma_a$	

3.	$\dashv \Sigma_i$	2; R_3
4.	$\vdash CK\mathbf{A}_m\Sigma_i\Sigma_i$	1; R_1, p_1/Σ_i, $p_2/\mathbf{A}_m$
5.	$\dashv K\mathbf{A}_m\Sigma_i$	3; 4; R_4

Case 6: $\dashv\Sigma_i$ is generated by the use of R_4 and two earlier steps in the sequence, $\dashv\Sigma_a$ and $\ulcorner\vdash C\Sigma_i\Sigma_a\urcorner$. Use the following proof to demonstrate $\ulcorner\dashv K\mathbf{A}_m\Sigma_i\urcorner$:

1.	$\vdash C\mathbf{A}_mC\Sigma_i\Sigma_a$	
2.	$\dashv K\mathbf{A}_m\Sigma_a$	
3.	$\vdash CCp_1Cp_2p_3Cp_1CNp_3Np_2$	**PC**
4.	$\vdash CC\mathbf{A}_mC\Sigma_i\Sigma_aC\mathbf{A}_mCN\Sigma_aN\Sigma_i$	3; R_1, $p_1/\mathbf{A}_m$, p_2/Σ_i, p_3/Σ_a
5.	$\vdash C\mathbf{A}_mCN\Sigma_aN\Sigma_i$	1; 4; R_2
6.	$\vdash CCp_1Cp_2p_3CCp_1p_2Cp_1p_3$	**PC**
7.	$\vdash CC\mathbf{A}_mCN\Sigma_aN\Sigma_iCC\mathbf{A}_mN\Sigma_aC\mathbf{A}_m\Sigma_i$	6; R_1, $p_1/\mathbf{A}_m$, p_2/Σ_a, p_3/Σ_i
8.	$\vdash CC\mathbf{A}_mN\Sigma_aC\mathbf{A}_m\Sigma_i$	5; 7; R_2
9.	$\vdash CCCp_1p_2Cp_1p_3CNCp_1p_3NCp_1p_2$	**PC**
10.	$\vdash CCC\mathbf{A}_mN\Sigma_aC\mathbf{A}_mN\Sigma_iCNC\mathbf{A}_mN\Sigma_iNC\mathbf{A}_mN\Sigma_a$	9; R_1, $p_1/\mathbf{A}_m$, $p_2/N\Sigma_a$, $p_3/N\Sigma_i$
11.	$\vdash CNC\mathbf{A}_mN\Sigma_iNC\mathbf{A}_mN\Sigma_a$	8; 10; R_2
12.	$\vdash CCNCp_1Np_2NCp_1Np_3CKp_1p_2Kp_1p_3$	**PC**
13.	$\vdash CCNC\mathbf{A}_mN\Sigma_iNC\mathbf{A}_mN\Sigma_aCK\mathbf{A}_m\Sigma_iK\mathbf{A}_m\Sigma_a$	12; R_2, $p_1/\mathbf{A}_m$, p_2/Σ_i, p_3/Σ_a
14.	$\vdash CK\mathbf{A}_m\Sigma_iK\mathbf{A}_m\Sigma_a$	11; 13; R_2
15.	$\dashv K\mathbf{A}_m\Sigma_i$	2; 14; R_4

Case 7: Σ_a is a substitution instance of Σ_i and $\dashv\Sigma_a$ is generated by use of R_4 and two earlier steps in the sequence, $\dashv\Sigma_b$ and $\ulcorner\vdash C\Sigma_a\Sigma_b\urcorner$. Use the following proof to demonstrate $\ulcorner\dashv K\mathbf{A}_m\Sigma_i\urcorner$:

1.	$\vdash C\mathbf{A}_mC\Sigma_a\Sigma_b$	
2.	$\vdash K\mathbf{A}_m\Sigma_b$	
3.	$\vdash CCp_1Cp_2p_3Cp_1CNp_3Np_2$	**PC**
4.	$\vdash CC\mathbf{A}_mC\Sigma_a\Sigma_bC\mathbf{A}_mCN\Sigma_bN\Sigma_a$	3; R_1, $p_1/\mathbf{A}_m$, p_2/Σ_a, p_3/Σ_b
5.	$\vdash C\mathbf{A}_mCN\Sigma_bN\Sigma_a$	1; 4; R_2
6.	$\vdash CCp_1Cp_2p_3CCp_1p_2Cp_1p_3$	**PC**
7.	$\vdash CC\mathbf{A}_mCN\Sigma_bN\Sigma_aCC\mathbf{A}_mN\Sigma_bC\mathbf{A}_mN\Sigma_a$	6; R_1, $p_1/\mathbf{A}_m$, p_2/Σ_b, p_3/Σ_a
8.	$\vdash CC\mathbf{A}_mN\Sigma_bC\mathbf{A}_mN\Sigma_a$	5; 7; R_2
9.	$\vdash CCCp_1p_2Cp_1p_3CNCp_1p_3NCp_1p_2$	**PC**
10.	$\vdash CCC\mathbf{A}_mN\Sigma_bC\mathbf{A}_mN\Sigma_aCNC\mathbf{A}_mN\Sigma_aNC\mathbf{A}_mN\Sigma_b$	9; R_1, $p_1/\mathbf{A}_m$, $p_2/N\Sigma_b$, $p_3/N\Sigma_a$
11.	$\vdash CNC\mathbf{A}_mN\Sigma_aNC\mathbf{A}_mN\Sigma_b$	8; 10; R_2
12.	$\vdash CCNCp_1Np_2NCp_1Np_3CKp_1p_2Kp_1p_3$	**PC**
13.	$\vdash CCNC\mathbf{A}_mN\Sigma_aNC\mathbf{A}_mN\Sigma_bCK\mathbf{A}_m\Sigma_aK\mathbf{A}_m\Sigma_b$	12; R_1, $p_1/\mathbf{A}_m$, p_2/Σ_a, p_3/Σ_b
14.	$\vdash CK\mathbf{A}_m\Sigma_aK\mathbf{A}_m\Sigma_b$	11; 13; R_2
15.	$\dashv K\mathbf{A}_m\Sigma_a$	14; 2; R_2
16.	$\dashv K\mathbf{A}_m\Sigma_i$	15; R_3

We shall thus be able to construct a new sequence the last line of which is $\ulcorner\dashv K\mathbf{A}_m\mathbf{B}\urcorner$. This sequence constitutes a demonstration of this last line from the premises $\vdash\mathbf{A}_1, \ldots, \vdash\mathbf{A}_{m-1}$.

NOTES

1. "A deduction theorem for rejection theses in Łukasiewicz's system of modal logic," *Notre Dame Journal of Formal Logic*, vol. XX (1979), pp. 461-464.

2. I refer to the system developed by Łukasiewicz in "A system of modal logic," *The Journal of Computing Systems*, vol. 1 (1953), pp. 111-149.

3. *Ibid*., p. 145.

4. *Ibid*., p. 114.

5. *Ibid*.

6. I use '**PC**' to stand for 'propositional calculus.' I feel free to make use of theses of the propositional calculus, since all of the axioms, and *a fortiori* all of the theses, of that calculus are demonstrable in Łukasiewicz's system. (*cf*. page 124 of "A system of modal logic".)

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