Notre Dame Journal of Formal Logic
Volume XX, Number 2, April 1979
NDJFAM

α-NAMING AND α-SPEEDUP THEOREMS

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*Introduction** The study of computation over infinite ordinals has its roots evolving from various areas of investigation. Takeuti [33] was concerned with the problem of reducing consistency of set theory to a theory of computation on ordinal numbers. Machover [21] sought to generalize model and recursion theoretic notions to the study of infinitary languages. Jensen and Karp [11] developed a theory of primitive recursive set and ordinal functions to investigate questions of a set theoretic vein. The first author's motivation was the need of a tool for considering various levels of Gödel's constructible hierarchy [5]; the second, from lines closely related to those of Machover. From the study of definability theory and its relation to high order logics and languages evolved the work of Kreisel [14]. Later, in collaboration with Sacks (in [15]), this work blossomed into metarecursion theory, the study of computation of Church-Kleene's ω_1^{ck}. It was Kripke [16] (and independently Platek [24]) who first isolated the key notion of admissible ordinal. The study of computation over such ordinals (unifying those aforementioned cases) became known as α-recursion theory. Kripke was able to develop enough α-recursion theory to yield an infinite analogue to the Kleene T-predicate. From this he then asserted that the major results of unrelativized recursion theory (as found in Kleene [13]) held in α-recursion theory. Sacks and Simpson [27] developed relativization and consequently the priority argument in α-recursion theory. They blended recursion and model theoretic ideas in showing that the Friedberg-Muchnik solution to Post's problem generalized to α. Following this, several of Sack's students and coworkers succeeded in proving that the major results of relativized recursion theory also lift. Particularly, interesting are the works of Lerman [19], Shore [29], [30], Lerman and Simpson [20], Leggett and Shore [18]. A rather well written survey of

*The results in this paper appear in the author's Ph.D. dissertation supervised by Professor Martin D. Davis at the Courant Institute of Mathematical Sciences, New York University. This research was supported by the National Science Foundation, Grant NSF-DCR71-029039-AOB.

Received June 18, 1977

α-recursion theory (from this point of view) may be found in Shore [31]. Another subarea of ordinary recursion theory, abstract computational complexity theory, has a flavor similar to that of relativized recursion theory. Founded by Blum [1] (after inspiration by Rabin [25]), the study is based on several axioms common to most interesting measures of computation. The parallel, however, exists in the way deeper results are proven; namely, via development of a generally complex construction followed by a rather detailed verification.

In [8], the author initiated the study of α-computational complexity by setting down infinite analogues to Blum's axioms. The motivation was (i) an investigation of set, model and recursion theoretic characterizations of complexity oriented notions, and (ii) the isolation of a proper domain for implementation of a generic forcing type (*Cf.* [32]) approach to complexity-theoretic proof constructions.

A first step in the above program, however, is the verification that little or nothing is lost upon generalization to α. An extensive survey of key results of abstract complexity theory (at the ω-level) can be found in [7]. It is shown in [8], [9] and [10] as well as this paper, that all of these results generalize, in one form or another, to α-recursion theory.

An outline of the paper is as follows. In section **0**, we review the fundamental notions of α-recursion theory and α-complexity theory necessary for understanding the paper. This is followed, in section **1**, by a discussion of the differences existing between ordinary ($\alpha = \omega$) and α-recursion theory. We observe how these distinctions alter a study of computation and, in particular, the various constructions used in complexity proofs. In section **2** we lift to α the well known Naming or Honesty result of McCreight and Meyer [22]. Namely, that for any α-complexity measure there exists α-measured sets of functions which name every α-complexity class. In section **3** we prove a generalization to Blum's famous Speedup [1] phenomenon. That is, given an α-recursive r, there exists a 0-1 value α-recursive f such that for any means (index) for computing f, a faster one, having speedup determined by r, exists. We conclude the paper in section **4** with a series of suggestions for further research.

0 *Preliminaries* We employ standard set theoretic notion. Namely, $\bigcup A$ for union of A; $A \cap B$ for intersection of A and B; A-B for set difference of A and B; f/B for mapping f restricted to B; $f[B]$ for the range of f/B; $\delta \in B$ for δ an element of B; $A \subseteq B$ for A a subset of B; $\emptyset$ for the empty set; $\mathrm{dom}(f)$, $\mathrm{rng}(f)$ for domain and range of f; $A \times B$ for Cartesian product of A and B; $f\colon A \to B$ for f a map from A to B.

A von Neumann ordinal, or simply an ordinal, is identified with the set of all smaller ordinals. We use lower case Greek letters to denote ordinals $(\alpha, \beta, \gamma, \ldots)$, lower case Latin letters for functions on the ordinals $(f, g, h, \ldots)$ and upper case Latin letters for sets of ordinals $(A, B, C, \ldots)$. Let L_α be the collection of sets obtained from Gödel's transfinite hierarchy of constructible sets before α. α is Σ_1 admissible if L_α satisfies the replacement axiom schema of **ZF** for Σ_1 formula of set theory (*Cf.* [5]). *From this point on α is some fixed Σ_1 admissible ordinal.*

A set $A \subseteq \alpha$ is α-recursively enumerable (α-r.e.) if it has a Σ_1 definition over L_α (with parameters in L_α). A partial function $f: \alpha \to \alpha$ is α-partial recursive if its graph is α-r.e. Such a map is α-recursive if its domain is α, while subsets of α are α-recursive if their characteristic functions are. (Since there exists a well-known one-one α-recursive map of α into L_α, it suffices to consider only functions on α and subsets of α instead of those on L_α).

We call a set $A \subseteq \alpha$ α-bounded (or bounded since α is fixed) if there exists a $\beta < \alpha$ so that $\sigma \epsilon A \to \sigma < \beta$. Sets which are both α-recursive and bounded are α-finite. Kripke [17] shows that the α-finite sets are precisely the sets of ordinals in L_α. An immediate consequence is

0.1 Fact *Let f be α-partial recursive and K an α-finite subset of the domain of f. Then $f[K]$ is α-finite.*

Following Sacks [26] we have a standard enumeration $\{K_\eta | \eta < \alpha\}$ of all the α-finite subsets of α. A set $B \subseteq \alpha$ is an α-finite function if (i) B is an α-finite set, (ii) B is the (α-recursive) encoding of the graph of an α-partial recursive f, (iii) $\mathrm{dom}(f)$ is a proper initial segment of α. From Sacks' enumeration one immediately obtains a canonical enumeration $\{G_\varepsilon | \varepsilon < \alpha\}$ of all α-finite functions.

The key fact about admissible ordinals α is that one can do Δ_1 (α-recursive) recursions in L_α. In particular, one may Gödel number the α-r.e., subsets of α, $\{R_\varepsilon | \varepsilon < \alpha\}$ and consequently the α-partial recursive functions $\{\phi_\varepsilon | \varepsilon < \alpha\}$. In fact, Kripke [17] develops an equation calculus (analogous to Kleene [13]) and establishes enough α-recursion theory to formulate an α analogue to the Kleene T-predicate. (It can be shown that the T-predicate can be built from Jensen-Karp [11] primitive recursive ordinal functions and thus independent of α). From this, Kripke was able to assert that all the major results of unrelativized recursion theory hold in α-recursion theory. Included among these are the α-Universal Function (or Enumeration) Theorem, α-S_n^m-Theorem and the α-Recursion Theorem. We make use of α-recursive pairing functions, π_1 and π_2 and $<, >$ obtaining a one-one correspondence between α and $\alpha \times \alpha$ (*Cf.* [5]). These possess the typical properties: for $\beta_1, \beta_2, \gamma < \alpha$, (i) $\langle \pi_1(\gamma), \pi_2(\gamma) \rangle = \gamma$; $\pi_1(\langle \beta_1, \beta_2 \rangle) = \beta_1$ and $\pi_2(\langle \beta_1, \beta_2 \rangle) = \beta_2$; and $\pi_i(\gamma) = \beta$ for $i = 1, 2$, $\beta < \alpha$ for an unbounded set of γ.

A generalization to Blum's [1] notion of abstract computational computational complexity measure is given by,

Definition: An *α-complexity measure* Φ is an enumeration (in α) of the α-partial recursive function $\{\phi_\varepsilon | \varepsilon < \alpha\}$ to which are associated the α-partial recursive *α-step counting* functions $\{\Phi_\varepsilon | \varepsilon < \alpha\}$ for which the following axioms hold:

(1) $\phi_\varepsilon(\beta)$ is defined iff $\Phi_\varepsilon(\beta)$ is defined.
(2) The predicate $\mathrm{M}(\varepsilon,\beta,\gamma) \leftrightarrow \Phi_\varepsilon(\beta) = \gamma$ is α-recursive.

(3) The α-recursive versions of the Universal, S_n^m and Recursion Theorems hold for the enumerations $\{\phi_\varepsilon\}$ and $\{\Phi_\varepsilon\}$.

Implicit in (3) is the capability to retrieve, given any index ε, both the function ϕ_ε, in the form of an 'algorithm', and the α-step counter, Φ_ε. Clearly, when $\alpha = \omega$, the notion of α-complexity measure reduces to Blum's notion. Several examples of α-complexity measures are found in [8].

A sequence of functions $\{f_\varepsilon | \varepsilon < \alpha\}$ is said to be α-recursively enumerable if the function $m(\varepsilon,\beta) = f_\varepsilon(\beta)$ is α-partial recursive; alternatively, if there is an α-recursively enumerable set of 'algorithms' so that for each f_ε at most one algorithm appears in the listing. A sequence of functions $\{f_\varepsilon | \varepsilon < \alpha\}$ which is α-recursively enumerable and for which it can be α-recursively determined for all ε, β, $\delta < \alpha$ whether $f_\varepsilon(\beta) = \delta$ is called an *α-measured set*. The usual example is the set of α-step-counting functions of an α-complexity measure.

For a fixed α-complexity measure Φ and α-recursive function s, the α-complexity class bounded by s is the set $C_s^\Phi = \{\phi_\sigma | \phi_\sigma$ is total and $\Phi_\sigma(\beta) \leq s(\beta)$ for all but an α-finite set of $\beta\}$. Thus, C_s^Φ, or simply C_s, when Φ is understood, is the set of all α-recursive functions whose α-complexity is bounded by s on all but an α-finite subset of α. The projectum of α, denoted α^*, is the least ordinal $\beta \leq \alpha$ such that there is a one-one α-recursive mapping from α into β (not necessarily onto). Throughout this paper $\dagger$ will serve as the corresponding α-recursive projection map. Since $\dagger$ is one-one α-recursive, we let $\dagger^{-1}: \alpha^* \to \alpha$ be its α-partial recursive inverse. For some α-complexity measure Φ, let $\varepsilon_0 < \alpha$ be such that $\phi_{\varepsilon_0} = \dagger^{-1}$. Then $\dagger_\sigma^{-1}$ is the approximation to $\dagger^{-1}$ through stage $\sigma < \alpha$ defined by

$$\dagger_\sigma^{-1}(\beta) = \begin{cases} \phi_{\varepsilon_0}(\beta) \text{ if } \Phi_{\varepsilon_0}(\beta) \leq \sigma \\ \uparrow \text{ otherwise} \end{cases}$$

Clearly, $\lim_{\sigma \to \alpha} \dagger_\sigma^{-1} = \dagger^{-1}$ and for σ, $\beta < \alpha$ the predicate $N(\sigma,\beta) \leftrightarrow \dagger_\sigma^{-1}(\beta)\downarrow$ is α-recursive. Unlike the situation at ω, for many admissibles $\alpha > \omega$, sets which are bounded and α-r.e. are not necessarily α-finite. However, α-finiteness occurs if the bound is taken small enough.

0.2 Fact *If $\eta < \alpha^*$ and A is an α-recursively enumerable subset of η, then A is α-finite.*

In this paper complexity constructions are developed in which priorities are based, not upon natural order ($<\alpha$), but rather on $\dagger[\alpha] \subseteq \alpha^*$. The above will, therefore, enable us to generate priorities as well as confirm various convergence results about our constructions.

1 *A comparison* To grossly capsulize the situation, one may say that α-recursion theory serves as a magnification device for studying fundamental concepts of computation. As a result, the generalization to α permits us to carefully analyze and scrutinize notions that would otherwise be bypassed. In this section we highlight several of these and point out their consequences in the study of α-computational complexity. A first source of interest concerns the divergence of concepts on the α-level which at the ω-level are coexistent. For instance, in ordinary recursion theory, the notions of finite, bounded and less than ω all coincide, while in α-recursion theory their analogues split. Furthermore, sets which are automatically computable, just because they are bounded, do not necessarily enjoy this property in the generalized setting.

This phenomenon was observed in the lift to α (in [9]) of the Blum-Rabin complex partial recursive function theorem. At ω the result tells us that for partial recursive g, a partial recursive f can be constructed where (i) dom f = dom g, and (ii) any program for f has complexity exceeding g on all but a finite set of input. The generalization to α lifts in the natural way except that finite becomes bounded (instead of α-finite). The difficulty is that the excluded set $\{\beta \mid \Phi_\varepsilon(\beta) > g(\beta)\}$ is not α-recursive for non-total g.

In this paper we almost exclusively employ the analogue α-finite for finite. For example, an α-complexity class, C_t, is a set of α-recursive functions with complexity below t on all but an α-finite subset (of α). In the α-Speedup Theorem, the set where speedup fails for the constructed function is α-finite. The reason for this is that in both situations we deal with α-recursive functions; hence, any excluded set turns out α-recursive.

A second facet is inherited from the fact that we deal with infinite ordinals; namely, the possibility of mapping all of α one-one into a proper initial segment. As a result, constructions used to prove recursion or complexity theoretic results which are founded upon priority schemes require, upon generalization to α, extensive overhaul. For example, consider the typical complexity cancellation construction where at stage $e < \omega$, cancellation occurs to the object with highest priority (smallest value) eligible for concellation. In the subsequent verification, one would then argue that any algorithm (index) which unboundedly qualifies for cancellation, will ultimately be cancelled.

In an α-recursion theoretic construction, an index $\varepsilon < \alpha$ may be cancellable at each of an α-unbounded set of stages. However, there is no reason why, at each of these stages, an index with higher priority (smaller value) beats it out. That is, the set of stages at which an index $\varepsilon' < \varepsilon$ intercedes in the cancellation of ε, is cofinal with α. The way this problem is handled in α-recursion theory is to base priorities upon the order induced by the Σ_1 projection map t, instead of natural order. Namely, index ε_1 has higher priority than index ε_2 ($\varepsilon_1, \varepsilon_2 < \alpha$) iff $t(\varepsilon_1) < t(\varepsilon_2) < \alpha^*$. By the α-effective nature of the construction the collection $E_{t(\varepsilon)}$ of "pseudo" indices below $t(\varepsilon)$ which are ultimately cancelled is α-r.e., and bounded below α^*, hence by Fact **0.2**, α-finite. Further, a Σ_1 (in L_α) map may be constructed mapping a cancelled pseudo index into its stage of cancellation. It then follows from admissibility (Fact **0.1**) that a stage $\sigma < \alpha$ bounds all stages at which indices of higher priority than ε are cancelled. This last part assures the ultimate cancellation of ε.

A third aspect derives from the fact that ω is Σ_n-admissible for all $n < \omega$ while α is just Σ_1-admissible. One of the consequences is that a simple and often taken for granted property may not hold; namely, that a finite union of finite subsets is finite. An easy counterexample, provided by Shore [31], is where α is $\aleph_\omega$, the ωth infinite cardinal. Although $\aleph_\omega$ is admissible (Kriple [16]) and ω and $\aleph_n$, $n < \omega$ are α-finite, the union $\bigcup_{n<\omega} \aleph_\omega$ blows up to $\aleph_\omega$ (= α).

This problem occurs frequently in α-recursion theory, in particular, in the lifting of injury arguments of ordinary recursion theory. It also appears in the generalization to α of the McCreight-Meyer Union Theorem in [10] (which entails a no-injury atop a finite injury priority construction) and is touched upon in the proof of the α-Speedup Theorem in this paper.

In any multiple priority argument, the mapping ψ of an index ε to the stage σ_ε which bounds its injury set (set of stages at which the index is injured) is at best a Σ_2 mapping. Hence, although a set E of pseudo indices below a $\dagger(\varepsilon) < \alpha^*$ is α-finite, the set $\psi[E]$ may not be since α is only Σ_1-admissible. A solution to this is to make the priority listing even smaller than α^* with an additional implementation of a blocking strategy (*Cf.* Shore [29]). The idea is that constructions have to work harder to make up for non-Σ_2 admissibility. A similar consideration arises in the proof of α-Speedup appearing in this paper. Namely, at stage σ a search is made through an α-finite set of indices where at each index a subcomputation is performed. The question arises as to whether or not an α-finite union of α-finite subcomputations, each of which converges (we know this by induction), will ultimately converge. Shore's illustration again provides us with a counterexample. Specifically, for $\alpha = \aleph_\omega$, perform ω subcomputations where the nth ($<\omega$) takes $\aleph_n$ steps.

However a very subtle and interesting distinction exists between ours and the $\aleph_\omega$ counterexample. In our construction, the α-finite set of indices E representing the subcomputations, together with the ordinals bounding each subcomputation is Σ_1 in L_α (hence, can be effectively obtained within the model L_α). Consequently, an α-partial recursive map can be defined from $\varepsilon' \in E$ to the ordinal encoding the totality of the subcomputations obtained through ε'. Thus, we employ admissibility of α to ensure a bound on the sum of these subcomputations. The reason the $\aleph_\omega$ example fails is that although ω is α-finite, the map $n \to \aleph_n$ is not Σ_1, hence, cannot be computed in L_α.

2 *The α-Naming Theorem* The α-Naming or α-Honesty Theorem tells us that for arbitrary α-computational complexity measures, there exists α-measured sets of functions naming all α-complexity classes. The construction in the proof provides a means of finding, for each α-partial recursive function, an α-partial recursive function so that the new collection is α-measured. Further, for α-recursive functions of the first list, the constructed functions are α-recursive bounding the same α-complexity classes.

2.1 Theorem *Let Φ be an α-computational complexity measure. Then there exists an α-measured set naming every α-complexity class.*

Our proof is a lift of Moll and Meyer's [23a] simplification of the original McCreight [22] proof. Here, for any α-partial recursive ϕ_τ we produce an α-partial recursive $\phi_{g(\tau)}$ such that

$$M(\tau,\beta,\gamma) \leftrightarrow \phi_{g(\tau)}(\beta) = \gamma \text{ is } \alpha\text{-recursive.}$$

Consequently, the set $\{\phi_{g(\tau)} \mid \tau < \alpha\}$ is α-measured. However, the resultant $\phi_{g(\tau)}$ may not necessarily be total even for ϕ_τ total. Therefore, we show how to obtain an α-recursive g' from g such that

(1) ϕ_τ total $\Rightarrow \phi_{g'(\tau)}$ total,
(2) $M(\tau,\beta,\gamma) \Leftrightarrow \phi_{g'(\tau)}(\beta) = \gamma$ is an α-recursive predicate,

and

(3) $C_{\phi_\tau} = C_{\phi_{g'(\tau)}}(\beta)$ for all total ϕ_τ.

The construction described below for computing $\phi_{g(\tau)}(\kappa)$ makes use of several auxiliary sets. The set Q^σ, at stage σ of the construction represents a "queue" on which elements are popped out and moved further along the queue. The elements of Q^σ are encodings of triples having the form $\langle\nu,\varepsilon,\rho\rangle$. The $\nu < \alpha^*$ represent the *priority* of index $\varepsilon < \alpha$; that is, ε's relative position on the queue. An index $\varepsilon < \alpha$ is made *poppable* when it is discovered that for some $\beta < \alpha$, $\Phi_\varepsilon(\beta) > \phi_\tau(\beta)$. The value of ρ is either 1 or 0 depending on whether or not ε is poppable. If index ε is later *popped* then an assignment of value to $\phi_{g(\tau)}(\beta)$ is made making $\Phi_\varepsilon(\beta) > \phi_{g(\tau)}(\beta)$ for some β. At that point index ε once again becomes *unpoppable*.

The set TO^σ, at stage σ, acts as a collection of triples from Q^τ, $\tau \leq \sigma$, which have been discarded. Whenever a triple of the form $\langle\nu,\varepsilon,\rho\rangle$ is ejected from Q^σ, it is done so by placing it into TO^σ. This is done so that all sets being accumulated are increasing rather than pulsating (as $\sigma \to \alpha$). The set GT^σ, at stage σ, represents the function $\phi_{g(\tau)}$ being constructed. A pair $\langle\beta,\gamma\rangle$ is placed into GT^σ at some stage σ if and only if $\phi_{g(\tau)}(\beta) = \gamma$. $Q^{<\sigma}$ is Q^σ just prior to stage σ, that is, $Q^{<\sigma} = \bigcup_{\tau<\sigma} Q^\tau$. Similarly, for $TO^{<\sigma}$ and $GT^{<\sigma}$.

2.2 *The construction* The construction that computes $\phi_{g(\tau)}(\kappa)$ is defined, by induction on stages $\sigma < \alpha$.

Stage 0. Set $Q^0 = TO^0 = GT^0 = \emptyset$.

Stage σ. Set $\beta = \pi_1(\sigma)$, $\mu = \pi_2(\sigma)$ and $Q^\sigma = Q^{<\sigma} \cup \{\langle t(\sigma),\sigma,0\rangle\}$.

We are placing index σ into the queue with priority $t(\sigma)$ in an unpoppable state. If $\Phi_\tau(\beta) = \mu$ then set

$$TO^\sigma = TO^{<\sigma} \cup \{\langle\gamma,\varepsilon,0\rangle \mid \langle\gamma,\varepsilon,0\rangle \in Q^\sigma - TO^{<\sigma} \,\&\, \Phi_\varepsilon(\beta) > \phi_\tau(\beta)\}$$

and

$$Q^\sigma = Q^\sigma \cup \{\langle\gamma,\varepsilon,1\rangle \mid \langle\gamma,\varepsilon,0\rangle \in Q^\sigma - TO^{<\sigma} \,\&\, \Phi_\varepsilon(\beta) > \phi_\tau(\beta)\}.$$

Only if it is discovered that $\Phi_\tau(\beta)\downarrow$, do we make poppable all those indices ε which take more than $\phi_\tau(\beta)$ steps to compute on input β.

If $\langle\beta,\theta\rangle \in GT^{<\sigma}$ for some $\theta < \alpha$ we set $GT^\sigma = GT^{<\sigma}$ and proceed to stage

$\sigma+1$. Thus we discover that $\phi_{g(\tau)}(\beta)$ has already been defined. If $\langle\beta,\theta\rangle \notin \mathrm{GT}^{<\sigma}$ for all $\theta < \alpha$, then

$$\nu = \min_{\nu}\{\exists\varepsilon < \alpha(\langle\nu,\varepsilon,1\rangle \in \mathrm{Q}^{\sigma} - \mathrm{TO}^{\sigma} \,\&\, \Phi_{\varepsilon}(\beta) > \mu) \\ \& \;\forall\lambda \leqslant \nu[(\langle\lambda,\varepsilon',0\rangle \in \mathrm{Q}^{\sigma} - \mathrm{TO}^{\sigma}) \rightarrow \Phi_{\varepsilon'}(\beta) \leqslant \mu]\}$$

That is, ε is the poppable index of highest priority, (i) taking more than μ steps on input β, and
(ii) all unpoppable indices of higher priority take no more than μ steps on input β.

If $\nu = 0$ then set $\mathrm{GT}^{\sigma} = \mathrm{GT}^{<\sigma}$ and proceed to stage $\sigma + 1$ Thus, no such index exists at this point. Otherwise, set

$$\varepsilon = \min_{\varepsilon}\{\langle\nu,\varepsilon,1\rangle \in \mathrm{Q}^{\sigma} - \mathrm{TO}^{\sigma} \,\&\, \Phi_{\varepsilon}(\beta) > \mu \\ \& \;\forall\varepsilon'\forall\lambda < \nu(\langle\lambda,\varepsilon',0\rangle \in \mathrm{Q}^{\sigma} - \mathrm{TO}^{<\sigma} \rightarrow \Phi_{\varepsilon'}(\beta) \leqslant \mu)\}$$

Here ε is taken as the *smallest* candidate having priority ν. Next, set $\mathrm{GT}^{\sigma} = \mathrm{GT}^{<\sigma} \cup \{\langle\beta,\mu\rangle\}$, $\mathrm{TO}^{\sigma} = \mathrm{TO}^{\sigma} \cup \{\langle\nu,\varepsilon,1\rangle\}$ and $\mathrm{Q}^{\sigma} = \mathrm{Q}^{\sigma} \cup \{\langle \dagger(\sigma),\varepsilon,0\rangle\}$. Thus, we are defining $\phi^{\alpha}_{g(\tau)}(\beta) = \mu$ and popping index ε to an unpoppable state into the $\dagger(\sigma)$-th position of the queue. If $\kappa = \beta$ then halt the procedure and output μ. Otherwise, proceed to stage $\sigma + 1$. This concludes the construction.
Q.E.D.

2.3 *The verification* It is clear that the function created is an α-partial recursive function of two arguments, $\theta(\tau,\kappa)$. By the α-S^m_n-Theorem (*Cf.* [16]) we acquire an α-recursive g such that $\phi_{g(\tau)}(\kappa) = \theta(\tau,\kappa)$. From the details of the construction if $\phi_{g(\tau)}(\beta)$ is defined at stage σ, then $\sigma = \langle\beta,\phi_{g(\tau)}(\beta)\rangle$. Hence, to decide if $\phi_{g(\tau)}(\beta) = \gamma$, we simply run the α-effective construction through stage $\langle\beta,\gamma\rangle$. It follows that $\{\phi_{g(\tau)}\}$ forms an α-measured set. We next take our construction one step further by defining,

$$\phi_{g'(\tau)}(\beta) = \min\{\phi_{g(\tau)}(\beta),\ \phi_{\tau}(\beta) + \Phi_{\tau}(\beta)\}$$

Again, since the right side of the above is α-partial recursive in β and τ, the α-S^m_n-Theorem justifies our notation.

Since $\{\phi_{g(\tau)}\}$ is α-measured, it can be α-recursively decided whether or not $\phi_{g(\tau)}(\beta) = \gamma$. Further $\{\Phi_{\varepsilon}\}$ is also α-measured and since $\phi_{\tau}(\beta)\downarrow \leftrightarrow \Phi_{\tau}(\beta)\downarrow$, it can be similarly determined if $\phi_{\tau}(\beta) + \Phi_{\tau}(\beta) = \gamma$. Thus, if either should equal γ, we can test to see if the other is $\leqslant\gamma$. It follows that $\{\phi_{g'(\tau)}\}$ is an α-measured set. It is clear from its definition that ϕ_{τ} total implies $\phi_{g'(\tau)}$ total. The remainder of the proof is concerned with showing that for ϕ_{τ} total, $C_{\phi_{\tau}} = C_{\phi_{g'(\tau)}}$. Since both ϕ_{τ} and $\phi_{g'(\tau)}$ are total this reduces to

2.3 Lemma *For ϕ_{τ} total and $\varepsilon < \alpha$*

$$\{\beta \mid \Phi_{\varepsilon}(\beta) > \phi_{\tau}(\beta)\}$$

is bounded if and only if

$$\{\beta \mid \Phi_{\varepsilon}(\beta) > \phi_{g'(\tau)}(\beta)\}$$

is bounded.

Before proceeding we define some useful terminology regarding the construction of $\phi_{g(\tau)}$. An index ε is said to be *stable* if there exists a stage σ_0 in the construction of $\phi_{g(\tau)}$ such that for some $\gamma < \alpha^*$, $\rho \in \{0,1\}$, (1) $\langle \gamma, \varepsilon, \rho \rangle \in Q^{\sigma_0} - TO^{\sigma_0}$, and (2) $\sigma' > \sigma_0 \to \langle \gamma, \varepsilon, \rho \rangle \in Q^{\sigma'} - TO^{\sigma'}$. Index ε is *stable at* 0 if the ρ above is 0; otherwise, ε is *stable at* 1. In either case, we say that index ε becomes *stabilized after* σ_0.

An index is said to be *unstable* if it is not stable. Precisely, ε is unstable if for any stage $\sigma < \alpha$, there exists a stage $\sigma' > \sigma$ such that for $\gamma, \gamma' < \alpha^*$, $\rho, \rho' \in \{0,1\}$ (1) $\langle \gamma, \varepsilon, \rho \rangle \in Q^{\sigma} - TO^{\sigma}$ and (2) $\langle \gamma', \varepsilon, \rho' \rangle \in Q^{\sigma'} - TO^{\sigma'}$ where either $\gamma \neq \gamma'$ or $\rho \neq \rho'$. For a fixed $\varepsilon < \alpha$, an ε-*triple* is a triple of the form $\langle \gamma, \varepsilon, \rho \rangle$ with $\gamma < \alpha^*$ and $\rho \in \{0,1\}$.

The proof of Lemma **2.3** divides into three subcases; namely, if ε is unstable, stable at 0 and stable at 1.

2.3.1 Lemma *Let ϕ_τ be total. Then for all indices $\varepsilon < \alpha$, if ε is unstable then*

$$A = \{\beta \mid \Phi_\varepsilon(\beta) > \phi_\tau(\beta)\}$$

and

$$A' = \{\beta \mid \Phi_\varepsilon(\beta) > \phi_{g'(\tau)}(\beta)\}$$

are unbounded.

Proof: Since ε is unstable, an easy induction on σ shows that

(1) $B = \{\sigma \mid$ an ε-triple goes from an unpoppable to a poppable state at stage $\sigma\}$,

and

(2) $B' = \{\sigma \mid$ and ε-triple gets popped at stage $\sigma\}$,

are both unbounded.

Suppose that A were bounded and consider the set $C = \{\beta \mid \sigma = \langle \beta, \Phi_\tau(\beta) \rangle \in B\}$. First, $C \subseteq A$ since for any $\beta \in C$, $\sigma = \langle \beta, \Phi_\tau(\beta) \rangle$ is a stage at which an ε-triple goes from an unpoppable to a poppable state. By the construction, this occurs only if $\Phi_\varepsilon(\beta) > \phi_\tau(\beta)$, and hence $\beta \in A$.

From the above, since A is bounded C is bounded. Further, since the construction is α-effective and Φ_τ is α-recursive, C is α-recursive, thus α-finite. Consider the α-recursive map $f(\beta) \equiv \langle \beta, \Phi_\tau(\beta) \rangle$ to see that $B \subseteq f[C]$. For if $\sigma \in B$ then at stage σ an ε-triple goes from an unpoppable to a poppable state. By the construction $\sigma = \langle \beta, \Phi_\tau(\beta) \rangle$ for some β and hence $\sigma = f(\beta) \in f[C]$. Since C is α-finite, by Fact **0.1**, $f[C]$ is α-finite, and thus bounded below α. However this implies B is bounded, contradicting (1). Therefore, A must be unbounded.

For the second part assume A' is bounded. Since A' is α-recursive, it is α-finite. Consider the map $k(\beta) \equiv \langle \beta, \phi_{g(\tau)}(\beta) \rangle$ to see that $B' \subseteq k[A']$. For if $\sigma \in B'$ then at stage σ, an ε-triple is popped. By the construction,

$\sigma = \langle\beta,\phi_{g(\tau)}(\beta)\rangle$ for some β. The above inclusion will be demonstrated upon showing that $\beta \in A'$. However, we know $\Phi_\varepsilon(\beta) > \phi_{g(\tau)}(\beta)$ from the construction and by the definition of $\phi_{g'(\tau)}$, $\phi_{g(\tau)} \geqslant \phi_{g'(\tau)}(\beta)$. Hence, $\Phi_\varepsilon(\beta) > \phi_{g'(\tau)}(\beta)$. Since A' is α-finite, by Fact **0.1** so is $k[A']$. However, we know $B' \subseteq k[A']$ which implies B' is bounded contradicting (2). Hence, A' must be unbounded.
Q.E.D.

2.3.2 Lemma *Let ϕ_τ be total. Then for all indices $\varepsilon < \alpha$, if ε is stable at* 0 *then*

$$A = \{\beta \mid \Phi_\varepsilon(\beta) > \phi_\tau(\beta)\}$$

and

$$A' = \{\beta \mid \Phi_\varepsilon(\beta) > \phi_{g'(\tau)}(\beta)\}$$

are α-finite.

Proof: For the first part consider the set

$$C = \{\sigma \geqslant \varepsilon \mid \text{during stage } \sigma,\ \varepsilon \text{ is poppable}\} \cup \varepsilon + 1.$$

Since ε is stable at 0, and the construction is α-effective, C is α-finite. Thus, by Fact **0.1**, $\pi_1[C]$ is also α-finite. To see that $A \subseteq \pi_1[C]$, consider $\beta \in A$. Since $\phi_\tau(\beta)$ is defined, there exists a stage $\sigma_0 = \langle\beta,\Phi_\tau(\beta)\rangle$. If $\sigma_0 < \varepsilon$, then $\beta = \pi_1(\sigma_0) \in \pi[C]$. If $\sigma_0 \geqslant \varepsilon$ then at stage σ_0 of the construction it is found that $\Phi_\varepsilon(\beta) > \phi_\tau(\beta)$ and consequently ε must be poppable. Therefore, $\beta = \pi_1(\sigma_0) \in \pi_1[C]$ also. Since $A \subseteq \pi_1[C]$, A is bounded and α-recursive, hence α-finite.

In order to prove the second part we assert the following.

Claim: If ε is stable then there exists a stage σ_0 such that after stage σ_0, (i) ε has already become stabilized, and (ii) all indices with higher priority than ε which stabilize (either at 0 or 1) have done so already.

Assuming this proved, consider the set $F = \{\beta \mid \langle\beta,0\rangle > \sigma_0\}$ where σ_0 is the stage of the claim. Since $\langle,\rangle$ is increasing in the second argument, all β's in F appear in the construction of $\phi_{g(\tau)}$ for the first time after stage σ_0. Consider $\beta \in F$ to see

$$(*) \qquad F \subseteq \{\beta \mid \Phi_\varepsilon(\beta) \leqslant \phi_{g'(\tau)}(\beta)\}.$$

By definition of $\phi_{g'(\tau)}$, this divides into two cases, namely, when $\phi_{g'(\tau)}(\beta)$ is $\phi_{g(\tau)}(\beta)$ or $\phi_\tau(\beta) + \Phi_\tau(\beta)$. In the first case, by the construction, assignments to $\phi_{g(\tau)}(\beta)$ made after stage σ_0 are done via indices of priority value higher than ε. Further, these assignments are made such that $\phi_{g(\tau)}(\beta) \geqslant \Phi_\varepsilon(\beta)$. For the second, assume to the contrary that $\phi_\tau(\beta) + \Phi_\tau(\beta) < \Phi_\varepsilon(\beta)$. Then $\phi_\tau(\beta) < \Phi_\varepsilon(\beta)$ and by stage $\sigma = \langle\beta,\Phi_\tau(\beta)\rangle$, ε would be poppable contradicting our original assumption.

To see that $D = \{\beta \mid \langle\beta,0\rangle \leqslant \sigma_0\}$ is bounded, consider

$$f(\sigma) = \begin{cases} \beta & \text{if } \sigma \leqslant \sigma_0 \ \&\ \sigma = \langle\beta,0\rangle \\ \uparrow & \text{otherwise} \end{cases}$$

and

$$H = \{\sigma \mid \sigma \leqslant \sigma_0 \ \& \ \sigma = \langle \beta, 0 \rangle\}$$

Since $H = \mathrm{dom}(f)$, is α-recursive and bounded, H is α-finite. Also $f[H] = D$ and by Fact **0.1**, D is α-finite, hence bounded. From (*) it follows that

$$A' = \{\beta \mid \phi_{g'(\tau)}(\beta) < \Phi_\varepsilon(\beta)\} \subseteq \{\beta \mid \langle \beta, 0\rangle \leqslant \sigma_0\}.$$

The α-finiteness of A' follows from its α-recursiveness and the boundedness of D.

All that remains is to verify the Claim. Before doing this we introduce some useful notation. For $\nu < \alpha^*$, consider the sets

F_ν^{0-1} $= \{\rho < \alpha^* \mid \rho \in \mathrm{range}(t)$ & $\rho < \nu$ and at some stage ρ is the priority of a triple which goes from an unpoppable to a poppable state$\}$.

F_ν^{1-p} $= \{\rho < \alpha^* \mid \rho \in \mathrm{range}(t)$ & $\rho < \nu$ at some stage ρ is the priority of a triple which gets popped$\}$.

S_ν^{0-1} $= \{\rho < \alpha^* \mid \rho \in \mathrm{range}(t)$ & $\rho < \nu$ & ρ is the priority of *two* different triples which both go from unpoppable to poppable states$\}$.

S_ν^{1-p} $= \{\rho < \alpha^* \mid \rho \in \mathrm{range}(t)$ & $\rho < \nu$ & ρ is the priority of *two* different triples which are popped$\}$.

For $\delta < \alpha^*$, $\sigma < \alpha$ and $i = 1, 2$ consider the functions:

$k_i^{0-1}(\delta) = \sigma \Longleftrightarrow \sigma$ is the stage at which a triple with priority δ goes from an unpoppable to a poppable state for the ith time.

$k_i^{1-p}(\delta) = \sigma \Longleftrightarrow \sigma$ is the stage at which a triple with priority δ gets popped from the ith time.

The sets F_ν^{0-1}, F_ν^{1-0}, S_ν^{0-1}, S_ν^{1-0}, $\nu < \alpha^*$ are α-recursively enumerable and bounded below α^*, hence by Fact **0.2**, they are α-finite. The mappings k_i^{0-1}, k_i^{1-0} are α-partial recursive due to the α-effectiveness of the construction. Thus by Fact **0.1**, $k_1^{0-1}[F_\nu^{0-1}]$, $k_2^{0-1}[S_\nu^{0-1}]$, $k_1^{1-p}[F_\nu^{1-p}]$ and $k_2^{1-p}[S_\nu^{1-p}]$ are all α-finite. Let σ' bound all four sets.

We now proceed to verify the claim. Since ε is stable there must be a stage σ'' after which ε does stabilize. The σ_0 required of Claim **2.3.4** is simply $\max\{\sigma', \sigma''\}$. By the details of the construction a priority $< \alpha^*$ can only be assigned to, at most, two triples having different second components. Therefore, no action may occur following stage σ_0 involving index ε or any index with higher priority than ε. Q.E.D.

Finally, the last case.

2.3.3 Lemma *Let ϕ_τ be total. Then if ε is stable at* 1, *then*

$$A = \{\beta \mid \Phi_\varepsilon(\beta) > \phi_\tau(\beta)\}$$

and

$$A' = \{\beta \mid \Phi_\varepsilon(\beta) > \phi_{g'(\tau)}(\beta)\}$$

are α-finite.

Proof: By the claim used in the proof of Lemma **2.3.2**, if ε is stable at 1 then there exists a stage σ_0 so that after stage σ_0, (i) ε has already stabilized at 1, and (ii) all indices with higher priority than ε which stabilize, have done so already. Further, for any β such that $\langle\beta,0\rangle > \sigma_0$, since $<, >$ is increasing in the second argument, prior to stage σ_0 no mention of β occurs in the construction.

Claim: For ε, τ as in Lemma **2.3.3**, and for β such that $\langle\beta,0\rangle > \sigma_0$, (i) $\Phi_\varepsilon(\beta) \leq \phi_\tau(\beta)$, and (ii) $\Phi_\varepsilon(\beta) \leq \phi_{g'(\tau)}(\beta)$.

From this claim it follows that $A \subseteq \{\beta \mid \langle\beta,0\rangle \leq \sigma_0\}$ and $A' \subseteq \{\beta \mid \langle\beta,0\rangle \leq \sigma_0\}$. Hence, the previously established boundedness of $\{\beta \mid \langle\beta,0\rangle \leq \sigma_0\}$ and the α-recursiveness of A and A' imply the result.

We conclude by verifying the claim. Let κ be an unpoppable index with smaller priority value than ε after stage σ_0 and β be such that $\langle\beta,0\rangle > \sigma_0$. We will see that $\Phi_\kappa(\beta) \leq \min\{\phi_\tau(\beta), \phi_{g'(\tau)}(\beta)\}$. Since κ has smaller priority value than ε, κ must be stable at 0. If follows by the proof of Lemma **2.3.2**, that since $\langle\beta,0\rangle > \sigma_0$ that $\Phi_\kappa(\beta) \leq \phi_{g'(\tau)}(\beta)$. On the other hand, $\Phi_\kappa(\beta) \leq \phi_\tau(\beta)$ for otherwise, κ would become poppable contrary to our assumption.

Regard β as fixed where $\langle\beta,0\rangle > \sigma_0$ and let $\mu = \sup\{\Phi_\kappa(\beta) \mid \kappa$ has smaller priority value than ε & κ is unpoppable after $\sigma_0\}$. We first observe that $\sigma = \langle\beta,\mu\rangle$ is the earliest stage at which $\phi_{g(\tau)}(\beta)$ could be defined. Certainly, it cannot occur prior to $\langle\beta,0\rangle$, since β never appears. If $\phi_{g(\tau)}(\beta)$ is defined at stage $\sigma = \langle\beta,\delta\rangle$ for $\delta < \mu$, then since there exists a γ such that $\delta < \gamma \leq \mu$ and $\gamma = \Phi_\kappa(\beta)$, we have that $\Phi_\kappa(\beta) > \delta = \phi_{g(\tau)}(\beta)$. But this could not occur by the details of the construction. To see that $\Phi_\varepsilon(\beta) \leq \mu$ assume to the contrary that $\Phi_\varepsilon(\beta) > \mu$. By the definition of μ, $\Phi_\varepsilon(\beta) \leq \mu$ for all unpoppable κ with priority value smaller than ε. From the above, at stage $\sigma = \langle\beta,\mu\rangle$ index ε would qualify for popping and would be so, contradicting its stability after stage σ_0.

The claim now follows from the above two facts $\Phi_\varepsilon(\beta) \leq \mu$ and $\mu \leq \min\{\phi_\tau(\beta), \phi_{g'(\tau)}(\beta)\}$. Q.E.D.

Lemmas **2.3.1**, **2.3.2** and **2.3.3** together imply that for total $\phi_\tau\{\beta \mid \Phi_\varepsilon(\beta) \leq \phi_\tau(\beta)\}$ α-finite if and only if $\{\beta \mid \Phi_\varepsilon(\beta) \leq \phi_{g'(\tau)}(\beta)\}$ α-finite. That is,

$$C_{\lambda\beta\phi_\tau(\beta)} = C_{\lambda\beta\phi_{g'(\tau)}(\beta)}. \qquad \text{Q.E.D.}$$

3 *The α-Speedup Theorem* We prove in this section the α-analogue to Blum's speedup result. Namely, given any α-recursive function r of two arguments, a 0-1 valued α-recursive function f can be constructed so that for any way of computing f, a faster one (on almost all of α) exists. Further, this new method takes fewer than r of the complexity of the former method steps. Our proof is a 'lift' of a simplified proof due to Young [34] of the ω-Speedup Theorem.

3.1 Theorem *For any α-recursive function $r(\beta,\gamma)$ there exists a 0-1 value α-recursive f such that for any index ε for f there exists another index τ for f where $\Phi_\varepsilon(\beta) > r(\beta,\Phi_\tau(\beta))$ for all but an α-finite set of β.*

An α-partial recursive function will be defined in terms of a construction given below. This construction will depend upon four parameters μ, ν, λ and κ and will proceed in stages $\sigma < \alpha$. As the construction proceeds, we shall be accumulating two sets $A_{\lambda\mu\nu}$ and $F_{\lambda\mu\nu}$. The set $A^\sigma_{\lambda\mu\nu}$ at stage σ, will serve as an α-finite collection of pseudo indices which have been cancelled prior to or during stage σ. The collection $A^{<\sigma}_{\lambda\mu\nu}$ denotes those pseudo indices cancelled just prior to stage σ. That is, $A^{<\sigma}_{\lambda\mu\nu} = \bigcup_{\tau<\sigma} A^\tau_{\lambda\mu\nu}$. The set $F^\sigma_{\lambda\mu\nu}$ at stage σ will consist of the graph of the desired function accumulated by stage σ and $F^{<\sigma}_{\lambda\mu\nu}$ is this set just prior to σ. Let $\{K_\varepsilon | \varepsilon < \alpha\}$ and $\{G_\varepsilon | \varepsilon < \alpha\}$ be enumerations of the α-finite subsets of α and α-finite functions, respectively, as in section **0**.

3.2 *The construction* The construction that computes a function value for argument κ is defined by transfinite induction on σ. The output will also depend on three other parameters: λ, μ and ν.

Stage 0. Set $A^0_{\lambda\mu\nu} = F^0_{\lambda\mu\nu} = \emptyset$.

Stage σ. If $\sigma \in \mathrm{dom}(G_{\pi_1(\nu)})$ then set

$$\theta = \min\{1, G_{\pi_1(\nu)}(\sigma)\};\ F^\sigma_{\lambda\mu\nu} = F^{<\sigma}_{\lambda\mu\nu} \cup \{\langle\sigma,\theta\rangle\};$$

$A^\sigma_{\lambda\mu\nu} = A^{<\sigma}_{\lambda\mu\nu}$ and go to check. We examine whether the $\pi_1(\nu)$-th α-finite function will give us a value on σ.

(*) Otherwise, compute the set

$$V = \{\varepsilon \mid \varepsilon \in [t(\mu), t(\sigma)]\ \&\ t_\sigma^{-1}(\varepsilon)\downarrow\ \&\ t_\sigma^{-1}(\varepsilon) \notin K_{\pi_2(\nu)}\ \&\ \Phi_{t_\sigma^{-1}(\varepsilon)}(\sigma) \le r(\sigma, \Phi_\lambda(\Omega_\varepsilon,\nu,\sigma)),$$
$$\text{where } \Omega_\varepsilon = \min_\Omega \{t(\Omega) > \varepsilon\}\}.$$

This is the set of all pseudo indices $\varepsilon < \alpha^*$ which are *eligible for cancellation at stage* σ.

Next, if $A^{<\sigma}_{\lambda\mu\nu} \neq A^{<\sigma}_{\lambda\mu\nu} \cup t^{-1}[V]$ then set

$$\rho = \min\{\tau \in V\ \&\ t_\sigma^{-1}(\tau) \notin A^{<\sigma}_{\lambda\mu\nu}\};\ A^\sigma_{\lambda\mu\nu} = A^{<\sigma}_{\lambda\mu\nu} \cup \{t_\sigma^{-1}(\rho)\}$$

and $F^\sigma_{\lambda\mu\nu} = F^{<\sigma}_{\lambda\mu\nu} \cup \{\langle\sigma,\theta\rangle\}$ where

$$\theta = \begin{cases} 0 \text{ if } \phi_{t_\sigma^{-1}(\rho)}(\sigma) \neq 0 \\ 1 \text{ if } \phi_{t_\sigma^{-1}(\rho)}(\sigma) = 0 \end{cases}$$

and go to check.

Here we found pseudo indices which are eligible for cancellation and which were never before cancelled. We cancel that index with highest priority by placing its inverse image into $A^\sigma_{\lambda\mu\nu}$. We then assign to our constructed function a value different from the function having pseudo index just cancelled.

If $A^{<\sigma}_{\lambda\mu\nu} = A^{<\sigma}_{\lambda\mu\nu} \cup t^{-1}_{\sigma}[V]$ then set

$$A^{\sigma}_{\lambda\mu\nu} = A^{<\sigma}_{\lambda\mu\nu};\ \theta = 0;\ F^{\sigma}_{\lambda\mu\nu} = F^{<\sigma}_{\lambda\mu\nu} \cup \{\langle\sigma,\theta\rangle\}$$

and go to check. Since we cannot cancel any pseudo index, we arbitrarily assign the value 0 to our function for argument σ.

Check: If $\sigma = \kappa$ then output θ; otherwise go to stage $\sigma + 1$. This concludes the construction. Q.E.D.

3.3 *The verification* Let $A_{\lambda\mu\nu} = \bigcup_{\sigma<\alpha} A^{\sigma}_{\lambda\mu\nu}$, $F_{\lambda\mu\nu} = \bigcup_{\sigma<\alpha} F^{\sigma}_{\lambda\mu\nu}$ and define the function $f_{\lambda\mu\nu}$ resulting from the construction by

$$f_{\lambda\mu\nu}(\sigma) = \delta \leftrightarrow \langle\sigma,\delta\rangle \text{ is placed into } F_{\lambda\mu\nu} \text{ at stage } \sigma.$$

It follows from the α-effectiveness of the construction that for any λ, μ, ν, $f_{\lambda\mu\nu}$ is a well-defined, 0-1 valued α-partial recursive function. Further, since $f_{\lambda\mu\nu}(\kappa)$ depends on the four arguments λ, μ, ν and κ, it may be expressed as $\phi_{\varepsilon_0}(\lambda,\mu,\nu,\kappa)$ for some $\varepsilon_0 < \alpha$. By the α-S^m_n Theorem, there exist α-recursive s and s^1_2 such that

$$f_{\lambda\mu\nu}(\kappa) = \phi_{s(\lambda)}(\mu,\nu,\kappa) = \phi_{s^1_2(\lambda,\mu,\nu)}(\kappa).$$

Since s is α-recursive, by the α-Recursion Theorem, there exists a $\lambda_0 < \alpha$ such that $\phi_{\lambda_0}(\mu,\nu,\kappa) = \phi_{s(\lambda_0)}(\mu,\nu,\kappa)$. The function required by the theorem is precisely $f_{\lambda_0 00}(\kappa)$.

We first show $f_{\lambda_0\mu\nu}$ is α-recursive by arguing that every stage $\sigma < \alpha$ is α-recursively completed. Since the calculation of $f_{\lambda_0 00}(\beta)$, requires the running of the construction till stage β, α-recursiveness follows. By examining the details of the construction, it is seen that at any stage σ, the only step which may *not* necessarily be α-recursively performed is (*). This is the search for ε in $[t(\mu), t(\sigma)]$ where

$$\Phi_{t^{-1}_{\sigma}(\varepsilon)}(\sigma) \leq r(\sigma, \Phi_{\lambda_0}(\Omega_\varepsilon,\nu,\sigma)), \text{ with } \Omega_\varepsilon = \min_{\Omega} \{t(\Omega) > \varepsilon\}.$$

3.3.1 Lemma *For all* μ, $\sigma < \alpha$ *and* ε *in* $[t(\mu),t(\sigma)]$, $\min_{\Omega}\{t(\Omega) > \varepsilon\}$ *is defined.*

Proof: The predicate $t(\Omega) > \varepsilon$ is α-recursive since t is; hence, the proof reduces to showing that for each $\varepsilon \in [t(\mu),t(\sigma)]$, there exists an $\Omega < \alpha$ where $t(\Omega) > \varepsilon$. But this is a consequence of the facts $t(\sigma) < \alpha^*$ and t is the one-one projection map. Q.E.D.

3.3.2 Lemma *Let* σ, λ, μ, $\nu < \alpha$ *and suppose* $t(\mu) > t(\sigma)$. *If for all* $\sigma' < \sigma$, $f_{\lambda\mu\nu}(\sigma')$ *is defined, then* $f_{\lambda\mu\nu}(\sigma)$ *is defined.*

Proof: From the hypothesis all stages $\sigma' < \sigma$ have been performed. At stage σ, since $t(\mu) > t(\sigma)$, (*) of the construction is never considered. Hence, $f_{\lambda\mu\nu}(\sigma)$ must be defined. Q.E.D.

In particular,

3.3.3 Corollary *Let* σ, μ, $\nu < \alpha$ *and suppose* $t(\mu) > t(\sigma)$. *If for all* $\sigma' < \sigma$, $f_{\lambda_0\mu\nu}(\sigma')$ *is defined, then* $f_{\lambda_0\mu\nu}(\sigma)$ *is defined.*

3.3.4 Lemma $\phi_{\lambda_0}(\mu,\nu,1)$ *is defined for all* μ, $\nu < \alpha$.

Proof: By induction on $\delta = \min_\delta \{t(\mu) + \delta \geqslant t(1)\}$.

Case 1. $t(\mu) > t(1)$. By Corollary **3.3.3**, it follows that $\phi_{\lambda_0}(\mu,\nu,1)$ is defined for all ν.

Case 2. $\delta = 0$. It follows that $t(\mu) = t(1)$. Convergence of $\phi_{\lambda_0}(\mu,\nu,1)$ occurs if (*) of the computation is not performed. If it is, convergence depends on that of $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,1)$ where $\Omega_\varepsilon = \min_\Omega \{t(\Omega) > \varepsilon\}$ and $t(\mu) = t(1)$. Since $t(\Omega_\varepsilon) > t(1)$ $(= \varepsilon)$, again, by Corollary **3.3.3** $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,1)$ is defined. Hence, $\phi_{\lambda_0}(\mu,\nu,1)$ is defined.

Case 3. Assume $\phi_{\lambda_0}(\mu,\nu,1)$ converges for μ, ν where $\min_\delta\{t(\mu)+\delta \geqslant t(1)\} < \delta$, to see $\phi_{\lambda_0}(\mu,\nu,1)$ is defined for μ, ν where $\min_\delta\{t(\mu) + \delta \geqslant t(1)\} = \delta$. If in the computation of $\phi_{\lambda_0}(\mu,\nu,1)$, (*) is not performed, we are done; else the convergence will depend on those of $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,1)$ where $t(\mu) \leqslant \varepsilon \leqslant t(1)$ and $\Omega_\varepsilon = \min_\Omega\{t(\Omega) > \varepsilon\}$. But for all of these, either $t(\Omega_\varepsilon) > t(1)$ or $0 \leqslant \min_\delta\{t(\Omega_\varepsilon) + \delta \leqslant t(1)\} < \delta$. In the first case convergence is assured by Corollary **3.3.3**; in the second, by the induction hypothesis. By Fact **0.2** the set

$$E = \{\varepsilon \mid t(\mu) \leqslant \varepsilon \leqslant t(1)\}$$

is α-finite. Define an α-partial recursive mapping $p\colon E \to \alpha$ by $p(\varepsilon) = \beta + \bigcup_{\varepsilon' < \varepsilon} p(\varepsilon')$ where $\varepsilon \in [t(\mu),t(1)]$ & $t_1^{-1}(\varepsilon)\downarrow$ & β is the length of the α-finite computation of $r(\sigma,\Phi_{\lambda_0}(\Omega_\varepsilon,\nu,1))$, $\Omega_\varepsilon = \min_\Omega t(\Omega) > \varepsilon$; 0, otherwise. By Fact **0.1**, $p[E]$ is α-finite, implying the union of the subcomputations (*) is bounded. Hence, $\phi_{\lambda_0}(\mu,\nu,1)$ will be defined. Q.E.D.

3.3.5 Lemma *Let* μ, ν, $\sigma < \alpha$. *Suppose for all* $\sigma' < \sigma$ $\phi_{\lambda_0}(\mu,\nu,\sigma')$ *is defined. Then* $\phi_{\lambda_0}(\mu,\nu,\sigma)$ *is defined.*

Proof: By induction on $\delta = \min_\delta\{t(\mu) + \delta \geqslant t(\sigma)\}$.

Case 1. If $t(\mu) > t(\sigma)$ then by Corollary **3.3.3**, $\phi_{\lambda_0}(\mu,\nu,\sigma)$ is defined for all ν.

Case 2. If $\delta = 0$ then $t(\mu) = t(\sigma)$. Thus convergence of $\phi_{\lambda_0}(\mu,\nu,\sigma)$ occurs if (*) is not performed; else convergence hinges on that of $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,\sigma)$ where $t(\mu) = \varepsilon = t(\sigma)$ and $\Omega_\varepsilon = \min_\Omega\{t(\Omega) > \varepsilon\}$. From the details of the construction, it follows that $\phi_{\lambda_0}(\mu,\nu,\sigma')$, $\sigma' < \sigma$ defined implies $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,\sigma')$ is also where $t(\mu) \leqslant t(\Omega_\varepsilon)$ (since a possible shorter search is performed in the latter). By Corollary **3.3.3**, since $t(\Omega_\varepsilon) > t(\sigma)$ $(= \varepsilon)$, $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,\sigma)$ is defined.

Case 3. Suppose the result holds for all μ,ν where $\min_\delta\{t(\mu) + \delta = t(\sigma)\} < \delta$, to see that $\phi_{\lambda_0}(\mu,\nu,\sigma)$ is defined for μ, ν where $\min_\delta\{t(\mu) + \delta = t(\sigma)\} = \delta$. Again, if (*) is not performed at stage σ, we are done; else the convergence depends on those of $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,\sigma)$ where $t(\mu) \leqslant \varepsilon \leqslant t(\sigma)$ and $\Omega_\varepsilon = \min_\Omega\{t(\varepsilon) > \varepsilon\}$. Again, since $\phi_{\lambda_0}(\mu,\nu,\sigma')$ is defined for $\sigma' < \sigma$, by the details of the construction $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,\sigma')$ is defined. By either Corollary **3.3.3** or the induction hypothesis, all of the computations $\phi_{\lambda_0}(\Omega_\varepsilon,\nu,\sigma)$ are defined. Just as in the proof of Lemma **3.3.4**, the union of all these subcomputations is bounded below α, and convergence of $\phi_{\lambda_0}(\mu,\nu,\sigma)$ follows. Q.E.D.

3.3.6 Lemma *For all* μ, ν, $\sigma < \alpha$, $\phi_{\lambda_0}(\mu,\nu,\sigma)$ *is defined.*

Proof: The result comes via a double induction on σ and μ using Lemmas **3.3.4** and **3.3.5**. Q.E.D.

Recall that $f_{\lambda_0 00}(\beta) = \phi_{\lambda_0}(0,0,\beta)$. We can assume, without loss of generality that $D_{\pi_1(0)} = G_{\pi_1(0)} = \emptyset$ and that $t(0) = 0$. Further, for any $\mu, \nu, \beta < \alpha$, the *main computation* of $\phi_{\lambda_0}(\mu,\nu,\beta)$ refers to the main body computation and not to any recursive calls which may be produced. An immediate observation is

3.3.7 Lemma *Let $\mu, \nu, \beta < \alpha$. Then in the main computation of $\phi_{\lambda_0}(\mu,\nu,\beta)$, no pseudo index ε is ever cancelled more than once.*

For $\gamma < \alpha^*$, let

$$E_\gamma^{\lambda_0} = \{\rho < \alpha^* \mid \rho \in \text{range}(t) \ \&\ \rho < \gamma \ \&\ \rho \text{ gets cancelled in } A_{\lambda_0 00} \text{ during the construction of } F_{\lambda_0 00}\}.$$

Let k^{λ_0} be a map from $\alpha^* \to \alpha$ defined as: $k^{\lambda_0}(\varepsilon)$ = the stage of the construction of $F_{\lambda_0 00}$ at which pseudo index ε gets cancelled. By Lemma **3.3.7** and the α-effectiveness of the construction, it follows that k^{λ_0} is a well defined α-partial recursive function.

3.3.8 Lemma *For all $\gamma < \alpha^*$.*

(i) $E_\gamma^{\lambda_0}$ and $k^{\lambda_0}[E_\gamma]$ are α-finite,

and

(ii) There is a stage σ_0 such that following σ_0 no pseudo index smaller than γ will ever be cancelled by the construction of $F_{\lambda_0 00}$.

Proof: (i) Clearly $E_\gamma^{\lambda_0}$ is α-recursively enumerable and bounded below α^*; hence, $E_\gamma^{\lambda_0}$ is α-finite by Fact **0.2**. From this and the observation that $E_\gamma^{\lambda_0} \subseteq \text{dom } k^{\lambda_0}$, by Fact **0.1**, $k^{\lambda_0}[E_\gamma^{\lambda_0}]$ is also α-finite.

(ii) By definition, $k^{\lambda_0}[E_\gamma^{\lambda_0}]$ is the set of all stages at which pseudo indices below γ are cancelled. By (i), this set is α-finite, hence, bounded by some $\sigma_0 < \alpha$. Q.E.D.

3.3.9 Lemma *For all $\mu < \alpha$ there exist σ_μ, $\nu_\mu < \alpha$ $(\sigma_\mu > \mu)$ such that*

(i) *no* $\varepsilon < t(\mu)$ *ever gets cancelled in computing* $\phi_{\lambda_0}(0,0,\sigma)$ *for all* $\sigma > \sigma_\mu$,

and

(ii) $G_{\pi_1(\mu_\nu)}(\beta) = \phi_{\lambda_0}(0,0,\beta)/\sigma_\mu$,

and

$K_{\pi_2(\mu_\nu)} = \{t^{-1}(\varepsilon) \mid \varepsilon$ *is cancelled in computing* $\phi_{\lambda_0}(0,0,\beta)$ *for some* $\beta \leq \sigma_\mu\}$.

Proof: (i) By Lemma **3.3.8** there exists a stage σ_0 such that no pseudo index smaller than $t(\mu)$ will ever be cancelled after stage σ_0. Choose $\sigma_\mu = \max\{\sigma_0, \mu\} + 1$.

(ii) Let σ_μ be as in (i). Then since $\phi_{\lambda_0}(0,0,\beta)/\sigma_\mu$ is an α-finite function, $G_\delta(\beta) = \phi_{\lambda_0}(0,0,\beta)/\sigma_\mu$ for some $\delta < \alpha$. Further, the set

$$D = \{\varepsilon \mid \varepsilon \text{ is cancelled in computing } \phi_{\sigma(\lambda_0)}(0,0,\beta) \text{ for some } \beta \leq \sigma_\mu\}$$

is α-recursive and bounded below $\alpha^* \leq \alpha$. From Fact **0.1**, $\mathfrak{t}^{-1}[D]$ is α-finite; hence for some $\eta < \alpha$,

$$K_\eta = \{\mathfrak{t}^{-1}(\varepsilon) \mid \varepsilon \text{ is cancelled in computing } \phi_{\lambda_0}(0,0,\beta) \text{ for some } \beta \leq \sigma_\mu\}.$$

The result follows with $\nu_\mu = \langle \delta, \eta \rangle$. Q.E.D.

An immediate consequence of the details of the construction and the definitions of σ_μ and ν_μ is

3.3.10 Corollary *For all* $\beta < \alpha$,

$$\phi_{\lambda_0}(\mu,\nu_\mu,\beta) = \phi_{\lambda_0}(0,0,\beta) = \phi_{s_2^1(\lambda_0,0,0)}(\beta) = f_{\lambda_0 00}(\beta).$$

3.3.11 Lemma *For any index* ε *for* $f_{\lambda_0 00}$, $\Phi_\varepsilon(\beta) > r(\beta, \Phi^\alpha_{s_2^1(\lambda_0,\Omega,\nu_\Omega)}(\beta))$ *for all but an* α*-finite set of* β, *where* $\Omega = \min_\Omega\{\mathfrak{t}(\Omega) > \mathfrak{t}(\varepsilon)\}$.

Proof: Suppose otherwise that

$$C = \{\beta \mid \Phi_\varepsilon(\beta) \leq r(\beta, \Phi_{s_2^1(\lambda_0,\Omega,\nu_\Omega)}(\beta))\}$$

is not α-finite. Since it is α-recursive, it must be unbounded. Let $\varepsilon' < \alpha^*$, where $\varepsilon' = \mathfrak{t}(\varepsilon)$, and $\sigma_1 < \alpha$ be such that $\mathfrak{t}^{-1}_{\sigma_1}(\varepsilon')\downarrow$. By Lemma **3.3.8** there exists a σ_0 such that after stage σ_0 no index of higher priority than $\mathfrak{t}(\varepsilon)$ is ever cancelled in the construction of $F_{\lambda_0 00}$. Since C is unbounded there must be a $\beta \in C$ where $\beta > \max\{\sigma_0, \sigma_1\}$ and ε is eligible for cancellation at stage β. It follows from the details of the construction of $F_{\lambda_0 00}$ that either at stage β or before, the pseudo index ε' will be cancelled implying, $\phi_\varepsilon \neq f_{\lambda_0 00}$. Q.E.D.

The remainder of the theorem follows immediately from the identity $\phi_{s_2^1(\lambda_0,\Omega,\nu_\Omega)}(\beta) = f_{\lambda_0 00}(\beta)$ of Corollary **3.3.10**. Q.E.D.

4 *Open problems* The *levelable* recursively enumerable sets are those which, for each badly behaved procedure (the resources required exceed a recursive function on an infinite set of inputs), have a more economical procedure (the resources are below a recursive function on the same infinite set of inputs). The *speedable* r.e. sets are those sets of natural numbers whose recognition procedures may be sped-up; those for which the faster procedures may be effectively obtained are the effectively speedable r.e. sets. It is known that levelable sets are speedable, and that speedable sets properly contain the effectively speedable ones. In fact, Blum and Marques [2] prove that the maximal r.e. sets of recursion theory (sets M where either $\omega - M$ is finite or for all r.e. A either $(\omega - A) \cap (\omega - M)$ is finite or $A \cap (\omega - M)$ is finite) are levelable, hence speedable, but not effectively speedable.

Lerman [19] considers a number of generalizations of maximal r.e. sets in α-recursion theory by replacing r.e. by α-r.e. and employing several analogues to finite (e.g., α-bounded, α-finite). He displays necessary and sufficient conditions on admissible α for existence of maximal

α-r.e. sets for some of the definitions. Maximal sets exist for some, but not all, countable admissibles, and fail to exist for uncountable ones. For these notations of maximal α-r.e.,

Q. Are there admissible α for which maximal effectively α-speedable sets exist?

Q. Are there admissible α for which speedable = effectively α-speedable?

Q. Are there admissible α for which α-levelable = α-speedable?

Q. Are there admissible α for which effectively α-levelable = effectively α-speedable?

If any of these result in the affirmative, find necessary and sufficient conditions characterizing such α?

Two of the classic results of abstract complexity theory, Blum's Speedup Theorem and Borodin's Gap Theorem [3], have been successfully generalized to effective operator versions. A map $F[\ ]$, of the unary partial recursive functions into itself is an effective operator, if there is an effective total algorithm taking any program for partial recursive ϕ into a program for $F[\phi]$. F is said to be a total effective operator if it maps the recursive functions into itself. We ask,

Q. Does Constable's [4] extension of Borodin's result lift to α?

Q. Does Meyer and Fisher's [23] generalization of Blum's Speedup Theorem further generalized to α-complexity theory?

If either of the above two questions is answered in the negative, we would naturally be interested in necessary and sufficient conditions on admissible α, α-complexity measure and total effective F characterizing such situations.

The proofs of the α-Naming Theorem and α-Speedup exploited the facts that the accompanying constructions were uniform in certain parameters. Namely, the α-Naming construction depended upon $\tau < \alpha$, while that of α-Speedup hinged upon λ, μ and ν. A very subtle point, in both cases, is that the constructions were *not* entirely uniform in *all* its parameters. Specifically, the two depended upon a particular infinite parameter, namely α. For in both, priorities were founded upon the projectum α^* and the projection mapping t, each of which presupposes the existence of α. In ordinary recursion theory, this problem would never enter one's mind, since we always assume a fixed ω.

Q. Does there exist a construction establishing the α-Naming Theorem which is uniform in both τ and α?

Q. Does there exist a construction uniform in λ, μ, ν and τ establishing the α-Speedup Theorem?

The answers to these are probably both yes and the solutions should be along the lines of Shore's [28] uniform solution to Post's problem. Namely, introduce approximations to α^* together with a projection map by making use of parameterless Σ_1 Skolem functions throughout the construction.

Harrington [6] provides a technique for translating results in α-recursion theory into analogous ones within the theory of higher type objects (*Cf.* [12]). His basic stipulation is that constructions in α-recursion theory be uniform; that is, they should mention *no* infinite parameters. Therefore, uniform solutions to the above two questions would automatically generate analogues to the Naming and Speedup Theorems in higher type theory.

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