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A GENERAL PROPOSITIONAL LOGIC OF CONDITIONALS

SCOTT K. LEHMANN

Many English conditionals are not truth-functional. It follows that the theory of logical consequence embodied in truth-functional formal logic has limited application to ordinary English arguments. For example, it cannot even explain the validity of modus ponens (**MP**) or the hypothetical syllogism (**HS**) when the conditionals involved are, say, subjunctive. A theory of wider application is obviously desirable. I propose here to develop a very general propositional theory of this sort.

Non-truth-functional accounts of the conditional are not of course new. Perhaps the best known are Lewis' systems of strict implication [1]. However, it would appear that Lewis' theories have even less to do with ordinary English than does truth-functional logic, for there are hardly any English 'if p, then q's that mean "'p' implies 'q'". Nonetheless a variant of this implicational account seems to me correct (perhaps in virtue of its imprecision). Someone asserting 'if p, then q' generally makes tacit appeal to some set of conditions which, together with the truth of 'p', would yield the truth of 'q'. If these tacit conditions are expressed by a set Γ of sentences and 'yield' is taken to mean 'implies', we obtain the following semantics for the conditional: 'if p, then q' is true iff 'q' is a consequence of $\Gamma \cup \{`p'\}$ and the sentences of Γ are true. In section 1 a formal propositional logic of conditionals **C** is developed from this semantics. **C**'s Gentzen-style proof apparatus is shown to be complete in section 2. Section 3 concludes the paper with some metalogical remarks.

1 *Syntax and Semantics of* **C** The symbols of **C** are

$$- \quad \vee \quad \rightarrow \quad , \quad (\quad)$$

plus an infinite decidable set of symbols distinct from these which shall be called 'sentence letters' but not further specified. An *expression* of **C** is any finite array of symbols of **C**. *Sentences* of **C** are defined through the following sequence of clauses:

1. Sentence letters are sentences of level 0.
2. If σ is a sentence of level k, $\ulcorner -\sigma \urcorner$ is a sentence of level k.

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3. If σ_i is a sentence of level k_i, $\ulcorner(\sigma_1 \vee \sigma_2)\urcorner$ is a sentence of level $\max(k_1,k_2)$.
4. The empty expression is an empty sentence-sequence of level 0.
5. A sentence of level k is a non-empty sentence-sequence of level k.
6. If Γ is a non-empty sentence-sequence of level k and σ is a sentence of level j, $\ulcorner\Gamma,\sigma\urcorner$ is a non-empty sentence sequence of level $\max(k,j)$.
7. If σ_i is a sentence of level k_i and Γ is a sentence-sequence of level $j \leq \max(k_1,k_2)$, $\ulcorner(\sigma_1 \rightarrow_\Gamma \sigma_2)\urcorner$ is a sentence of level $\max(k_1,k_2) + 1$.
8. A sentence of level k is a sentence.

Henceforth I shall use 'σ' with or without subscripts as a syntactical variable ranging over sentences of **C**. To avoid proliferating notation I shall use 'Γ' sometimes to refer to a sentence-sequence of appropriate level and sometimes to refer to the set of sentences derived in the obvious way from such a sentence-sequence.

An *interpretation* I of **C** is an assignment of truth values to its sentence letters. The *value* of σ under an interpretation I of **C** is defined as follows:

1. If σ is a sentence letter, its value under I is the truth value assigned it by I.
2. If $\sigma = \ulcorner(\sigma_1 \rightarrow_\Gamma \sigma_2)\urcorner$, the value of σ under I is **T** iff (a) the value under I of each sentence of Γ is **T** and (b) there is no interpretation of **C** under which the value of each sentence of Γ is **T**, the value of σ_1 is **T**, and the value of σ_2 is **F**.
3. If $\sigma = \ulcorner(\sigma_1 \vee \sigma_2)\urcorner$ or $\ulcorner -\sigma_1\urcorner$, the value of σ under I is defined from the values of σ_1 and σ_2 under I in the usual way.

The notions of satisfaction and consequence are now defined as usual. (2) just above can then be expressed: the value of $\ulcorner(\sigma_1 \rightarrow_\Gamma \sigma_2)\urcorner$ under I is **T** iff I satisfies Γ and σ_2 is a consequence of $\Gamma \cup \{\sigma_1\}$.

The most convenient development of a formal inference apparatus for **C** utilizes rules and axioms based upon sequents. Accordingly the symbols '$\vdash$' and '$\dashv$' are added to the vocabulary of **C** and the required syntactical categories defined as follows:

1. A sentence-sequence of level k is a sentence-sequence.
2. If Δ_1 and Δ_2 are sentence-sequences, $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$ and $\ulcorner\Delta_1 \dashv \Delta_2\urcorner$ are sequents.

Henceforth 'Δ_1' and 'Δ_2' will be used like 'Γ'. $\ulcorner\Delta_1,\sigma \vdash \Delta_2\urcorner$ shall be understood to be $\ulcorner\sigma \vdash \Delta_2\urcorner$ when Δ_1 is empty, and so forth. The only semantic notion to be defined for sequents is validity: $\ulcorner\Delta_1 \dashv \Delta_2\urcorner$ is *valid* iff $\Delta_1 \cup \{\ulcorner -\sigma\urcorner | \sigma \in \Delta_2\}$ is satisfiable; $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$ is *valid* iff $\ulcorner\Delta_1 \dashv \Delta_2\urcorner$ is not valid. The *axioms* of **C** are the sequents $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$, where $\Delta_1 \cap \Delta_2 \neq \emptyset$, and the sequents $\ulcorner\Delta_1 \dashv \Delta_2\urcorner$, where $\Delta_1 \cap \Delta_2 = \emptyset$ and the Δ_i are empty or consist only of sentence letters.

The *rules of inference* of **C** are the following:

	($\vdash$ - rules)	($\dashv$ - rules)
1.	$\dfrac{\ulcorner\Delta_1 \vdash \sigma, \Delta_2\urcorner}{\ulcorner\Delta_1, -\sigma \vdash \Delta_2\urcorner}$	$\dfrac{\ulcorner\Delta_1 \dashv \sigma, \Delta_2\urcorner}{\ulcorner\Delta_1, -\sigma \dashv \Delta_2\urcorner}$
2.	$\dfrac{\ulcorner\Delta_1, \sigma \vdash \Delta_2\urcorner}{\ulcorner\Delta_1 \vdash -\sigma, \Delta_2\urcorner}$	$\dfrac{\ulcorner\Delta_1, \sigma \dashv \Delta_2\urcorner}{\ulcorner\Delta_1 \dashv -\sigma, \Delta_2\urcorner}$
3.	$\dfrac{\ulcorner\Delta_1, \sigma_1 \vdash \Delta_2\urcorner \quad \ulcorner\Delta_1, \sigma_2 \vdash \Delta_2\urcorner}{\ulcorner\Delta_1, (\sigma_1 \vee \sigma_2) \vdash \Delta_2\urcorner}$	$\dfrac{\ulcorner\Delta_1, \sigma_1 \dashv \Delta_2\urcorner}{\ulcorner\Delta_1, (\sigma_1 \vee \sigma_2) \dashv \Delta_2\urcorner}$ and $\dfrac{\ulcorner\Delta_1, \sigma_2 \dashv \Delta_2\urcorner}{\ulcorner\Delta_1, (\sigma_1 \vee \sigma_2) \dashv \Delta_2\urcorner}$
4.	$\dfrac{\ulcorner\Delta_1 \vdash \sigma_1, \sigma_2, \Delta_2\urcorner}{\ulcorner\Delta_1 \vdash (\sigma_1 \vee \sigma_2), \Delta_2\urcorner}$	$\dfrac{\ulcorner\Delta_1 \dashv \sigma_1, \sigma_2, \Delta_2\urcorner}{\ulcorner\Delta_1 \dashv (\sigma_1 \vee \sigma_2), \Delta_2\urcorner}$
5.	$\dfrac{\ulcorner\Gamma, \sigma_1 \dashv \sigma_2\urcorner}{\ulcorner\Delta_1, (\sigma_1 \rightarrow_\Gamma \sigma_2) \vdash \Delta_2\urcorner}$ and $\dfrac{\ulcorner\Gamma, \Delta_1 \vdash \Delta_2\urcorner}{\ulcorner\Delta_1, (\sigma_1 \rightarrow_\Gamma \sigma_2) \vdash \Delta_2\urcorner}$	$\dfrac{\ulcorner\Gamma, \sigma_1 \vdash \sigma_2\urcorner \quad \ulcorner\Gamma, \Delta_1 \dashv \Delta_2\urcorner}{\ulcorner\Delta_1, (\sigma_1 \rightarrow_\Gamma \sigma_2) \dashv \Delta_2\urcorner}$
6.	$\dfrac{\ulcorner\Delta_1 \vdash (\mathscr{C}\Gamma), \Delta_2\urcorner \quad \ulcorner\Gamma, \sigma_1 \vdash \sigma_2\urcorner}{\ulcorner\Delta_1 \vdash (\sigma_1 \rightarrow_\Gamma \sigma_2), \Delta_2\urcorner}$	$\dfrac{\ulcorner\Delta_1 \dashv (\mathscr{C}\Gamma), \Delta_2\urcorner}{\ulcorner\Delta_1 \dashv (\sigma_1 \rightarrow_\Gamma \sigma_2), \Delta_2\urcorner}$ and $\dfrac{\ulcorner\Delta_1 \dashv \Delta_2\urcorner \quad \ulcorner\Gamma, \sigma_1 \dashv \sigma_2\urcorner}{\ulcorner\Delta_1 \dashv (\sigma_1 \rightarrow_\Gamma \sigma_2), \Delta_2\urcorner}$

where $(\mathscr{C}\Gamma)$ is the conjunction of the sentences of Γ.

In addition, there are the usual right and left rearrangement rules (for $\vdash$ and $\dashv$) and, somewhat anomalously, the $\vdash$-rule for expansion on the right. *Proofs* may now be defined in the usual way as tree structures constructed from the axioms by the rules of inference.

2 *Completeness* It is easy to verify that the axioms are valid and that the rules of inference preserve validity. Hence any provable sequent is valid. To prove the converse I show how to construct from an unprovable sequent a "tree" of sequents containing a substructure whose "axioms" are non-valid sequents and whose "rules of inference" preserve non-validity. The individual steps in the construction are effected through *construction rules* corresponding to applications in reverse of the rules of inference (modulo rearrangement and expansion).

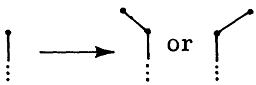
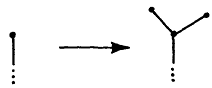
In the classical case the "tree" so constructed misses being a proof by containing at least one branch terminating in a sequent $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$, where $\Delta_1 \cap \Delta_2 = \emptyset$ and the Δ_i are empty or consist only of sentence letters. Here the situation is slightly complicated by there being in some cases (the $\dashv$-rules (3) and (6), and the $\vdash$-rules (5)) *two* inference rules for the introduction of a compound. Consequently the constructed "tree" will not correspond so nicely to a failed proof, and the role of branches is assumed by more complicated branching structures, which I shall call 'fans'.

The construction rules will be written like the inference rules, with the understanding that they are to be applied *upward*. A rule is defined at $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$ or $\ulcorner\Delta_1 \dashv \Delta_2\urcorner$ only when $\Delta_1 \cap \Delta_2 = \emptyset$. The "eliminated sentence" is the leftmost of its form on the indicated side. In each case I explain how a fan is extended when a rule is applied to a sequent in which one of its branches terminates.

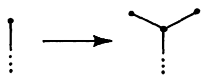
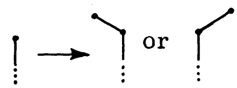
The construction rules (1′), (2′), and (4′) corresponding to the inference rules (1), (2), and (4) respectively are written just like (1), (2), and (4) respectively. A fan is thereby extended in the obvious way.

($\vdash$-rules) ($\dashv$-rules)

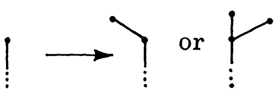
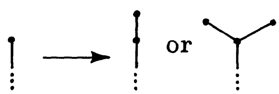
3′. $$\frac{\ulcorner\Delta_1,\sigma_1 \vdash \Delta_2\urcorner \quad \ulcorner\Delta_1,\sigma_2 \vdash \Delta_2\urcorner}{\ulcorner\Delta_1,(\sigma_1 \vee \sigma_2) \vdash \Delta_2\urcorner} \qquad \frac{\ulcorner\Delta_1,\sigma_1 \dashv \Delta_2\urcorner \quad \ulcorner\Delta_1,\sigma_2 \dashv \Delta_2\urcorner}{\ulcorner\Delta_1,(\sigma_1 \vee \sigma_2) \dashv \Delta_2\urcorner}$$

Fan:  Fan: 

5′. $$\frac{\ulcorner\Gamma,\sigma_1 \dashv \sigma_2\urcorner \quad \ulcorner\Delta_1,\Gamma \vdash \Delta_2\urcorner}{\ulcorner\Delta_1,(\sigma_1 \rightarrow_\Gamma \sigma_2) \vdash \Delta_2\urcorner} \qquad \frac{\ulcorner\Gamma,\sigma_1 \vdash \sigma_2\urcorner \quad \ulcorner\Delta_1,\Gamma \dashv \Delta_2\urcorner}{\ulcorner\Delta_1,(\sigma_1 \rightarrow_\Gamma \sigma_2) \dashv \Delta_2\urcorner}$$

Fan:  Fan: 

6′. $$\frac{\ulcorner\Delta_1 \vdash (\mathcal{E}\Gamma),\Delta_2\urcorner \quad \ulcorner\Delta_1 \vdash \Delta_2\urcorner \quad \ulcorner\Gamma,\sigma_1 \vdash \sigma_2\urcorner}{\ulcorner\Delta_1 \vdash (\sigma_1 \rightarrow_\Gamma \sigma_2),\Delta_2\urcorner} \qquad \frac{\ulcorner\Delta_1 \dashv (\mathcal{E}\Gamma),\Delta_2\urcorner \quad \ulcorner\Delta_1 \dashv \Delta_2\urcorner \quad \ulcorner\Gamma,\sigma_1 \dashv \sigma_2\urcorner}{\ulcorner\Delta_1 \dashv (\sigma_1 \rightarrow_\Gamma \sigma_2),\Delta_2\urcorner}$$

Fan:  Fan: 

The construction procedure is simply specified: work up from the given sequent S as far as possible, applying the construction rules in some fixed order to, say, leftmost branches. This process will obviously terminate after a finite number of steps, having produced a tree T whose branches terminate either in axioms (such branches are *closed*), or in sequents $\ulcorner\Delta_1 \dashv \Delta_2\urcorner$ with $\Delta_1 \cap \Delta_2 \neq \emptyset$ or $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$ with $\Delta_1 \cap \Delta_2 = \emptyset$ and the Δ_i empty or consisting only of sentence letters (such branches are *open*). T will contain as substructures fans generated from S in the manner indicated. A fan is *open* if all of its branches are open. Evidently each branch of an open fan in T terminates in a non-valid sequent. Furthermore, it is clear that the fan structure preserves non-validity downward. For example, consider the $\vdash$-rule (6′). If $\ulcorner\Delta_1 \vdash (\mathcal{E}\Gamma),\Delta_2\urcorner$ is not valid, neither is $\ulcorner\Delta_1 \vdash (\sigma_1 \rightarrow_\Gamma \sigma_2),\Delta_2\urcorner$ by one of the $\dashv$-rules (6). If $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$ and $\ulcorner\Gamma,\sigma_1 \vdash \sigma_2\urcorner$ are not valid, neither is $\ulcorner\Delta_1 \vdash (\sigma_1 \rightarrow_\Gamma \sigma_2),\Delta_2\urcorner$ by the other $\dashv$-rule (6). Thus it need only be established that if S is not provable, T contains an open fan. This amounts to showing that non-provability is preserved *upward* in some fan, which is also assured by the fan structure. For example, consider the $\vdash$-rule (6′). If $\ulcorner\Delta_1 \vdash (\sigma_1 \rightarrow_\Gamma \sigma_2),\Delta_2\urcorner$ is not provable, neither is $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$ by the expansion rule. Furthermore, by the $\vdash$-rule (6) either $\ulcorner\Delta_1 \vdash (\mathcal{E}\Gamma),\Delta_2\urcorner$ is not provable or $\ulcorner\Gamma,\sigma_1 \vdash \sigma_2\urcorner$ is not provable. Thus either $\ulcorner\Delta_1 \vdash (\mathcal{E}\Gamma),\Delta_2\urcorner$ is not provable or neither $\ulcorner\Delta_1 \vdash \Delta_2\urcorner$ nor $\ulcorner\Gamma,\sigma_1 \vdash \sigma_2\urcorner$ are provable. This completes the proof of completeness and therewith the formal discussion of **C**.

3 *Metalogical Remarks* While **C** may have some theoretical interest in providing a general picture of the logical behavior of conditionals, its applicability to particular English arguments is quite limited:

1. In most cases we have only a very general idea to what conditions Γ someone who asserts 'if *p*, then *q*' makes tacit appeal.
2. Furthermore, in some cases (e.g., **MP** and **HS**) a simple semantic argument establishes validity without any information whatever about Γ; however, this argument is not mirrored in a syntactic proof. Thus while the axioms and rules are in some sense the best possible, it is convenient to supplement them with other "admissible" rules. In particular:

$$\frac{\ulcorner\Delta_1,\Gamma\vdash\sigma_1,\Delta_2\urcorner \quad \ulcorner\Delta_1,\Gamma,\sigma_2\vdash\Delta_2\urcorner}{\ulcorner\Delta_1,(\sigma_1\rightarrow_\Gamma\sigma_2)\vdash\Delta_2\urcorner} \quad \text{and} \quad \frac{\ulcorner\mathsf{Cond}_\Gamma(\Delta_1),\Gamma,\sigma_1\vdash\sigma_2\urcorner}{\ulcorner\Delta_1\vdash(\sigma_1\rightarrow_\Gamma\sigma_2),\Delta_2\urcorner}$$

where $\mathsf{Cond}_\Gamma(\Delta_1)$ is the sentence-sequence of "Γ-conditionals" in Δ_1.
3. Appeal to infinitely many conditions cannot be represented in **C** as developed.
4. **C** is only a propositional, not a quantificational system.

In fact, from (3) and (4) it is hard to see that **C** can claim even to embody a correct *general* picture. Clearly the semantical side of **C** can be modified to remedy this: we need merely add the variables, quantifiers, and predicates of first-order logic together with an apparatus for forming notations for sets of formulae (e.g., Gödel numberings and arithmetic formulae); the truth definitions can easily be formulated. However, it will no longer be possible to characterize the concept of logical consequence so obtained syntactically as provability from axioms by rules of inference. We cannot really deal syntactically with infinite sets of formulae. Furthermore, the construction procedure of the completeness proof would here supply a decision procedure for (ordinary) logical consequence in violation of Church's Theorem (specifically, a ⊣-rule for ∃-introduction on the right cannot be formulated).

This limitation may possibly illuminate some remarks of Quine. Quine has characterized Lewis' systems of strict implication as founded on a confusion of use and mention [2]. Doubtless the same criticism would be made of the system **C**. Exactly what is its force? We are attempting to characterize the logical behavior of conditionals through a syntactic system of axioms and rules of inference. Is Quine maintaining that this cannot be done? Or is it that an "extensionally" correct description of this sort—one giving the right answers to questions of validity in all clear cases—may yet, through embodying use-mention "confusions", be rejected on "intensional" grounds? It is hard to see just what Quine intends, but the former claim is the clearer and, as we see, has the advantage of being clearly correct.

Mention has been made above of applying formal logic to natural language, but as usual little has been said of what this consists in. I shall conclude with some brief remarks on this matter. What is it to have a correct theory of logical consequence for (a portion of) some natural language? I take it that it is at least to possess a device for obtaining correct answers to questions of validity. We may want to require also that these answers follow from a theory of truth. The familiar devices are formal logical systems with (largely implicit) natural language/formal language translation manuals. The idea is to translate a natural-language argument sentence by sentence into a formal-language argument, apply the procedures of the formal system for establishing logical consequence to this formal-language argument, and claim that any answer forthcoming holds also for the natural-language argument. What supports this claim? Suppose for some reason the given natural-language argument is not obviously valid or obviously invalid; why should we accept an answer obtained by this method? Specifically, is there anything in the method that *guarantees* a correct answer, or are we reduced to citing the inductive evidence of certifiably correct answers supplied by it in the past? Unfortunately, as I shall argue, the "guarantees" are only inductive.

Let us assume that the formal methods give the right answers to questions about the validity of formal arguments, so that any problems lie with translation. Now a translation is in one sense merely a mapping of sentences onto sentences. One might hope to define this mapping in a purely syntactic way; in fact this looks like the only way to obtain an *effective* manual. But there will be nothing in such a syntactic definition that can guarantee right answers. The obvious remedy is to require that a sentence and its translation have the same truth conditions. Of course this is not likely to be very helpful in particular cases; often the problem is just that we do not know the truth conditions of natural-language sentences. Furthermore, although it is not very clear what a truth condition is and hence what their identity criteria are, it seems quite implausible that formal- and natural-language sentences generally have the *same* truth conditions. For example, speakers of English simply are not talking about sets when they ascribe a property to something; nor are they discussing possible worlds when they claim that somebody does not know something. What can be required is something weaker: that the truth conditions of the natural-language sentence and its formal-language translation *can be represented* in the same way, namely by the truth conditions of the formal-language sentence. But now how can one discover whether a translation is correct? To say that the truth conditions can be represented in the same way can here mean only that the consequence structures of the natural and formal languages are isomorphic under translation. But by supposition we do not know this. We may have good inductive grounds for believing it; in particular, the consequence structure of certain simple portions of the natural language may be shown to be the same as that of their images under translation. But there are no "guarantees".

I have referred to the "method" of answering questions about validity afforded by formal logical systems, but it is clear that there really is no method without an explicit and effective translation manual (and equally clear that no such manual exists at present for any large portion of any natural language). It is sometimes said [3] that a formal logical system (say, a formal first-order theory) embodies a logical *analysis* of ordinary language (say, English). This is true in the sense that the semantics of formal first-order languages establishes via the "intended" readings "standard" meanings for what Quine [4] calls the 'logical particles' of English and "standard" patterns of semantic behavior for certain categories of terms (e.g., predicates). These "standard" meanings and patterns are fixed insofar as the meanings of the words in the statement of the truth conditions of formal-language sentences are fixed. The problem of applying formal first-order methods to English arguments then reduces to that of *expressing* ordinary English in "standard" English. But of course the problem remains: the theory tells us what to look for, how to attempt to construe ordinary English sentences; it does not tell us how to succeed.

It should now be clear that the conclusion expressed at the outset of this paper concerning the limitations of truth-functional logic was premature. For it presupposed that one must establish that a natural-language sentence and its formal-language translation have essentially the same truth conditions before one can confidently employ any formal methods to answer questions about validity. We have seen, however, that the basis of confidence in formal methods must be the inductive evidence of their past success. Truth-functional methods certifiably give the right answers in many cases (e.g., **MP** and **HS** for any conditionals whatever) where they have no "basis" for doing so. Their failures (e.g., the paradoxes of material implication) appear to be well-known and avoidable. In short, their inductive support is good. I certainly could not recommend replacing classical propositional systems with the system **C**.

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The University of Connecticut
Storres, Connecticut