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AXIOMATICS FOR IMPLICATION

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This paper presents a basic axiomatic and two increments thereto, for propositional systems with implication as the sole functor. The basic axiomatic gives exactly the set of Modus Ponens formulae (defined below); addition of the first increment gives Positive Logic; and with addition of the second increment we reach the complete Classical Logic.
After some preliminaries in section 1, the axiomatics are presented in section 2. Section 3 establishes their properties.

1 *Preliminaries. Modus ponens formulae* Lower case Greek letters, with and without subscripts, are used for well-formed propositional formulae whose only functor is implication. Braces—'{' and '}'—form ordered sets of such formulae. '$\sim$' denotes a relationship between an ordered set of formulae and a single formula which is defined below.

Definition 1 $\{\alpha_1, \ldots, \alpha_n\}$ *closes* β (written $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$) is defined inductively in two steps.

I. Let there be some α_i $(1 \leq i \leq n)$ such that $\alpha_i = \beta$: then $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$.

II. Let there be some γ such that $\{\alpha_1, \ldots, \alpha_n\} \sim C\gamma\beta$ and $\{\alpha_1, \ldots, \alpha_n\} \sim \gamma$: then $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$.

Definition 2 $C\alpha_1 \ldots C\alpha_n\beta$ $(n \geq 1)$ is a *Modus Ponens formula* iff β is elementary and $\{\alpha_1, \ldots, \alpha_n\} \sim \beta$.

Examples of Modus Ponens formulae are: *Cpp, CpCqp, CCpCqrCCpqCpr*. Formulae which are not Modus Ponens formulae are: *CCCpqrCqr*, *CCCprsCCCqprs*.

2 *Axiomatics* The three axiomatic systems are based on a single axiom, and—including substitution—six inference rules. Axiom and rules are as follows.

Axiom. *Cpp*

Rule 1. Where $[x/\beta]\alpha$ is the result of replacing every occurrence of the variable x in α by β. $\alpha \vdash [x/\beta]\alpha$

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Rule 2. $C\alpha_1 \ldots C\alpha_n C\beta\gamma \vdash C\alpha_1 \ldots C\beta C\alpha_n\gamma$ $(n \geq 1)$
Rule 3. $\alpha \vdash C\beta\alpha$
Rule 4. $C\alpha C\beta\gamma \vdash C\alpha CC\alpha\beta\gamma$
Rule 5. $C\alpha\beta, \alpha \vdash \beta$
Rule 6. $C\alpha\beta \vdash CCC\alpha\gamma\alpha\beta$

System **1** is defined by the axiom and rules 1 through 4.
System **2** is obtained from system **1** by the addition of rule 5.
System **3** is obtained from system **2** by the addition of rule 6.

3 *Properties of the systems* The following theorems establish the properties of the three systems.

Theorem 1 *α is a thesis of System* **1** *iff α is a Modus Ponens formula.*

Proof: (1) All theses of system **1** are Modus Ponens formulae: Cpp is a Modus Ponens formula, and this class of formulae is closed under the operations in rules 1 through 4.

(2) Every Modus Ponens formula is a thesis of system **1**:
Let $C\alpha_1 \ldots C\alpha_n\beta$ be a Modus Ponens formula, then either there is some α_i $(1 \leq i \leq n)$ such that $\alpha_i = \beta$, or there is some $\alpha_i = C\gamma_1 \ldots C\gamma_m\beta$, where for every γ_j $(1 \leq j \leq m)$, $\{\alpha_1, \ldots, \alpha_n\} \sim \gamma_j$. In the former case we can use rule 1 to obtain $C\beta\beta$ from the axiom, and then by means of rules 3 and 2 insert and order all the remaining α-antecedents. In the latter case we obtain $CC\gamma_m\beta C\gamma_m\beta$ from the axiom, and then for each γ_j $(1 \leq j < m)$ in decreasing succession, using rules 3 and 4 when there is no $\gamma_k (k > j)$ such that $\gamma_j = \gamma_k$ and rules 2 and 4 otherwise, we obtain $C\gamma_j CC\gamma_j C\gamma_{j+1} \ldots C\gamma_m\beta C\delta_1 \ldots C\delta_l\beta$ where every γ_p $(j + 1 \leq p \leq m)$ is identical with some δ_q $(1 \leq q \leq l)$ and there is no δ_r $(1 \leq r \leq l$ and $q \neq r)$ such that $\delta_q = \delta_r$. From this, rule 2 yields $CC\gamma_j C\gamma_{j+1} \ldots C\gamma_m\beta C\gamma_j C\delta_1 \ldots C\delta_l\beta$. When $j = 1$ we will have $CC\gamma_1 \ldots C\gamma_m\beta C\delta_1 \ldots C\delta_l\beta$ where every γ_p $(1 \leq p \leq m)$ is identical to some δ_q $(1 \leq q \leq l)$ and there is no δ_r $(1 \leq r \leq l$ and $q \neq r)$ such that $\delta_q = \delta_r$. From the fact that for every γ_j, $\{\alpha_1, \ldots, \alpha_n\} \sim \gamma_j$, it follows that for every δ_q, either there is some $\alpha_i = \delta_q$ or there is some $\alpha_i = C\varepsilon_1 \ldots C\varepsilon_s\delta_q$ where for every ε_k $(1 \leq k \leq s)$, $\{\alpha_1, \ldots, \alpha_n\} \sim \varepsilon_k$. In the former case no action is required. In the latter case we proceed with each ε_k exactly as we proceeded with each γ_j $(1 \leq j < m)$. When for every δ_q every ε_k has been dealt with, the number of α-antecedents in our formula will have increased by $t \geq 0$. The reasoning applied to δ_q applies to any such new antecedent ξ, and we can use on every ξ the procedure we used for each δ_q. Since $C\alpha_1 \ldots C\alpha_n\beta$ is finite, the process must eventually terminate, and we can use rule 3 to insert any remaining α-antecedents, and rule 2 to order the formula.

Theorem 2 *α is a thesis of system* **2** *iff α is a thesis of Positive Logic.*

Proof: (1) All theses of system **2** are theses of Positive Logic: Cpp is a thesis of Positive Logic, and rules 1 through 5 are valid Positive Logic rules.

(2) Every thesis of Positive Logic is a thesis of system **2**:

Both *CpCqp* and *CCpCqrCCpqCpr* are Modus Ponens formulae and, a fortiori, theses of system **2**. These two formulae with rules 1 and 5 constitute a known Positive Logic base.

Theorem 3 *α is a thesis of system* **3** *iff α is a thesis of Classical Implicational Logic.*

Proof: (1) All theses of system **3** are theses of Classical Logic: *Cpp* is a thesis of Classical Logic, and rules 1 through 6 are valid rules in that system.

(2) All theses of Classical Implicational Logic are theses of system **3**: From the axiom we can obtain *Cpp* which by rule 6 gives *CCCpqpp*. This thesis and the two Modus Ponens formulae *CCpqCCqrCpr* and *CpCqp* together with rules 1 and 5 constitute the Tarski-Bernays base for Classical Implicational Logic.

It is perhaps worth pointing out that the proof of *CCCpqpp* just given illustrates a general mechanism for replacing an axiom $C\alpha\beta$ by the axiom *Cpp* and the rule $C\beta\gamma \vdash C\alpha\gamma$, in any system where Modus Ponens holds, and both *Cpp* and *CCpqCCqrCpr* are theses.

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