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LOCAL AND GLOBAL OPERATORS AND MANY-VALUED MODAL LOGICS

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1 *Motivation* In formal contexts, we may distinguish between two types of conditional operators.* The truth value of a *local* conditional is defined in terms of the truth values of its antecedent and consequent. The truth value of a *global* conditional is defined in terms of the possible truth-values of its antecedent and consequent. Global conditionals are usually formed by applying a modal operator to a local conditional. For example, strict implication is defined by applying the necessity operator to material implication. That is, "$p \prec q$" is defined as "$L(p\ C0\ q)$", where $C0$ is the standard two-valued material implication and the properties of the necessity operator "L" are determined by the particular modal logic being employed.

There has been some move to formally treat ordinary language counterfactual conditionals as global conditionals of a certain sort. (See for example [1] and [3]). Viewed from this perspective, the local conditional involved is not the standard two-valued material implication, but is rather a three-valued operator.

We may use T, F, and I for "true", "false", and "indeterminate", respectively. We may then define the conditional $C1$ as follows (contrasting it with material implication $C0$):

$C0$	T	F
T	T	F
F	T	T

$C1$	T	I	F
T	T	I	F
I	I	I	I
F	I	I	I

Some possible examples of conditional statements which may perhaps be appropriately translated by $C1$ are: "If you trespass, you will be

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prosecuted." "If you attack me, I will kill you." In cases in which the antecedent of these conditionals is not fulfilled, it may plausibly be argued that we cannot determine that the statements are either true or false; they remain indeterminate.

Now, consider a counterfactual statement: "If we were on the moon, we would see a grey, barren landscape." We know that the antecedent is false, and yet we do not want to say the truth value of the statement is indeterminate—we would instead claim it to be true. One possible way of formally treating such expressions would be to render them as global conditionals, perhaps compounded from *C1*. Letting V be the semantic valuation function, and employing the standard "possible worlds" interpretation of modal operators, a possible semantic definition would be:

$$V(\text{if } p \text{ then } q, w_i) = \begin{cases} T, & \text{if for all } w_j \text{ such that } w_iRw_j,\ V(p\ C1\ q, w_j) \neq F \\ F, & \text{otherwise} \end{cases}$$

In other words, we could semantically define the monadic operator L as follows:

$$V(L(p), w_i) = \begin{cases} T, & \text{if for all } w_j \text{ such that } w_iRw_j\ V(p, w_j) \neq F \\ F, & \text{otherwise} \end{cases}$$

Then the global conditional could simply be defined as $L(p\ C1\ q)$.

In a many-valued system, there are a variety of possible definitions for the operator L and for the local conditional. Each pair of definitions will yield in general another possible global conditional. It would be useful if there were a systematic method for investigating the formal properties of such global conditionals.

In this paper we present a scheme for treating a large class of global conditionals via many-valued modal logics. We will deal with the class of many-valued logics treated in [4]. We will make use of only one modal logic, namely **T** (see [2]). Our purpose in treating only one modal logic is pedagogical—most any modal system could be used. In addition, although we will for the sake of clarity restrict our treatment to the propositional case, these results may be extended to the predicate calculus.

Our procedure for constructing a many-valued modal logic is quite simple—we begin with a many-valued system, add a standard modal system, and then add a few axioms for completeness.

We will use $\vdash$ with and without subscripts in the usual manner to indicate provability in specific systems. In particular, we will use $\vdash_2$ to indicate provability in a standard complete two-valued system; the two-valued system is assumed to use the same sentence letters and the same connectives $\supset$, &, $\vee$, and $\sim$, as the multi-valued and modal systems, although of course the semantics for these syntactical symbols will not be the same for two-valued logic as for many-valued or modal logic. Thus, for our purposes, the same syntactical expression may occur in the two-valued system, in the n-valued system, $n > 2$, and in the modal system, but of course the interpretation of the expression will differ from system to system.

2 *A many-valued system* The many-valued system we will employ is treated in detail in [4]. For the semantics, we assume integer values $1, 2, \ldots, M$, where $2 \leqslant M$. A certain value S, $1 < S \leqslant M$, is assumed to be "critical". The values greater than S are said to be undesignated (false) and those less than or equal to S are said to be designated (true). We assume that a certain number of "truth-functional" connectives F_i are given. We use P, Q, and R as meta-expressions, with or without subscripts and superscripts. If F_i is a truth function of n components then if $P_1, \ldots, P_n$ are expressions, the truth value of the expression $F_i(P_1, \ldots, P_n)$ is denoted by $f_i(p_1, \ldots, p_n)$, where each p_j is the truth value of P_j. The specific object language symbols we shall employ include the monadic sentence operators J_k, $1 \leqslant k \leqslant M$. The expression $J_k(P)$ intuitively means "P takes value k". We assume that $F_1, \ldots, F_M$ refer to $J_1, \ldots, J_M$, respectively. We use $\supset$, &, $\vee$, and $\sim$ as conditional, conjunction, disjunction, and negation, respectively. We will use $F_{M+1}, \ldots, F_{M+4}$ to refer to these specific connectives. We assume the following "standard conditions" for these operators:

(1) The value of $P \& Q$ is designated iff both P and Q are designated.
(2) The value of $P \vee Q$ is undesignated iff both P and Q are undesignated.
(3) The value of $P \supset Q$ is undesignated iff P is designated and Q is undesignated.
(4) The value of $\sim P$ is designated iff P is undesignated.
(5) The value of $J_k(P)$ is designated iff the value of P is k.

Any matrix may be assigned to these operators as long as conditions (1)-(5) are satisfied. Any other operators may be introduced via the axioms to be given below. Additional operators will be referred to by F_j, for $j > M + 4$. We will also use the following notational conveniences:

$$\sum_{i=r}^{s} P_i = P_r \vee \ldots \vee P_s$$

$$\prod_{i=r}^{s} P_i = P_r \& \ldots \& P_s$$

$$\mathop{\Gamma}_{i=r}^{s} P_i Q = \begin{cases} Q & \text{if } r < s \\ P_s \supset \mathop{\Gamma}\limits_{i=r}^{s-1} P_i Q & \text{if } r \geqslant s \end{cases}$$

The axiom schemes for our multi-valued logic may then be given as follows:

(*A1*) $Q \supset (P \supset Q)$
(*A2*) $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$
(*A3*) $(P \supset Q) \supset ((Q \supset R) \supset (P \supset R))$
(*A4*) $(J_k(P) \supset (J_k(P) \supset Q)) \supset (J_k(P) \supset Q)$ for $1 \leqslant k \leqslant M$
(*A5*) $\mathop{\Gamma}\limits_{i=1}^{M} (J_i(P) \supset Q) Q$
(*A6*) $J_i(P) \supset P$, for $1 \leqslant i \leqslant s$

(*A7*) $\prod_{k=1}^{n} J_{p_k}(P_k)\, J_{f_i(p_1,\ldots,p_n)}(F_i(P_1, \ldots, P_n))$, for each n-place operator F_i, $1 \leqslant i$.

The only rule of proof is modus ponens (*mp*): from P and $P \supset Q$, infer Q. If an expression P is provable in the system we will write $\vdash_{\overline{mv}} P$. A word of clarification may be in order concerning axiom schema (*A7*). Suppose for example that $M = 3$, and consider the following dyadic operator:

*	1	2	3
1	1	2	3
2	1	2	3
3	2	3	1

Then (*A7*) would yield nine axiom schemes, one for each entry in the matrix. For example, the following 3 axioms correspond to the first column:

$$J_1(P_1) \supset (J_1(P_2) \supset J_1(P_1 * P_2))$$
$$J_2(P_1) \supset (J_1(P_2) \supset J_1(P_1 * P_2))$$
$$J_3(P_1) \supset (J_1(P_2) \supset J_2(P_1 * P_2))$$

By means of (*A7*), any truth functional operator desired may be introduced into the system.

Each choice of M and S will yield in general a different system. However, for each system it is possible to prove the following:

Theorem 1 *For any expression* P, $\vdash_{\overline{mv}} P$ *iff* P *always takes a designated value.* (See [4], Chapter III.)

Theorem 2 *Consider an expression* P *using only the operators* $\supset$, &, v, *and* $\sim$. *Then* $\vdash_2 P$ *iff* $\vdash_{\overline{mv}} P$.

Proof: If we replace "designated" and "undesignated" in the standard conditions by "true" and "false", we obtain the normal two-valued semantics for the operators. The result then follows from the completeness of the two-valued system and Theorem 1. Q.E.D.

3 *The addition of the modal system* For the sake of simplicity, we will here discuss only the system **T**; we will presuppose the treatment given in [2]. Any modal system formed by adding special axioms and rules of proof to a complete two-valued base could have been used. The addition of the modal portion is a device to introduce the desired "possible world" semantics into our system. We will now begin to be a bit more formal. We will initially assume that only one monadic modal operator L is to be introduced. We add the following axiom schemes to (*A1*)-(*A7*):

(*A8*) $L(P) \supset P$
(*A9*) $L(P \supset Q) \supset (L(P) \supset L(Q))$

In addition, we have the rule of necessitation (*nec*): From P infer $L(P)$. We assume then that the syntax consists of &, v, $\supset$, $\sim$, $J_1, \ldots, J_M$, and L. In addition we assume that we are given a set A of atomic sentence letters $A_1, A_2, \ldots$.

For the semantics, an interpretation I consists of a non-empty set W of worlds $w_1, w_2, \ldots$, a subordination relation R on the worlds, and a mapping V from $A \times W$ into the integers $\{1, \ldots, M\}$. For T, the only requirement on R is that it be reflexive. The valuation function V is extended to more complex expressions in two different ways. For the truth-functional operators F_i, discussed in the last section, we define:

$$V(F_i(P_1, \ldots, P_n), w_j) = f_i(V(P_1, w_j), \ldots, V(P_n, w_j))$$

where F_i is an n-ary operator. Thus each of the truth-functional operators is a *local* operator, in the sense that in each world the value of an expression whose major operator is *local* can be determined from the values of its constituents in that same world.

The operator L is a *global* rather than a local operator. That is, in world w_i the determination of the value of an expression $L(P)$ depends on the values that P takes in worlds other than w_i; in particular, it depends on the values that P takes in the worlds subordinate to w_i. That is, $V(L(P), w_i)$ will be a function of all values $V(P, w_j)$ such that $w_i R w_j$. We will be more specific at this point.

Intuitively, $L(P)$ means that P is necessary, or "true" in all possible worlds. Therefore, we will require that the definition of L must meet the following standard condition:

(6) The value of $L(P)$ in any world w_i is designated iff in every world w_j such that $w_i R w_j$ the value of P is designated.

Further, each of the standard conditions (1)-(5) are to be applied to each world. That is, each condition should be preceded by the phrase "In each world . . ." and followed by the phrase ". . . in that world". The operator L could be defined in many different ways, depending on the values of M and S. To say that P never takes the value k in any world (subordinate to the world in which the statement is made, say w_i) can be expressed by $L({\sim}J_k(P))$. That is, ${\sim}J_k(P)$ is designated in every world subordinate to w_i; which means that $J_k(P)$ is undesignated in every world subordinate to w_i; which means that P does not take the value k in any world subordinate to w_i. Thus, to say that P takes the value k in some world subordinate to the world in which the statement is made can be expressed by ${\sim}L({\sim}J_k(P))$. We can then express all possible such situations by combinations of these expressions. For example, suppose $M = 3$, and consider the following set of expressions:

(*SD1*) ${\sim}L({\sim}J_1(P))\ \&\ {\sim}L({\sim}J_2(P))\ \&\ {\sim}L({\sim}J_3(P))$
(*SD2*) $L({\sim}J_1(P))\ \&\ {\sim}L({\sim}J_2(P))\ \&\ {\sim}L({\sim}J_3(P))$
(*SD3*) ${\sim}L({\sim}J_1(P))\ \&\ L({\sim}J_2(P))\ \&\ {\sim}L({\sim}J_3(P))$
(*SD4*) $L({\sim}J_1(P))\ \&\ L({\sim}J_2(P))\ \&\ {\sim}L({\sim}J_3(P))$
(*SD5*) ${\sim}L({\sim}J_1(P))\ \&\ {\sim}L({\sim}J_2(P))\ \&\ L({\sim}J_3(P))$
(*SD6*) $L({\sim}J_1(P))\ \&\ {\sim}L({\sim}J_2(P))\ \&\ L({\sim}J_3(P))$

$(SD7)$ $\sim L(\sim J_1(P))$ & $L(\sim J_2(P))$ & $L(\sim J_3(P))$
$(SD8)$ $L(\sim J_1(P))$ & $L(\sim J_2(P))$ & $L(\sim J_3(P))$

Each of the expression schemes $(SD1)$-$(SD8)$ will be called a possible world state description (*pwsd*). The first one, $(SD1)$, says intuitively: there is a subordinate world in which P takes value 1, and a subordinate world in which P takes value 2, and a subordinate world in which P takes value 3. Note that $(SD8)$ is actually contradictory, since it says that P takes no value at all.

We assume that the value of $L(P)$ in a given world w_i depends only on the value that P takes in worlds subordinate to w_i. Thus the specific semantic definition of L can be expressed by using the *pwsd*'s. To elaborate the example above, suppose $S = 2$, and suppose we want L to be defined in the following way:

$$V(L(P), w_i) = \begin{cases} 1 \text{ iff for all } w_j \text{ such that } w_iRw_j,\ V(P, w_j) \neq 3 \\ 3 \text{ otherwise} \end{cases}$$

Such a definition satisfies standard condition (6). We may express this definition by a series of expression schemes of the language as follows:

$(SD1) \supset J_3(L(P))$
$(SD2) \supset J_3(L(P))$
$(SD3) \supset J_3(L(P))$
$(SD4) \supset J_3(L(P))$
$(SD5) \supset J_1(L(P))$
$(SD6) \supset J_1(L(P))$
$(SD7) \supset J_1(L(P))$
$(SD8) \supset J_1(L(P))$

It is easy to generalize from this example. The total number of *pwsd*'s depends on M. There will be 2^M *pwsd*'s. If we let $f(SDi)$ represent the value that $L(P)$ takes under the conditions described by *pwsd* (SDi), then we can express the semantics for L by a set of axiom schemes:

$(A10)$ $(SDi) \supset J_{f(SDi)}(L(P))$ for $1 \leq i \leq 2^M$

Alternatively, to lessen the number of axioms, we could use a method closer to the partial normal forms of [4]. This procedure amounts to considering each k, $1 \leq k \leq M$, and selecting all and only those (SDi) for which the expression takes value k. We would then have M axiom schemes of the form

$$\sum (SDi) \supset J_k(L(P)).$$

But the results are equivalent. The addition of axiom scheme $(A10)$ is necessary to guarantee the completeness of the system. We will write $\vdash$ with no subscript to indicate provability in the combined system.

The standard system **T** is formed by adding axioms $(A8)$ and $(A9)$ and the rule *nec* to a complete system of standard propositional calculus. We will use $\vdash_{\mathbf{T}}$ to indicate provability in standard two-valued **T**. Formed in this way, the system **T** can be shown to be complete with respect to the usual two-valued possible world semantics in which R is required only to be reflexive. We obtain useful relations between **T**, the **mv**-system, and the combined system by the following theorems:

Theorem 3 *For any expression P, if* $\vdash P$ *then in every world in every interpretation, P takes a designated value.*

Proof: The proof is the familiar one. It is an easy matter to verify that the theorem is true of any instance of any of the axioms and that the inference rules preserve the desired property. Both of these facts depend on our assumption that the connectives satisfy standard conditions. Q.E.D.

Theorem 4 *If P is any expression which contains no occurrence of L; then* $\vdash_{\mathbf{mv}} P$ *iff* $\vdash P$.

Proof: First suppose $\vdash_{\mathbf{mv}} P$. Since the axioms and rules of proof for the multi-valued system are contained in the combined system, the **mv** proof of P also serves to show $\vdash P$. On the other hand, suppose $\vdash P$. Then by Theorem 3, P takes a designated value in every world in every interpretation. But that means P always takes a designated value in the **mv** semantics. Then by Theorem 1, $\vdash_{\mathbf{mv}} P$. Q.E.D.

Theorem 5 *Let P be any expression whose only connectives are* $\supset$, &, v, $\sim$, *and L. Then* $\vdash P$ *iff* $\vdash_{\mathbf{T}} P$.

Proof: First, suppose $\vdash P$. Then by Theorem 3, in every world in every interpretation, P takes a designated value. If we replace the terms "designated" and "undesignated" in standard conditions (1)-(6) by the terms "true" and "false", respectively, we obtain the statement of the semantics for these connectives in **T**. Thus P is designated in every world in every interpretation in the combined system iff P is true in every world in every interpretation in the system **T**. But **T** is complete, so $\vdash_{\mathbf{T}} P$. On the other hand, suppose $\vdash_{\mathbf{T}} P$. Then by Theorems 2 and 4 and the fact that all the axioms and rules of proof of **T** are in the combined system, we have $\vdash P$. Q.E.D.

Theorem 6 *Let* P_1 *and* P_2 *be any two expressions and* A_i *be any sentence letter. Let* P_1' *be the result of uniformly substituting* P_2 *for* A_i *in* P_1. *If* $\vdash P_1$, *then* $\vdash P_1'$.

Proof: We transform the proof of P_1 by substituting P_2 for A_i everywhere. Since we are using axiom schemes, the property of being an instance of an axiom scheme is preserved. Then for any step in the proof of P_1, the rules used to justify that step may also be used to justify the corresponding step in the transformed proof. Q.E.D.

4 *Completeness of the combined system* We will construct a Henkin style completeness proof. Thus we will first discuss consistent sets and maximally consistent sets. A finite set of expressions $\{P_1, \ldots, P_n\}$ is said to be consistent iff not $\vdash \sim(P_1 \,\&\, \ldots \,\&\, P_n)$; an infinite set of expressions is consistent iff every finite subset is consistent. A set of expressions is said to be maximally consistent iff (a) it is consistent, and (b) if any expression not in the set is added to it, the resulting set is inconsistent. The proofs of the following theorems are directly parallel to the proofs in [2], Chapter 9.

Theorem 7 *If $\mathscr{P}$ is a maximally consistent set then there is no expression P such that both P and $\sim P$ are members of $\mathscr{P}$.*

Theorem 8 *If $\mathscr{P}$ is a maximally consistent set then for every expression P, either P or $\sim P$ is in $\mathscr{P}$.*

Theorem 9 *If $\mathscr{P}$ is a maximally consistent set then for any expressions, P and Q, if $P \supset Q$ is in $\mathscr{P}$ and P is in $\mathscr{P}$ then Q is in $\mathscr{P}$.*

Theorem 10 *Any consistent set of expressions $\mathscr{P}$ may be extended to a maximally consistent set $\mathscr{P}'$ such that $\mathscr{P} \subseteq \mathscr{P}'$.*

Theorem 11 *For any expressions $P, P_1, \ldots, P_n$, if $\{L(P_1), \ldots, L(P_n), \sim L(\sim P)\}$ is a consistent set then so is $\{P_1, \ldots, P_n, P\}$.*

For the completeness proof, we will also need the following theorem:

Theorem 12 *If $\mathscr{P}$ is a maximally consistent set then for every expression P there is exactly one k such that $J_k(P)$ is a member of $\mathscr{P}$.*

Proof: Let P and $\mathscr{P}$ be given. We first show that there is such a k. By Theorem 8, for each k such that $1 \leqslant k \leqslant M$, either $J_k(P)$ is in $\mathscr{P}$ or $\sim J_k(P)$ is in $\mathscr{P}$. But $\sim J_1(A_i) \mathbin{\&} \ldots \mathbin{\&} \sim J_M(A_i)$ is an **mv**-contradiction. Hence by Theorem 1, $\models_{\mathbf{mv}} \sim(\sim J_1(A_i) \mathbin{\&} \ldots \mathbin{\&} \sim J_M(A_i))$. But then by Theorem 4 and Theorem 6, $\vdash \sim(\sim J_1(P) \mathbin{\&} \ldots \mathbin{\&} \sim J_M(P))$. Thus, since $\mathscr{P}$ is consistent, it cannot be the case that for every k, $1 \leqslant k \leqslant M$, $\sim J_k(P)$ is in $\mathscr{P}$. Hence for some k, $J_k(P)$ is a member of $\mathscr{P}$. To show that k is unique, suppose for $k_1 \neq k_2$, $J_{k_1}(P)$ is in $\mathscr{P}$ and $J_{k_2}(P)$ is in $\mathscr{P}$. Note that $J_{k_1}(A_i) \mathbin{\&} J_{k_2}(A_i)$ is an **mv** contradiction, and hence $\models_{\mathbf{mv}} \sim(J_{k_1}(A_i) \mathbin{\&} J_{k_2}(A_i))$. Thus $\vdash \sim(J_{k_1}(P) \mathbin{\&} J_{k_2}(P))$. But this contradicts the assumption that $\mathscr{P}$ is consistent. Hence k must be unique. Q.E.D.

Theorem 13 *Let $\mathscr{P}$ be a maximally consistent set and let P be any expression. Then P is in $\mathscr{P}$ iff for some k, $1 \leqslant k \leqslant S$, $J_k(P)$ is in $\mathscr{P}$.*

Proof: First, suppose P is in $\mathscr{P}$. By Theorem 12, we know there is some k such that $J_k(P)$ is in $\mathscr{P}$. Suppose $k > S$. Then $A_i \mathbin{\&} J_k(A_i)$ is an **mv** contradiction. Hence $\vdash \sim(P \mathbin{\&} J_k(P))$. But this contradicts the consistency of $\mathscr{P}$, and thus $1 \leqslant k \leqslant S$. On the other hand, suppose for some k, $1 \leqslant k \leqslant S$, $J_k(P)$ is in $\mathscr{P}$ but that P is not in $\mathscr{P}$. Then $\sim P$ is in $\mathscr{P}$. But then $\sim A_i \mathbin{\&} J_k(A_i)$ is an **mv** contradiction, and thus $\vdash \sim(\sim P \mathbin{\&} J_k(P))$. But this contradicts the consistency of $\mathscr{P}$, and hence P must be in $\mathscr{P}$. Q.E.D.

Theorem 14 *If $\mathscr{P}$ is any maximally consistent set and P is any expression such that $\vdash P$, then P is in $\mathscr{P}$.*

Proof: It is not difficult to show that if $\vdash P$ then $\vdash \sim\sim P$. Hence if $\sim P$ were in $\mathscr{P}$, $\mathscr{P}$ could not be consistent. But then by Theorem 8, P is in $\mathscr{P}$. Q.E.D.

We now proceed to our main result, the completeness theorem for the combined system.

Theorem 15 *For any expression P, if P takes a designated value in every world in every interpretation, then* $\vdash P$.

Proof: By the standard Henkin style argument, it is sufficient to show that for any consistent set of formulas, there is an interpretation which assigns a designated value to each formula of the set. Suppose $\mathscr{K}$ is our consistent set. By Theorem 10, we extend $\mathscr{K}$ to a maximally consistent set $\mathscr{B}_0$. Consider an arbitrary expression of the form $\sim L(\sim P)$ in $\mathscr{B}_0$, and consider the set $\mathscr{K}' = \{P\} \cup \{Q: L(Q) \text{ is in } \mathscr{B}_0\}$. It follows from Theorem 11 that $\mathscr{K}'$ is consistent. Hence we can extend $\mathscr{K}'$ to a maximally consistent set $\mathscr{B}_1$. In this fashion, we construct a family of maximally consistent sets $\mathscr{B}_i$ such that (a) $\mathscr{K} \subseteq \mathscr{B}_0$, and (b) for any $i > 0$, there is some j such that $\mathscr{B}_i$ is formed on the basis of $\mathscr{B}_j$ by extending a set of the form $\mathscr{K}'$ above, and (c) for every $\mathscr{B}_i$ and every expression of the form $\sim L(\sim P)$ in $\mathscr{B}_i$, there is a $\mathscr{B}_j$ formed by extending a set of the form $\mathscr{K}'$, above. To each set $\mathscr{B}_i$ we associate a world w_i. We set w_iRw_j, and for $i \neq j$, we set w_iRw_j iff $\mathscr{B}_j$ was formed on the basis of $\mathscr{B}_i$. In each w_i, we assign every sentence letter A_j the value k for which $J_k(A_j)$ is in $\mathscr{B}_i$; Theorem 12 guarantees the possibility of making such an assignment. We now show that in every world w_i, for any expression P, $V(P, w_i) = k$ iff $J_k(P)$ is in $\mathscr{B}_i$. The proof is by induction on the complexity of P.

Case 1: Suppose P is a sentence letter, A_j. Then the theorem is true by construction of the interpretation.

Case 2: Suppose P is of the form $F_k(P_1, \ldots, P_n)$ where F_k is a local n-ary syntactical operator. We know that for any w_i, for each P_j, there is a k_j such that $J_{k_j}(P_j)$ is in w_i. By induction, $V(P_j, w_i) = k_j$. Further, by Theorems 9 and 14 and axiom scheme (*A7*), we know $J_{f_k(k_1,\ldots,k_n)}(P)$ is in $\mathscr{B}_i$. But by definition of F_k, $V(F_k(P_1, \ldots, P_n), w_i) = f_k(k_1, \ldots, k_n)$.

Case 3: Suppose P is of the form $L(Q)$. Consider an arbitrary world w_i. For each k, $1 \leq k \leq M$, consider the expression $L(\sim J_k(Q))$; either it or its negation must be in w_i. Form the *pwsd* (*SDi*) by conjoining the expressions $L(\sim J_k(Q))$ which are in $\mathscr{B}_i$ and the expressions $\sim L(\sim J_k(Q))$ which are in $\mathscr{B}_i$. It is not difficult to show that for any expressions

$$R_1, \ldots, R_n, \vdash R_1 \supset \left(R_2 \supset \left(\ldots \supset \left(R_n \supset \left(\prod_{i=1}^{n} R_i\right)\right)\right)\right).$$

Hence by Theorems 14 and 9, (*SDi*) is in $\mathscr{B}_i$. But then by Theorems 14 and 9 and axiom scheme (*A10*), $J_{f(SDi)}(L(Q))$ is in $\mathscr{B}_i$. We must show that $L(Q)$ takes value $f(SDi)$ in w_i. Consider each conjunct in (*SDi*). (a) If it is of the form $L(\sim J_k(Q))$, then by construction, $\sim J_k(Q)$ is in every set $\mathscr{B}_j$ constructed from $\mathscr{B}_i$; then $J_k(Q)$ is in no set $\mathscr{B}_j$ constructed from $\mathscr{B}_i$; further, by axiom scheme (*A8*), $J_k(Q)$ is not in $\mathscr{B}_i$; thus by induction Q never takes value k in any world w_j such that w_iRw_j. (b) If it is of the form $\sim L(\sim J_k(Q))$, then by construction, some set $\mathscr{S}_j$ was constructed from $\mathscr{S}_i$, and $\mathscr{S}_j$ contains $J_k(Q)$; hence by induction, Q takes value k in w_j, and since $\mathscr{S}_j$ was constructed from $\mathscr{S}_i$, we know $w_i R w_j$. Thus each of the conditions specified in (SDi) is satisfied. Hence by definition, $V(L(Q), w_i) = f(SDi)$.

Thus we know that in every world w_i, for any expression P, $V(P, w_i) = k$ iff $J_k(P)$ is in $\mathscr{S}_i$. In particular then, $V(P, w_0) = k$ iff $J_k(P)$ is in $\mathscr{S}_0$. But then by Theorem 13, every expression in $\mathscr{S}_0$ is assigned a designated value. Thus in particular, every formula of the original consistent set $\mathscr{W}$ is assigned a designated value. Q.E.D.

5 *Applications* Using the techniques presented here, it is possible to investigate the properties of a wide variety of global conditionals and to investigate their interrelationships. The first step is to decide what structure one wishes to impose on the possible worlds. (In the case sketched above, we placed no restrictions on the number of worlds and required only that the relation R be reflexive.) We then attach the axioms required to impose that structure to the many-valued logic of our choice. There are at least three ways then of obtaining different global operators within the same system.

The first way of obtaining different global operators is to introduce a multiplicity of local operators. For example, consider the conditionals defined by the following matrices:

$C2$	T	I	F
T	T	F	F
I	I	I	I
F	I	I	I

$C3$	T	I	F
T	T	F	F
I	I	I	F
F	I	I	I

Varied global operators may then be expressed by applying the operator L to the local operators. For example, we may wish to compare $L(P\ C1\ Q)$, $L(P\ C2\ Q)$, and $L(P\ C3\ Q)$.

A more direct route requires the introduction of another axiom scheme. Suppose we wish to introduce several global operators K_m^n, where K_m^n is the mth n-ary operator. We need to slightly generalize the notion of a *pwsd*. Consider an expression of the form $J_{k_1}(P_1)\ \&\ \ldots\ \&\ J_{k_n}(P_n)$; there are M^n such expressions—we will call them local state descriptions (*lsd*'s). A possible world state description consists of a statement of which of the *lsd*'s are realized and which are not. Thus there are 2^{M^n} *pwsd*'s. The following axiom scheme must then be added to our system:

$$(A11)\ (SDi) \supset J_{f_m^n(SDi)}\ (K_m^n(P_1, \ldots, P_n)) \qquad \text{for } 1 \leqslant i \leqslant 2^{M^n}$$

In this way, *any* global operator may be introduced into the system, as long as its semantics may be defined in terms of *pwsd*'s. The completeness proof remains much the same as before, with an additional step in the induction for K_m^n; such a step would closely parallel the step for the L operator. In fact, axiom $(A10)$ is just a special case of axiom $(A11)$.

The problem with the above two methods is that they presuppose the same possible world structure for all global operators. We may wish to introduce one global operator with the possible world structure of **T** and another with the possible world structure of S4, for example. Our third approach is then to add separate modal axioms for each operator to be introduced. However, the completeness of a system so constructed is an open question.

In conclusion, it has been demonstrated for the propositional case that once a number M of truth values, a critical value S, and a possible worlds structure have been selected, it is possible to construct a many-valued modal logic with the desired semantics. There seems to be no barrier to extending these results to the predicate calculus as well.

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