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# ADMISSIBLE SETS AND RECURSIVE EQUIVALENCE TYPES

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Recently there has been much interest in admissible sets. Part of this is due to the fact that the constructive notions of finiteness and recursion can be extended to include infinite sets and operations. In such a structure, recursive equivalence types can be defined which correspond to the classical ones. We shall show that the Cantor-Bernstein Theorem and the Tarski Cancellation Law hold in a straightforward manner. However, a satisfactory definition of an isol depends upon the admissible set. We shall exclude projectible admissible sets which have elements that include large $\Sigma_1$ definable subsets. Also, we shall need a weak uniformizing procedure to tie together recursive enumerability and $\Sigma_1$ definability. With these conditions some of the equivalences that hold for isols can be extended to admissible sets. We shall conclude with a stronger definition of an isol which preserves a cancellation law similar to that for the ordinary isols.*

**1** *Definitions and propositions* The following definition and Propositions 1-6 are due to Jensen [3]. The definition will give great flexibility in defining functions. The proofs of the propositions are elementary and can be found in [1]. Throughout we shall consider a non-empty transitive set $\mathfrak{M}$. Our language contains the predicates = and $\epsilon$ with their usual interpretation and constant symbols for each $x \epsilon \mathfrak{M}$; we shall use the same symbol for both object and name. We allow *bounded quantification* $\forall x \epsilon y\varphi$ and $\exists x \epsilon y\varphi$. Those formulas which contain only bounded quantifiers are called $\Sigma_0$ *predicates*. They are closed under the operations $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$, and $\forall x \epsilon y$, $\exists x \epsilon y$. We are particularly interested in the $\Sigma_1(\Pi_1)$ *predicates* of the form $\exists x\varphi(\forall x\varphi)$ where $\varphi$ is $\Sigma_0$. We say $\mathfrak{M}$ is *admissible* if $\mathfrak{M}$ satisfies the following axioms (called **PZF**):

(1) Axioms of the empty set, pairing, and union.

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(2) $\Sigma_0$ separation: $\forall x \exists y \forall z [z \in y \leftrightarrow z \in x \wedge \varphi]$, where $\varphi$ is $\Sigma_0$ and only $z$ is free in $\varphi$.
(3) $\Sigma_0$ replacement or collection:

$$\forall x \exists y \varphi \rightarrow \forall u \exists v \forall x \in u \exists y \in v \varphi,$$

where $\varphi$ is $\Sigma_0$ and only $x$, $y$ are free in $\varphi$.

If $x \in \mathfrak{M}$, we say $x$ is *metafinite*. Let $\mathfrak{M}^k$ denote the set of $k$-tuples with elements in $\mathfrak{M}$. A set or relation $\alpha \subset \mathfrak{M}^k$ is $\Sigma_1(\Pi_1)$ *definable* if there is a $\Sigma_1(\pi_1)$ predicate $\varphi$ with $k$ free variables such that

$$(x_1, \ldots, x_k) \in \alpha \leftrightarrow \varphi(x_1, \ldots, x_k).$$

We call $\alpha$ a $\Sigma_1(\Pi_1)$ *set*. A set $\alpha$ is $\Delta_1$ *definable* if it is both $\Sigma_1$ and $\Pi_1$ definable; we call $\alpha$ a $\Delta_1$ *set*. For clarity we shall use the following conventions: An admissible set is denoted by $\mathfrak{M}$. Lower case English letters $a$, $b$, $c$, $u$, $v$, $w$, $x$, $y$, $z$ will denote elements of $\mathfrak{M}$; the remainder will denote functions. Lower case Greek letters $\alpha$, $\beta$, $\gamma$ etc. will denote subsets of $\mathfrak{M}$; but we shall reserve the use of $\varphi$ and $\psi$ for predicates. Also dom $f$, rng $f$, $f''a$, $f \restriction a$; $f \circ g$ will denote the domain of $f$, range of $f$, the image of $a$ under $f$, the restriction of $f$ to $a$ and the composition of $f$ and $g$.

Proposition 1 ($\Delta_1$ separation) *The intersection of a metafinite set and a $\Delta_1$ set is a metafinite set.*

Proposition 2 ($\Sigma_1$ collection) *Let $\varphi(x,y)$ be a $\Sigma_1$ predicate and suppose $\forall x \exists y \varphi(x,y)$ holds in $\mathfrak{M}$. Then for any $a \in \mathfrak{M}$ there is a $b \in \mathfrak{M}$ such that $\forall x \in a \exists y \in b \varphi(x,y)$.*

Proposition 3 *If $\varphi(x)$ is $\Sigma_1$, then so is $\forall x \in a \varphi(x)$ and $\exists x \in a \varphi(x)$.*

We shall say a partial function $f$ from $\mathfrak{M}^k$ into $\mathfrak{M}$ is *partial recursive* (p.r.) if the relation $f(x_1, \ldots, x_k) = y$ is $\Sigma_1$; a p.r. function is *recursive* if the domain of $f$ is all of $\mathfrak{M}$. We note that the domain and range of a p.r. function are $\Sigma_1$ sets.

Proposition 4 *Let $f$ be a* p.r. *function and $a \in \mathfrak{M}$. If $a \subset$ dom $f$ then $f''a \in \mathfrak{M}$.*

Proposition 5 *The composition of (partial) recursive functions is (partial) recursive. Also, $\bigcup x$, $x \cup y$, $\{x_1, \ldots, x_k\}$, $\langle x_1, \ldots, x_k \rangle$,* dom $x$, rng $x$, $x''y$, *etc. are recursive functions.*

Proposition 6 (Recursion Theorem) *Let $h(y, x_1, \ldots, x_k)$ be a recursive function such that $\{\langle z,y \rangle \mid z \in h(y, x_1, \ldots, x_k)\}$ is a well-founded relation for all $x_1, \ldots, x_k$ in $\mathfrak{M}$. Let $g(y, x_1, \ldots, x_k, w)$ be a* p.r. *function. Then there exists exactly one* p.r. *function $f(y, x_1, \ldots, x_k)$ such that*

$$f(y, x_1, \ldots, x_k) \simeq g(y, x_1, \ldots, x_k, \langle f(z, x_1, \ldots, x_k) \mid z \in h(y, x_1, \ldots, x_k) \rangle).$$

Proposition 7 *Let* On *denote the ordinals in $\mathfrak{M}$. Then* On *is a $\Sigma_0$ set.*

We shall use the convention that if $x$, $y$ are ordinals then $x < y$ means $x \in y$.

Proposition 8 (Choice function for metafinite sets of ordinals) *For* $x \in \mathfrak{M}$, *let* $\mathrm{inf}(x) = y$ *if* $x \subset \mathrm{On}$ *and* $y$ *is the smallest element in* $x$, *and* $\mathrm{inf}(x) = 0$ *otherwise. Then* $\mathrm{inf}(x)$ *is a recursive function.*

**2** *Simple arithmetic on admissible sets* We say $\varepsilon$ is an *admissible ordinal* if $\varepsilon$ is the first ordinal not in an admissible set $\mathfrak{M}$. We shall consider subsets of $\varepsilon$ and assume we are in some fixed admissible set $\mathfrak{M}$. A p.r. function $f(x)$ whose domain is $\varepsilon$ will also be called *recursive*. (We can make $\mathrm{dom}\, f$ equal $\mathfrak{M}$ by letting $f(x) = 0$; if $x \notin \mathrm{On}$.) The following definitions are standard and can be found in Dekker and Myhill [2]. Denote by $\mathcal{P}(\varepsilon)$ the set of all subsets of $\varepsilon$. We say $\alpha, \beta \in \mathcal{P}(\varepsilon)$ are *recursively equivalent* ($\alpha \simeq \beta$) if there is a 1-1 p.r. function $f(x)$ such that $\alpha \subset \mathrm{dom}\, f$ and $f''\alpha = \beta$. It follows that $\simeq$ is an equivalence relation; thus $\simeq$ partitions $\mathcal{P}(\varepsilon)$ into equivalence classes. For $\alpha \in \mathcal{P}(\varepsilon)$, define $\langle\alpha\rangle = \{\beta \mid \beta \simeq \alpha\}$. If $A = \langle\alpha\rangle$, $A$ is called the *recursive equivalence type* (**RET**) of $\alpha$. Let $\Omega$ be the class of **RET**s of members of $\mathcal{P}(\varepsilon)$. If $\alpha \in A \in \Omega$ we call $\alpha$ a *representative* of $A$.

We know from the ordinary theory of **RET**s that to define addition we must put a separability condition on the representatives of **RET**s. If $\alpha, \beta \in \mathcal{P}(\varepsilon)$, then $\alpha$ is *separable* from $\beta (\alpha | \beta)$ if there exist $\Sigma_1$ sets $\mu, \nu \in \mathcal{P}(\varepsilon)$ such that $\alpha \subset \mu$, $\beta \subset \nu$, and $\mu \cap \nu = \emptyset$. Given any two **RET**s we can always find separable representatives in the even and odd ordinals.

Proposition 9 *If* $\alpha_1 \simeq \alpha_2$ *and* $\beta_1 \simeq \beta_2$ *such that* $\alpha_1 | \beta_1$ *and* $\alpha_2 | \beta_2$, *then* $\alpha_1 \cup \beta_1 \simeq \alpha_2 \cup \beta_2$.

Let $A, B \in \Omega$ and $\alpha \in A$, $\beta \in B$, where $\alpha | \beta$. We define the *sum* of $A$ and $B$ by $A + B$, where $A + B = \langle\alpha \cup \beta\rangle$. From the above, addition can always be performed and is well defined. If $A, B \in \Omega$ we shall say $A \leqslant B$ if there is a $C \in \Omega$ such that $A + C = B$.

Theorem 1 (Cantor-Bernstein Theorem) *Let* $A, B$ *be* **RET**s *of an admissible ordinal* $\varepsilon > \omega$, *the first countable ordinal. Then* $A \leqslant B$ *and* $B \leqslant A$ *implies* $A = B$.

*Proof:* We note that a proof of the theorem for the case $\varepsilon = \omega$ uses the facts that (1) $\Sigma_1$ sets are recursively enumerable, (2) a computation can be halted before it is completed, and (3) finite sets can be well ordered by a recursive function. We do not assume these facts; however, there is enough structure to permit an iteration a countable number of times. The proof will follow the classical one, modifying where necessary to make it effective. Since $A \leqslant B$ and $B \leqslant A$, there exist **RET**s $C$ and $D$ such that $A + C = B$ and $B + D = A$. We can assume that there exist representatives $\alpha, \beta, \gamma, \delta$ of $A, B, C, D$, respectively, such that $\alpha$ and $\beta$ are contained in the even ordinals and $\gamma$ and $\delta$ are contained in the odd ordinals. Moreover, there exist two p.r. functions $p(x)$ and $q(x)$ such that $\alpha \cup \gamma \subset \mathrm{dom}\, p$, $p''(\alpha \cup \gamma) = \beta$ and $(\beta \cup \delta) \subset \mathrm{dom}\, q$, $q''(\beta \cup \delta) = \alpha$. Let us also assume $0, 2 \notin \mathrm{dom}\, p$. Define the p.r. function $r(x)$ as follows: Let $r(0) = 0$, $r(2) = 2$. In the following if $q^{-1}(x)$ or $p^{-1}q^{-1}(x)$ is undefined, $r(x)$ is also undefined. For $x \neq 0, 2$ compute $q^{-1}(x)$; if $q^{-1}(x)$ is odd, let $r(x) = 0$. If $q^{-1}(x)$ is even, compute $z = p^{-1}q^{-1}(x)$. If $z$ is odd, let $r(x) = 2$; if $z$ is even, let $r(x) = z$. Note that for $x \in \alpha$, $r(x) = 0$ if $q^{-1}(x) \in \delta$, $r(x) = 2$ if $z \in \gamma$, and $r(x) = z$ if $z \in \alpha$. Define the p.r. function $g(y,x,a)$ as follows:

$$g(y,x,a) = z \leftrightarrow \text{'}a \text{ is a function'} \wedge \text{dom } a = y \wedge \text{rng } a \subset \text{On}$$
$$\wedge [[y = 0 \wedge z = r(x)] \vee [0 < y < \omega \wedge z = r(a_{y-1})] \vee [y = \omega \wedge z = \inf(\text{rng } a)]].$$

Here we use the fact that $\varepsilon$ is greater than $\omega$ and that the choice principle holds for a metafinite subset of $\varepsilon$. We define the function $f(y,x)$ by the recursion theorem. Let $h(y,x) = y$, if $y \in \text{On}$ and 0 otherwise. Then for any $x$ the relation $\{\langle z,y\rangle \mid z \in h(y,x)\} = \{\langle z,y\rangle \mid z < y\}$ is well-founded. Hence by the recursion theorem $f(y,x) \simeq g(y,x,\langle f(z,x) \mid z < y\rangle)$ is a p.r. function. By the construction of $g(y,x,a)$ we see that for $y < \omega$, $f(y,x) \simeq r^{y+1}(x)$, an iteration of $x$ by $r(x)$ for $y + 1$ times. If $x \in \alpha$, the function $f(y,x)$ is defined for every $y < \omega$ and hence, by a slight modification of Proposition 4, $\langle f(y,x) \mid y < \omega\rangle$ is a metafinite set. Then $f(\omega,x)$ is always defined for any $x \in \alpha$. Note that if $x \in \alpha$, $f(\omega,x) = 0$ means that the computations $q^{-1}(x)$, $p^{-1}q^{-1}(x)$, $q^{-1}p^{-1}q^{-1}(x)$, . . . eventually terminate in $\delta$; $f(\omega,x) > 0$ means that either the computations are always in $\alpha$ and $\beta$ or they eventually terminate in $\gamma$. Finally, define the 1-1 p.r. function $t(x)$ which will map $\alpha$ onto $\beta$: $t(x) = q^{-1}(x)$ if $f(\omega,x) > 0$ and $t(x) = p(x)$ if $f(\omega,x) = 0$, and is undefined otherwise. The function $t(x)$ corresponds to the classical equivalence. Q.E.D.

Theorem 2 (Tarski Cancellation Law) *Let $A$, $B$, $M$ be* **RET***s of an admissible ordinal $\varepsilon > \omega$. If $A + M = B + M$, then there exist* **RET***'s $A'$, $B'$, $N$ such that $A = A' + N$, $B = B' + N$ and $A' + M = M = B' + M$.*

*Proof:* Let $\alpha$, $\beta$, $\mu$ be representatives of $A$, $B$, $M$ respectively such that $\alpha$ and $\beta$ are contained in the even ordinals and $\mu$ is contained in the odd ordinals. We also assume that there exists a 1-1 p.r. function $p(x)$ such that $\alpha \cup \mu \subset \text{dom } p$ and $p''(\alpha \cup \mu) = \beta + \mu$. Let $p_0(x) = p(x)$ and $p_1(x) = p^{-1}(x)$. Define the p.r. function $g\ (y,x,a)$ for $i < 2$ as follows:

$$g_i(y,x,a) = b \leftrightarrow x,\ y \in \text{On} \wedge \text{'}a \text{ is a function'} \wedge$$
$$[[\,y = 0 \wedge b = p_i(x)] \vee [0 < y < \omega \wedge \text{dom } a = y \wedge$$
$$[[\text{'}a_{y-1} \text{ is even'} \wedge b = a_{y-1}] \vee [\ \text{'}a_{y-1} \text{ is odd'} \wedge$$
$$b = p_i(a_{y-1})]]] \vee [y = \omega \wedge b = \bigcup \text{rng } a]].$$

By the recursion theorem define the p.r. function

$$f_i(y,x) \simeq g(y,\ x, \langle f_i(z,x) \mid z < y\rangle)\,,$$

and $r_i(x) = f_i(\omega,x)$, for $i < 2$. For $x \in \alpha$, $r_0(x)$ is the union of an iteration of $p(x)$ until $p^y(x)$ is in $\beta$. We shall now define the 1-1 p.r. functions $s_i(x)$ that will set up our correspondence: For $i < 2$ and $x$ even, compute $r_i(x)$. If $r_i(x)$ has an even element $w$, let $s_i(x) = \omega$; otherwise let $s_i(x) = p_i(x)$. For $x$ odd, compute $r_i(x)$. If $r_i(x)$ has an even element let $s_i(x) = x$; otherwise let $s_i(x) = p_i(x)$. If $r_i(x)$ does not exist, $s_i(x)$ is left undefined. For $x \in \alpha$, $s_0(x)$ is a member of $\beta$ if the iteration of $p^y(x)$ ends up in $\beta$; otherwise $s_0(x)$ is equal to $p(x)$, an element in $\mu$. For $x \in \mu$, $s_0(x)$ is $x$ if the iteration ends up in $\beta$; otherwise $s_0(x) = p(x)$. In both cases $s_0(x)$ is in $\mu$. In view of the definition of the function $s_0(x)$, it is an easy though tedious exercise to show that $s_0(x)$ is 1-1. By symmetry, we see that $s_1(x)$ is 1-1. Moreover, if $x \in \alpha$ and $s_0(x) = y \in \beta$, we then have $s_1(y) = x$. Let $\gamma_{i0} = \{x | s_i(x) \text{ is even}\}$, $\gamma_{i1} = \{x | s_i(x) \text{ is odd}\}$, for $i < 2$. These sets are $\Sigma_1$ such that $\gamma_{i0}$, $\gamma_{i1}$ are disjoint. Let $\alpha_i = \alpha \cap \gamma_{0i}$ and $\beta_i = \beta \cap \gamma_{1i}$, for $i < 2$. We then have $\alpha_0 | \alpha_1$, $\beta_0 | \beta_1$; and $\alpha = \alpha_0 \cup \alpha_1$, $\beta = \beta_0 \cup \beta_1$. If $x \in \alpha_0$, $s_0(x)$ is in $\beta_0$; if $y \in \beta_0$ then $s_1(y) = s_0^{-1}(y) \in \alpha_0$. Hence $\alpha_0 \simeq \beta_0$. From the definition of $s_i(x)$, $\alpha_1$ and $\beta_1$ we see that $s_0''(\alpha_1 \cup \mu) = \mu$ and $s_1''(\beta_1 \cup \mu) = \mu$. Hence $\alpha_1 \cup \mu \simeq \mu$ and $\beta_1 \cup \mu \simeq \mu$. Letting $N = \langle\alpha_0\rangle = \langle\beta_0\rangle$, and $A_1 = \langle\alpha_1\rangle$, $B_1 = \langle\beta_1\rangle$, we have $A = N + A_1$, $B = N + B_1$, and $A_1 + M = B_1 + M = M$. Q.E.D.

From the constructions in the proof, we see that a cardinal theorem whose proof uses a definable countable iteration will generalize to **RET**s of an admissible set. For example, in [1], it is proved that if $n \cdot A = A + \ldots + A$ $n$ times, then $n \cdot A \leqslant n \cdot B$ implies $A \leqslant B$.

**3** *Recursively enumerable sets* We would like to generalize the concept of "having no infinite recursively enumerable subset" which plays such a vital role in the theory of isols. It should be clear that "infinite" now means "not metafinite." We have two problems: (1) In projectible admissible sets, there are elements which contain a large ($\Sigma_1$ but not metafinite) subset. Almost any standard theorem about recursively enumerable or immune sets will fail. (2) A priori, there may not be enough structure to perform a systematic search. For example, since the finite sets can be effectively enumerated, if a $\Sigma_1$ predicate $\exists y\varphi(x,y)$ holds we can effectively find a $z$ such that $\varphi(x,y)$ holds for $y = z$ but not for $y < z$. This searching enables us to enumerate a $\Sigma_1$ set by a recursive function. In an admissible set, there may be no definable well-ordering of its members; hence any analogue of recursive enumerability will fail. For our purposes we shall consider those admissible sets whose $\Sigma_1$ predicates $\exists y\varphi(x,y)$ can be uniformized by a $\Sigma_1$ function $f(x)$ such that if $\exists y\varphi(x,y)$ holds, so does $\varphi(x,f(x))$. Although weaker than well-ordering, we shall be able to generalize some basic propositions about recursive enumerability to admissible sets.

The following two definitions and Proposition 10 are due to Jensen [3]. An admissible set $\mathfrak{M}$ is *non-projectible* if it satisfies the stronger replacement axiom:

(A) $\forall u \exists v \forall x \in u[\exists y\varphi \leftrightarrow \exists y \in v\varphi]$,

where $\varphi$ is $\Sigma_0$ and only $x$ and $y$ are free in $\varphi$. Otherwise $\mathfrak{M}$ is *projectible*. We say the function $f(x)$ *uniformizes* the predicate $\varphi(x,y)$ if $\text{dom}\, f = \{x | \exists y\varphi(x,y)\}$ and $\forall x[\exists y\, \varphi(x,y) \leftrightarrow \varphi(x,f(x))]$. We say an admissible set is $\Sigma_1$ *uniformizable* if each $\Sigma_1$ predicate is uniformizable by a partial recursive function.

Proposition 10 *Let $\mathfrak{M}$ be a $\Sigma_1$ uniformizable admissible set. Then the following are equivalent to* (A):

(B) *If* $u \in \mathfrak{M}$ *and* $\alpha$ *is any* $\Sigma_1$ *set, then* $u \cap \alpha \in \mathfrak{M}$.
(C) *If* $u \in \mathfrak{M}$ *and* $f(x)$ *is a* p.r. *function, then* $f''u \in \mathfrak{M}$.

We note that (A) $\rightarrow$ (B) $\rightarrow$ (C) follows directly from the definition of (A); we need $\Sigma_1$ uniformizability to prove (C) $\rightarrow$ (A).

A non-empty set $\alpha$ contained in $\varepsilon$ is *recursively enumerable* (r.e.) if there is a recursive function $f(x)$ with domain $\varepsilon$ and range $\alpha$, i.e., $f''\varepsilon = \alpha$. Let us say that the empty set is r.e. A set $\alpha$ contained in $\varepsilon$ is *recursive* if $\alpha$ is a $\Delta_1$ set.

Proposition 11 *Any metafinite set is recursive. Any* r.e. *set is* $\Sigma_1$.

Theorem 3 *Any recursive set is recursively enumerable.*

*Proof:* Let $\alpha \subset \varepsilon$ be a recursive set; then there exist $\Sigma_0$ predicates $\varphi(x,u)$ and $\psi(x,v)$ such that $x \in \alpha \leftrightarrow \exists u \varphi(x,u)$ and $x \notin \alpha \leftrightarrow \exists v \psi(x,v)$. We define the p.r. function $g(x,d)$ as follows, where $x \in \mathrm{On}$ and $d \in \mathfrak{M}$ is a 1-1 strictly increasing function with domain $x$ and range contained in $\varepsilon$:

$$g(x,d) = z \leftrightarrow \forall w \in \mathrm{rng}\ d[z > w] \wedge \exists u \varphi(z,u) \wedge \\ \forall z' < z[\exists y < x[z' \leqslant d_y] \vee \exists v \psi(z',v)].$$

The element $g(x,d)$ is the first element of $\alpha$ not in the range of $d$. By the recursion theorem we can define a function $f(x)$ such that

$$f(x) \simeq g(x, \langle f(y) \mid y < x \rangle).$$

We note that if $f(x)$ is defined, then $f(x)$ is an element of $\alpha$, and that for any $y < x$, $y \in \alpha$ iff there is an $x' < x$ such that $f(x') = y$. Also, $f(x)$ is a 1-1 strictly increasing function whose domain is an ordinal $w \leqslant \varepsilon$. If $w < \varepsilon$, then it follows that $\alpha$ is metafinite. If we let $f(x) = f(0)$ for $x \geqslant w$, then $f(x)$ enumerates $\alpha$. If $w = \varepsilon$, then $\alpha$ cannot be metafinite. Moreover, if $y \in \alpha$, $y \leqslant f(y)$ and, from the above there is some $x \leqslant y$ such that $f(x) = y$. Thus $\alpha$ is the range of a strictly increasing recursive function. Q.E.D.

We would like to enumerate a $\Sigma_1$ set but, as mentioned earlier, we may not be able to do so. However, the following will be sufficient for our purposes.

Theorem 4 *Let* $\mathfrak{M}$ *be an admissible set which is non-projectible and* $\Sigma_1$ *uniformizable. Let* $\alpha \subset \varepsilon$ *be an infinite* (i.e., *not metafinite*) $\Sigma_1$ *set. Then* $\alpha$ *contains an infinite* r.e. *subset.*

*Proof:* Let $x \in \alpha \leftrightarrow \exists y \varphi(x,y)$ where $\varphi$ is $\Sigma_0$. Consider the $\Sigma_0$ formula $\varphi'(x,z)$ where $\varphi'(x,z) \leftrightarrow z = \langle v,w \rangle \wedge v \geqslant x \wedge \varphi(v,w)$. Given any $x \in \mathrm{On}$ there must be a $v \geqslant x$ which is in $\alpha$; otherwise $x \cap \alpha = \alpha$ would be in $\mathfrak{M}$. Since $\mathfrak{M}$ is non-projectible, this contradicts our hypothesis. Thus there is a $w \in \mathfrak{M}$ such that $\varphi(v,w)$ and hence a $z \in \mathfrak{M}$ such that $\varphi'(x,z)$. We have shown that $\exists z \varphi'(x,z)$ holds for $x \in \mathrm{On}$. Since $\mathfrak{M}$ is $\Sigma_1$ uniformizable there is a recursive function $f(x)$ such that $\forall x \varphi'(x,f(x))$. Define the familiar recursion functions $k(z)$, $l(z)$ by $z = \langle k(z),l(z) \rangle$ and 0 if $z$ is not an ordered pair. Let $h(x) = k(f(x))$. Then for any $x$ we have $h(x) \geqslant x$ and $\varphi(h(x), l(f(x)))$; therefore $\mathrm{rng}\ h$ is an unbounded subset of $\alpha$. Q.E.D.

Corollary *Assuming the hypotheses of* Theorem 4, $\alpha$ *contains an infinite recursive set.*

*Proof:* We claim $\beta = \text{rng } h$ is a recursive set. For $y \in \beta \leftrightarrow \exists x[h(x) = y]$ and $y \notin \beta \leftrightarrow \forall x \leqslant y \ \exists w[h(x) = w \wedge w \neq y]$. Q.E.D.

We conclude this section with a well-known theorem which also merges the ideas of non-projectibility and $\Sigma_1$ uniformizability. The proof uses a weaving technique and is a variation of that found in [2].

Theorem 5 *Let* $\mathfrak{M}$ *be an admissible set which is non-projectible and* $\Sigma_1$ *uniformizable. Let* $\alpha$ *and* $\beta$ *be infinite* $\Sigma_1$ *subsets* (*not necessarily* r.e.) *of* $\varepsilon$ *such that their complements* $\alpha'$, $\beta'$ *are recursively equivalent. Then there exists a* 1-1 *recursive function* $h(x)$ *such that* $h''\alpha = \beta$ *and* $h''\alpha' = \beta'$.

*Proof:* Assume $\alpha' \simeq \beta'$ by a 1-1 p.r. function $p(x)$. We first note that $\varepsilon = \text{dom } p \cup \alpha = \text{rng } p \cup \beta$ and that the relations $x \in \text{dom } p$, $x \in \alpha$, $x \in \text{rng } p$, $x \in \beta$ are $\Sigma_1$. Let

$$x \in \text{dom } p \leftrightarrow \exists z \varphi_1(x,z), \ x \in \alpha \leftrightarrow \exists z \varphi_2(x,z)$$

where $\varphi_1$ and $\varphi_2$ are $\Sigma_0$. Similarly, let $\psi_1$, $\psi_2$ be defined for $x \in \text{rng } p$ and $x \in \beta$. From the above, we have, for $x \in \text{On}$, $\exists z[\varphi_1(x,z) \vee \varphi_2(x,z)]$ and $\exists z[\psi_1(x,z) \vee \psi_2(x,z)]$. Since $\mathfrak{M}$ is $\Sigma_1$ uniformizable, there exist recursive functions $f_1(x)$, $f_2(x)$ such that for $x \in \text{On}$, $\varphi_1(x, f_1(x)) \vee \varphi_2(x, f_1(x))$ and $\psi_1(x,f_2(x)) \vee \psi_2(x,f_2(x))$. Thus, given $x$ we can decide whether $x \in \alpha$ or $x \in \text{dom } p$ (or sometimes both). The same applies for $x \in \beta$ and $x \in \text{rng } p$. Secondly, since $\alpha$ and $\beta$ are infinite $\Sigma_1$ sets and $\mathfrak{M}$ is non-projectible, $\alpha$ and $\beta$ contain infinite recursive subsets $\gamma$ and $\delta$ which can be enumerated by strictly increasing recursive functions $c_z$ and $d_z$. Let the p.r. function $g_1(x,a_1,a_2) = y$, where $x \in \text{On}$ and $a_1(x)$ and $a_2(x)$ are 1-1 functions with domains and ranges contained in the ordinals, be defined as follows: If $x \in \text{rng } a_2$, let $y = a_2^{-1}(x)$. If not, but $x \in \text{dom } a_1$, we let $y = a_1(x)$. If $x$ is in neither, we compute $f_1(x)$ and check the validity of $\varphi_1(x,f_1(x))$. If it is true, then we know $x \in \text{dom } p$; we compute $p(x)$ and see it is in $\text{rng } a_1 \cup \text{dom } a_2$. If $p(x)$ is not a member, let $y = p(x)$; if it is we let $y$ be equal to the first element in $\delta$ not in $\text{rng } a_1 \cup \text{dom } a_2$. If $\neg\varphi_1(x,f_1(x))$, let $y$ be the first element in $\delta$ not in $\text{rng } a_1 \cup \text{dom } a_2$. Similarly we can define $g_2(x,b_1,b_2)$, substituting $b_1$, $b_2$, $\psi(x,f_2(x))$, $p^{-1}(x)$, $c_z$ for $a_1$, $a_2$, $\varphi_1(x,f_1(x))$, $p(x)$, $d_z$, respectively.

We now weave the two functions together by defining the p.r. function $g_3(x,a) = \langle y_1,y_2\rangle$, where $a(x)$ maps ordinals into ordered pairs of ordinals and $k(a(x))$ and $l(a(x))$ are 1-1, as follows: $y_1 = g_1(x,k \circ a,l \circ a)$ and $y_2 = g_2(x,l \circ a,k \circ a \cup \{\langle x,y_1\rangle\})$. Apply the recursion theorem to obtain the p.r. function

$$g(x) \simeq g_3(x, \langle g(y) \mid y < x\rangle) = g_3(x,g \restriction x);$$

by transfinite induction we can conclude that $g(x)$ is in fact a recursive function. Finally, let $h(x) = k(g(x))$ and $h'(x) = l(g(x))$. We claim that $h(x)$ has all the desired properties and that $h^{-1}(x) = h'(x)$. Let us assume that $h \restriction x$ and $h' \restriction x$ are functions such that the following hold:

(1) $h \restriction x$ and $h' \restriction x$ are 1-1;
(2) for $y < x$, $y \in \alpha \leftrightarrow h(y) \in \beta$;
(3) for $y < x$, $y \in \beta \leftrightarrow h'(y) \in \alpha$;
(4) for $y, z < x$ then $h(y) = z \leftrightarrow h'(z) = y$.

We shall show that (1)-(4) hold with $x$ replaced by $x + 1$. From the definitions, we can see that $h(x) = g_1(x, h \restriction x, h' \restriction x)$; we shall examine the definition of $g_1$. Suppose $x$ is $h'(y)$ for $y < x$; then $h(x) = y$. If $h(x^*) = y$ for $x^* < x$, then by (4) $h'(y) = x^*$, a contradiction. By (3) $y \in \beta \leftrightarrow h'(y) = x \in \alpha$; hence $x \in \alpha \leftrightarrow h(x) = y \in \beta$. Hence (1), (2), (4) hold for $x + 1$ in this case.

Suppose $x$ is not in the range of $h' \restriction x$ and $\varphi_1(x, f(x))$ holds. We compute $p(x)$; if $p(x)$ is not in the $\text{rng}\, h \restriction x \cup \text{dom}\, h' \restriction x$, then $h(x) = p(x)$. Hence $h \restriction x + 1$ is 1-1 and $x \in \alpha \leftrightarrow h(x) = p(x) \in \beta$. If $p(x) = y$ is in the $\text{rng}\, h \restriction x \cup \text{dom}\, h' \restriction x$, then we let $h(x)$ be the first element of $\delta \subset \beta$ which is in neither. Thus $h \restriction x + 1$ is 1-1. If $y = h(x^*)$ for $x^* < x$, then $h(x^*)$ cannot equal $p(x)$ since $p(x)$ is 1-1. Either $x^*$ is a $c_z \in \alpha$ or $y$ is a $d_z \in \beta$. In either case we can conclude $p(x) = y \in \beta$ and hence $x \in \alpha$. If $y \in \text{dom}\, h' \restriction x = x$, then $h'(y) \neq p^{-1}(y) = x$ because $x$ is not in the range of $h' \restriction x$. Hence, as in the previous case $p(x) \in \beta$ and therefore $x \in \alpha$. Finally, if $\neg\varphi(x, f_1(x))$, then $\varphi_2(x, f_1(x))$ and so $x \in \alpha$ and $h(x)$ is an element of $\beta$ not in $\text{rng}\, h \restriction x$. In a similar manner we can show that $h' \restriction x + 1$ satisfies (1), (3), and (4).

By induction we can conclude that $h(x)$ is a 1-1 function which maps $\alpha$ into $\beta'$ and $\alpha'$ into $\beta'$. Moreover, given $x$ either $h \restriction x + 1$ maps an element into $x$ or $h'(x) > x$. But then $h(h'(x)) = x$. Therefore $h(x)$ maps $\alpha$ onto $\beta$ and $\alpha'$ onto $\beta'$. Q.E.D.

**4** *Immune sets and isols* We say $\alpha \subset \varepsilon$ is *immune* if $\alpha$ contains no infinite r.e. subset.

Proposition 12 *Let $\mathfrak{M}$ be non-projectible. If $a$ is metafinite, then $a$ is immune.*

*Proof:* The same argument is used as in the proof of Theorem 4. Q.E.D.

Theorem 6 *Let $\mathfrak{M}$ be non-projectible and $\Sigma_1$ uniformizable. Then the following are equivalent:*

(1) *$\alpha$ is immune;*
(2) *$\alpha$ contains no infinite $\Sigma_1$ set;*
(3) *$\alpha$ contains no infinite recursive set.*

*Proof:* (1) → (2) If $\alpha$ contained an infinite $\Sigma_1$ set, then by Theorem 4, $\alpha$ would contain an infinite r.e. subset.
(2) → (3) If $\alpha$ contained an infinite recursive set $\beta$, then by Theorem 3 and Proposition 11, $\beta$ would be r.e. and hence $\Sigma_1$.
(3) → (1) If $\alpha$ were not immune, $\alpha$ would contain an infinite r.e. subset. By Proposition 11 and the Corollary to Theorem 4, $\alpha$ then contains an infinite recursive set. Q.E.D.

Proposition 13 *Let $\alpha$, $\beta$ be subsets of $\varepsilon$ such that $\alpha \simeq \beta$. Then $\alpha$ immune iff $\beta$ immune.*

For the remainder of this paper, let us assume that we are in a non-projectible, $\Sigma_1$ uniformizable set $\mathfrak{M}$. An *isol* will be a **RET** of an immune set. Let $\Lambda$ denote the collection of all isols. By Proposition 13, every element of an isol is an immune set. Note that by Proposition 12 any **RET** of a metafinite set is an isol.

Theorem 7 *The following are equivalent for a* **RET** $A$:

(1) $A \notin \Lambda$;
(2) $\varepsilon \leqslant A$;
(3) $A + \varepsilon = A$.

*Proof:* (1) $\rightarrow$ (2) If $A \notin \Lambda$, then by Theorem 6, $A$ has a representative $\alpha$ which contains an infinite recursive set $\beta$. By Theorem 3, $\beta \simeq \varepsilon$. Moreover, $\beta | \alpha - \beta$ and $\alpha = (\alpha - \beta) \cup \beta$; hence, $A = \langle \alpha - \beta \rangle + \varepsilon$; i.e., $\varepsilon \leqslant \alpha$.
(2) $\rightarrow$ (3) We know $\varepsilon + \varepsilon = \varepsilon$. If $\varepsilon \leqslant A$, there is a **RET** $B$ such that $\varepsilon + B = A$. Thus $\varepsilon + A = \varepsilon + \varepsilon + B = \varepsilon + B = A$.
(3) $\rightarrow$ (1) If $A + \varepsilon = A$, there exist disjoint, separable representatives $\alpha$, $\varepsilon'$ of $A$ and $\varepsilon$ and a p.r. function $p(x)$ such that $p''(\alpha \cup \varepsilon') \supset \alpha$. Since $\varepsilon'$ is an infinite r.e. set $p''\varepsilon \subset \alpha$ is also an infinite r.e. set. Q.E.D.

The following two propositions can be proved in a standard manner.

Proposition 13 *If* $B \in \Lambda$ *and* $A \leqslant B$, *then* $A \in \Lambda$.

Proposition 14 *If* $A$, $B \in \Lambda$, *then* $A + B \in \Lambda$.

We shall try to establish a cancellation law for the isols.

Proposition 15 *If* $A$, $B \in \Lambda$ *and* $A + \varepsilon = B + \varepsilon$, *then there exists a* $Z < \varepsilon$ *such that* $A + Z = B$ *or* $B + Z = A$.

*Proof:* By Theorem 2, there exist **RET**s $U$, $V$, and $N$ such that $A = U + N$, $B = V + N$, and $U + \varepsilon = \varepsilon = V + \varepsilon$. Since $U \leqslant A$ and $V \leqslant B$, $U$ and $V$ must be isols. Hence $U < \varepsilon$ and $V < \varepsilon$, and thus $U$ and $V$ are **RET**s of metafinite sets. Following the proof of Theorem 3, we can map any metafinite set of ordinals into an initial segment of $\varepsilon$. Therefore, $U$ and $V$ are comparable; if $U \leqslant V$, there is a **RET** $Z < \varepsilon$ such that $U + Z = V$. Then $A + Z = U + N + Z = U + Z + N = V + N = B$. The other case is similar. Q.E.D.

For $\omega < \varepsilon$, we have $\omega + 1 = \omega$, but $1 \neq 0$; we cannot expect to have any absolute cancellation law for the isols. The most we can expect would result from an application of Theorem 3. We shall now strengthen our definition: We observe that if a metafinite set is mapped properly into itself, the remainder is metafinite and separable from the range; this property fails for $\varepsilon$. Let $\alpha \subset \varepsilon$ be *isolated* if whenever $h(x)$ is a 1-1 function such that $\alpha \subset \operatorname{dom} h$, $h''\alpha \subset \alpha$ and $h''\alpha | \alpha - h''\alpha$, then $\alpha - h''\alpha$ is metafinite. Thus $h''\alpha$ almost fills $\alpha$.

Proposition 16 *If* $\alpha \simeq \beta$ *and* $\alpha$ *is isolated, then* $\beta$ *is isolated.*

*Proof:* Let $\alpha \simeq \beta$ by a 1-1 partial recursive function $p(x)$. Suppose a 1-1 partial recursive function $h(x)$ mapped $\beta$ into itself and $\beta - h''\beta | h''\beta$. Then

$g(x) = p^{-1}(h(p(x)))$ maps $\alpha$ 1-1 into itself and $\alpha - g''\alpha | g''\alpha$. Hence the remainder $\gamma$ is metafinite and $\alpha = g''\alpha \cup \gamma$. Applying $p(x)$ to both sides of the equation and noting the definition of $g(x)$, we have $\beta = p''\alpha = h''\beta \cup p''\gamma$. But $p''\gamma$ is metafinite, hence $\beta$ is isolated. Q.E.D.

If $\alpha$ is isolated, then $A = \langle\alpha\rangle$ will be called a *Cancellation type* (C-type). By the previous proposition, every member of a C-type is an isolated set. Also, every metafinite set is a C-type.

Theorem 8 *The following are equivalent for a* **RET** $A$:

(1) *If* $\alpha \in A$, *then* $\alpha$ *is isolated;*
(2) *If* $A = A + B$, *then* $B < \varepsilon$.

*Proof:* (1) → (2) Suppose there are representatives $\alpha$, $\beta$ of $A$, $B$ such that $\alpha | \beta$ and a 1-1 p.r. function $h(x)$ such that $\alpha \cup \beta \subset \text{dom}\, h$ and $h''(\alpha \cup \beta) = \alpha$. Then $h''\alpha \subset \alpha$ and $h''\alpha | \alpha - h''\alpha$ because $\alpha - h''\alpha = h''\beta$. Since $\alpha$ is isolated, the remainder $h''\beta$ is metafinite, and hence also $\beta = h^{-1}(h''\beta)$; thus $B < \varepsilon$.
(2) → (1) Suppose a 1-1 p.r. function $h(x)$ maps $\alpha$ into itself such that $h''\alpha | \alpha - h''\alpha$. Then $\alpha \simeq h''\alpha$ and $\alpha = h''\alpha \cup (\alpha - h''\alpha)$. Taking types, $A = A + \langle\alpha - h''\alpha\rangle = A + B$. Because $B < \varepsilon$, the remainder is metafinite. Q.E.D.

Proposition 17 *If* $A$ *is a* C*-type, then* $A$ *is an isol.*

*Proof:* If $A$ is not an isol, then $A = A + \varepsilon$. Q.E.D.

The converse does not hold, as shown by the following example. Let $\alpha$ be an immune but not metafinite set of limit ordinals. Let $\beta = \{x + n | x \in \alpha \wedge n < \omega\}$. If $\beta$ contained an infinite r.e. subset $\gamma$, then $\gamma$ would be unbounded, since our admissible set is non-projectible. The projection of $\gamma$ into $\alpha$ by the recursive function $f(x + n) = x$, where $x$ is a limit ordinal, would also be an unbounded r.e. subset. If we let $g(x) = x + 1$, then $g(x)$ maps $\beta$ into itself with separable remainder $\alpha$. Hence $\beta \simeq g''\beta \cup \alpha$; taking types, $B = B + A$, but $A$ is not metafinite.

Theorem 9 (Cancellation Law for C-types) *If* $A$ *is a* C*-type and* $X$, $Y$ *are* **RET**'s *and* $X + A = Y + A$, *then there is a* **RET** $Z < \varepsilon$ *such that either* $X + Z = Y$ *or* $Y + Z = X$.

*Proof:* By Theorem 2, there exist **RET**'s $U$, $V$, and $N$ such that $X = U + N$, $Y = V + N$, and $U + A = A = V + A$. Since $A$ is a C-type, $U$ and $V$ must be metafinite **RET**s and hence comparable. If $U \leq V$, then there is a **RET** $Z < \varepsilon$ such that $U + Z = V$. Then $X + Z = U + N + Z = U + Z + N = V + N = Y$. The other case is similar. Q.E.D.

Theorem 10 *If* $A$ *and* $B$ *are* C*-types, then* $A + B$ *is a* C*-type.*

*Proof:* Suppose $A + B + X = A + B$. then $(B + X) + A = B + A$, and by the previous theorem there is a $Y < \varepsilon$ such that $B + X + Y = B$ or $B + X = B + Y$. In the first case, $X \leq X + Y < \varepsilon$ since $B$ is a C-type. In the second case, apply the theorem again so that for some $Z < \varepsilon$ either $X + Z = Y < \varepsilon$ or $X = Y + Z < \varepsilon$. In either case $X < \varepsilon$. Q.E.D.

Theorem 11 *If* $A \leq B$ *and* $B$ *is a* C*-type, then* $A$ *is a* C*-type.*

*Proof:* Let $A + C = B$ and suppose $A + X = A$. Then $A + C + X = A + X$, i.e., $B + X = B$. Q.E.D.

Theorem 12 *Let* $A$ *be a* C*-type and* $X$, $Y$ *be* **RET**s. *If* $A + X \leq A + Y$ *then* $X \leq Y + Z$ *where* $Z < \varepsilon$.

*Proof:* Let $A + X + V = A + Y$. By Theorem 9, there is a $U < \varepsilon$ such that $X + V + U = Y$ or $X + Y = Y + U$. In the first case let $Z = 0$; in the second let $Z = U$. Q.E.D.

A **RET** $A$ is an *ordinary* isol if $A \neq A + 1$. Then an ordinary isol is a C-type since, if $A = A + X$ and $X \neq 0$, then $A \leq A + 1 \leq A + X = A$ and hence $A = A + 1$. An isol $A$ is *indecomposable* if $A = B + C$ implies $B < \varepsilon$ or $C < \varepsilon$. An indecomposable isol $A$ is also a C-type. We therefore have finite sums of metafinite **RET**s, ordinary isols, and indecomposable isols which are C-types. It is not known if there are others.

A final comment should be made about the importance of non-projectibility. Suppose an element of $\mathfrak{M}$ contained an infinite r.e. subset $\pi$ of ordinals. Let us denote by $\varepsilon^*$ the smallest ordinal such that $\pi \subset \varepsilon^*$. Then $\varepsilon^*$ or any ordinal greater than it cannot be immune. Also $\pi$ cannot have an infinite recursive subset $\beta$; for if it did, $\beta = \varepsilon^* \cap \beta$ would be in $\mathfrak{M}$ by Proposition 1. This fact causes (3) $\rightarrow$ (1) of Theorem 6 and (1) $\rightarrow$ (2) of Theorem 7 to fail when we let $\alpha$ and $A$ equal $\varepsilon^*$ and $\langle \varepsilon^* \rangle$, respectively.

## REFERENCES


[1] Bredlau, C., *Recursive Equivalence Types and Admissible Sets*, Doctoral dissertation, Rutgers University, New Brunswick, New Jersey (1972).

[2] Dekker, J. C. E., and J. Myhill, "Recursive equivalence types," *University of California Publications in Mathematics (New Series)*, vol. 3 (1960), pp. 67-214.

[3] Jensen, R., Unpublished notes on admissible sets.

[4] Kreisel, G., and G. Sacks, "Metarecursive sets," *The Journal of Symbolic Logic*, vol. 30 (1965), pp. 318-337.

[5] Kripke, S., "Transfinite recursion on admissible ordinals, I, II" (abstracts), *The Journal of Symbolic Logic*, vol. 29 (1964), pp. 161-162.



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