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Recursively Saturated ω_1-like Models of Arithmetic

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Two models of PA are called *similar* if they are elementarily equivalent and have the same standard systems. In [3] it was shown than any two recursively saturated ω_1-like similar models are $L_{\infty\omega_1}$-equivalent, and that there is at least a continuum of pairwise nonisomorphic, recursively saturated, ω_1-like models which are similar to a given countable recursively saturated model of PA. In this paper we show that the number of models with the above properties is in fact $2^{\aleph_1}$, and we may also construct them to be mutually not elementarily embeddable.

Thus, it is natural to ask in what extensions of $L_{\omega\omega}$ it is possible to describe recursively saturated ω_1-like models up to isomorphism. Since we have $2^{\aleph_1}$ pairwise nonisomorphic models, countable languages are out of the consideration (at least when $2^{\aleph_0} < 2^{\aleph_1}$). This applies in particular to the stationary logic $L(aa)$. In Section 3 we take a look at finitely determinate structures, which were studied by Eklof and Mekler in [1]. The reason is that the proof of our theorem on the existence of $2^{\aleph_1}$ pairwise nonisomorphic models (Theorem 2.4) does not exclude the possibility that a stationary logic version of the isomorphism theorem is true for finitely determinate models. Theorem 3.5 shows that this is not the case. We still may have $2^{\aleph_1}$ pairwise nonisomorphic, recursively saturated, ω_1-like finitely determinate models which have the same standard systems and satisfy the same $L(aa)$ theories. Moreover, from a lemma due to Shelah, it follows that the models constructed are also $L_{\infty\omega_1}(aa)$-equivalent. The proof of Theorem 3.5 uses the $\diamond$ principle and the existence of Kurepa trees.

No knowledge of stationary logic, except for the Eklof-Mekler characterization of $L(aa)$-equivalence for finitely determinate structures and/or the Shelah lemma, is needed for our considerations. In fact, all of our results about

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ω_1-like models are just suitable translations of facts about countable, recursively saturated, models. The main role is played by the structures of the form (M, M_0), where M is countable recursively saturated model, $M_0 \prec_e M$, and M_0 is the union of intervals of the form $[0, a_n]$ for some special sequence $\{a_n\}_{n\in\omega}$ coded in M, called an *ascending sequence of skies.*

Structures of the above form are not recursively saturated but have properties surprisingly similar to those of recursively saturated structures (M, M_0) with $M_0 \prec_e M$.

In Section 3 we give a construction of finitely determinate, recursively saturated, ω_1-like models. The construction uses pairs of models of the first form described above. We have failed to use recursively saturated pairs for a similar kind of construction. We comment on this and state an open problem at the end of Section 3.

Section 4 was added later, after a conversation with Shelah.

1 Preliminaries We will use the terminology and notation of [7] and [9]. We also refer to these papers for all the notation and results not explained here.

Letters M and N with various subscripts will denote nonstandard models of PA, and we call them "models" for short.

As usual we choose one of the standard ways of coding finite sets and sequences in PA. If a is an element of M, then D_a is the set of elements of M coded by a, $(a)_i$ is the i-th term of the sequence coded by a, and $lh\, a$ is the length of this sequence.

If f is a function and A is a subset of the domain of f, then the image of A under f is denoted by $f * A$, and the restriction of f to A is denoted by $f \restriction A$.

End extensions and elementary end extensions, denoted respectively by $\subseteq_e$ and $\prec_e$, are always understood to be proper.

Many of the results we mention can be stated in stronger forms. However we have chosen the level of generality just appropriate for our purposes. This applies in particular to the following *basic isomorphism theorem* (cf. [7]). Recall that models are called *similar* if they are elementarily equivalent and have the same standard systems.

Theorem 1.1 *Any two countable, recursively saturated, similar models are isomorphic.*

The next theorem was stated first in [9]. It is also an easy corollary of the results of [3] which we mention in Section 3.

Theorem 1.2 (Smoryński) *If M and N are countable, recursively saturated, models and $I \subseteq_e M \prec_e N$ then there is an isomorphism of M with N which is the identity on I, and $(M, I) \prec (N, I)$.*

A model M is said to be ω_1*-like* if it is of power $\aleph_1$ but every initial segment of it is countable. Since every recursively saturated model is the union of a chain of recursively saturated countable elementary submodels, we have the following corollary.

Corollary 1.3 *If M is a recursively saturated ω_1-like model, M_0 is countable and recursively saturated, and $I \subseteq_e M_0 \prec_e M$, then $(M_0, I) \prec (M, I)$.*

Now let us mention the result of Eklof and Mekler characterizing $L(aa)$-equivalence of models of power $\aleph_1$ (of any first-order theory).

If M is of power $\aleph_1$, then every continuous chain of countable submodels of M whose union is M is called an *ω_1-filtration of M.* The following is a characterization of finitely determinate structures from [1]. We shall treat it as a definition.

Definition 1.4 A model M of power $\aleph_1$ is called *finitely determinate* if there is an ω_1-filtration $\{M_\alpha\}_{\alpha\in\omega_1}$ of M such that for all $k, n, r \in \omega$, $k \leq n$, if $(\alpha_0, \ldots, \alpha_n)$, $(\beta_0, \ldots, \beta_n) \in [\omega_1]^{<\omega}$, where $\alpha_j = \beta_j$ for $j < k$, $a_0, \ldots, a_r \in M_{\alpha_k} \cap M_{\beta_k}$, then

$$(M, M_{\alpha_n}, \ldots, M_{\alpha_0}, a_0, \ldots, a_r) \equiv (M, M_{\beta_n}, \ldots, M_{\beta_0}, a_0, \ldots, a_r) .$$

Theorem 1.5 (Eklof, Mekler) *Let M and N be finitely determinate structures of power $\aleph_1$. Then M is $L(aa)$-equivalent to N if and only if there are ω_1-filtrations of M and N, respectively, such that for every $(\alpha_0, \ldots, \alpha_n) \in [\omega_1]^{<\omega}$ we have*

$$(M, M_{\alpha_n}, \ldots, M_{\alpha_0}) \equiv (N, N_{\alpha_n}, \ldots, N_{\alpha_0}) .$$

2 *Nonisomorphic models*

In this section we give a quite general method for constructing nonisomorphic, recursively saturated, ω_1-like models. In a special case, we may also show that the models constructed cannot be elementarily embedded in one another.

All the constructions of families of pairwise nonisomorphic models in this paper will be based on the following model theoretic result. Its usefulness in our context was pointed out to me by Wilfrid Hodges.

Theorem 2.1 *Let M and N be models of power $\aleph_1$ (of any first-order theory), and let $\{M_\alpha\}_{\alpha\in\omega_1}$, $\{N_\alpha\}_{\alpha\in\omega_1}$ be their ω_1-filtrations. If f is an isomorphism of M onto N, then the set $\{\alpha \in \omega_1 : f * M_\alpha = N_\alpha\}$ is closed and unbounded in ω_1.*

This result can be proved by a back and forth procedure and is true also for κ-filtrations of models of power κ for every regular cardinal κ.

As will be clear from the proof, to obtain $2^{\aleph_1}$ pairwise nonisomorphic, ω_1-like, recursively saturated models which are similar to a given countable recursively saturated model, it is enough to know that for every countable recursively saturated model M, there are recursively saturated M_0, M_1 such that $M_0 \prec_e M$, $M_1 \prec_e M$, and the structures (M, M_0) and (M, M_1) are not isomorphic. In fact the following much stronger result is true. We will also need this result in Section 3.

Theorem 2.2 (Smoryński [8]) *For every countable recursively saturated model M there is a continuum of elementarily inequivalent structures of the form (M, M_0), where M_0 is recursively saturated and $M_0 \prec_e M$.*

The following simple but useful lemma is a straightforward application of the basic isomorphism theorem.

Lemma 2.3 *Let M_0, N_0 be countable recursively saturated, similar models. Then for every M such that $M_0 \prec_e M$, there is an N such that $N_0 \prec_e N$ and $(M, M_0) \cong (N, N_0)$.*

The proof of our main result on nonisomorphic models is a combination of Theorem 2.1 and the fact that ω_1 is the union of $\aleph_1$ disjoint stationary sets. Let $\mathbf{C}$ be a family of power $\aleph_1$ consisting of disjoint stationary subsets of ω_1. We denote the power set of $\mathbf{C}$ by $\mathcal{P}(\mathbf{C})$.

Theorem 2.4 *For every countable recursively saturated model M, there are $2^{\aleph_1}$ recursively saturated, ω_1-like, pairwise nonisomorphic models which are similar to M.*

Proof: For every $\chi \in \mathcal{P}(\mathbf{C})$, we will construct a continuous chain of countable recursively saturated elementary end extensions of M. Let $M_0 \prec_e M$, $M_1 \prec_e M$ be recursively saturated and such that $(M, M_0) \ncong (M, M_1)$. Let $M_0(\chi) = M$, and for limit ordinals $\lambda \in \omega_1$, let $M_\lambda(\chi) = \bigcup_{\alpha<\lambda} M_\alpha(\chi)$. The successor step looks as follows. If $\alpha \in \bigcup \chi$, then we take $M_{\alpha+1}(\chi)$ such that $(M_{\alpha+1}(\chi), M_\alpha(\chi)) \cong (M, M_0)$. Otherwise, take $M_{\alpha+1}(\chi)$ such that $(M_{\alpha+1}(\chi), M_\alpha(\chi)) \cong (M, M_1)$. For $\chi \in P(\mathbf{C})$, let $M(\chi) = \bigcup_{\alpha\in\omega_1} M_\alpha(\chi)$. We claim that if $\chi_1, \chi_2 \in \mathcal{P}(\mathbf{C})$ and $\chi_1 \neq \chi_2$, then $M(\chi_1)$ and $M(\chi_2)$ are not isomorphic.

Suppose f: $M(\chi_1) \to M(\chi_2)$ is an isomorphism. Take a stationary set S on which χ_1 and χ_2 differ. By Theorem 2.1, there are $\alpha, \beta \in S$, $\alpha < \beta$, such that $f * M_\alpha(\chi_1) = M_\alpha(\chi_2)$ and $f * M_\beta(\chi_1) = M_\beta(\chi_2)$. Hence, we obtain that $(M_\beta(\chi_1), M_\alpha(\chi_1)) \cong (M_\beta(\chi_2), M_\alpha(\chi_2))$. However, it follows from Theorem 1.2 that for every χ and every $\alpha < \beta$, $(M_\beta(\chi), M_\alpha(\chi)) \cong (M_{\alpha+1}(\chi), M_\alpha(\chi))$, which gives a contradiction.

Now we shall give a construction of $2^{\aleph_1}$ models which are ω_1-like, recursively saturated, and similar, but which are not elementarily embeddable in one another. The construction will be an elaborate version of the one we have given for the proof of Theorem 2.4. We will need the following two results. The first is an obvious corollary of the main result of [4]. Let us say that a subset $X \subseteq M$ is *inductive* if the structure (M, X) satisfies the induction schema in the language of PA with an additional predicate for X. Let us also say that X, $Y \subseteq M$ are *elementarily equivalent* if $(M, X) \equiv (M, Y)$.

Theorem 2.5 *Every countable recursively saturated model M of PA possesses an uncountable family of inductive subsets of M which are pairwise elementarily inequivalent, and each of them can be coded in some countable, recursively saturated elementary end extension of M.*

Theorem 2.6 (Kotlarski [5], Schmerl [6]) *If $M, N \models PA$, X is an inductive subset of M, and N is a cofinal extension of M (written $M \prec_{cof} N$), then there exists $\bar{X} \subseteq N$ such that $(M, X) \prec (N, \bar{X})$.*

Theorem 2.7 *For every countable recursively saturated model M, there is a family of $2^{\aleph_1}$ recursively saturated, ω_1-like models which are similar to M, such that no one of them can be elementarily embedded into another.*

Proof: Let $\{X_\alpha\}_{\alpha\in\omega_1}$ be a family of subsets of M given by Theorem 2.5, and let $T_\alpha = Th(M, X_\alpha)$. Let $\mathbf{C}$ be as in the proof of Theorem 2.4. For every $\chi \in \mathcal{P}(\mathbf{C})$ we construct $M(\chi) = \bigcup_{\alpha\in\omega_1} M_\alpha(\chi)$ as follows. Let $M_0(\chi) = M$, and let N be any countable, elementary end extension of M. For limit λ, we put $M_\lambda(\chi) = \bigcup_{\alpha<\lambda} M_\alpha(\chi)$.

Successor step:

Case 1: $\alpha \notin \bigcup \chi$.
In this case, we take $M_{\alpha+1}(\chi)$ such that $(M_{\alpha+1}(\chi), M_\alpha(\chi)) \cong (N, M)$.

Case 2: $\alpha \in \bigcup \chi$.
For every $S \in \chi$, we fix an enumeration $\{\gamma_\beta^S\}_{\beta\in\omega_1}$ of S. We take $S \in \chi$ such that $\alpha \in S$, and β such that $\alpha = \gamma_\beta^S$. There exists $X \subseteq M_\alpha(\chi)$ such that $(M_\alpha(\chi), X) \models T_\beta$ and X can be coded in a countable, recursively saturated elementary end extension of $M_\alpha(\chi)$. We take one such extension as $M_{\alpha+1}(\chi)$. We claim that if for $\chi_1, \chi_2 \in \mathcal{P}(\mathbf{C})$ there exists S such that $S \in \chi_1$ and $S \notin \chi_2$, then $M(\chi_1)$ is not elementarily embeddable in $M(\chi_2)$.

Suppose that f is an embedding of $M(\chi_1)$ into $M(\chi_2)$. Then $\{\alpha \in \omega_1 : f * M_\alpha(\chi_1) \prec_{cof} M_\alpha(\chi_2)\}$ is closed and unbounded in ω_1. Hence, there is a set $A \subseteq S$ of power $\aleph_1$ such that for all $\alpha \in A$, $f * M_\alpha(\chi_1) \prec_{cof} M_\alpha(\chi_2)$.

Take $\alpha \in A$. We have $(f * M_\alpha(\chi_1), f * X) \models T_\beta$, where X is the set used in the construction of $M_{\alpha+1}(\chi_1)$. By Theorem 2.7, there is $\bar{X} \subseteq M_\alpha(\chi_2)$ such that $(M_\alpha(\chi_2), \bar{X}) \models T_\beta$ and $\bar{X}$ is coded in $M_{\alpha+1}(\chi_2)$ (to verify the second part of this last statement consult [5] or [6]). But, since $\alpha \notin \bigcup \chi_2$, $(M_{\alpha+1}(\chi_2), M_\alpha(\chi_2))$ is isomorphic to (N, M). The family of subsets of M coded in N is countable. In particular, we have only countable many complete theories of subsets of M coded in N. This gives a contradiction and finishes the proof.

3 Nonisomorphic finitely determinate $L(aa)$-equivalent models Theorem 2.2 says that for every countable recursively saturated model M, there is a continuum of distinct theories of structures of the form (M, M_0), where $M_0 \prec_e M$ and M_0 is recursively saturated. Using the basic isomorphism theorem, for every such a theory T, we can construct a continuous chain of recursively saturated models $\{M_\alpha(T)\}_{\alpha\in\omega_1}$, such that for every $\alpha \in \omega_1$, $(M_{\alpha+1}(T), M_\alpha(T)) \models T$. Let $M(T) = \bigcup_{\alpha\in\omega_1} M_\alpha(T)$. If $T_1 \neq T_2$, then clearly $M(T_1)$ and $M(T_2)$ cannot be $L(aa)$-equivalent. So for every countable recursively saturated model M, we obtain a continuum of recursively saturated ω_1-like models which are similar to M and hence are all $L_{\infty\omega_1}$-equivalent, but they are pairwise $L(aa)$-inequivalent.

Our next aim will be to show that (at least assuming $V = L$) there are $2^{\aleph_1}$ finitely determinate, recursively saturated, ω_1-like similar models which are pairwise nonisomorphic but $L(aa)$-equivalent.

First let us recall some terminology and facts from [9] and [3]. We say that $I \subseteq_e M$ *has cofinality* ω *in* M if there is an $a \in M$ which codes an increasing sequence of a nonstandard length and $I = \bigcup_{n\in\omega} [0, (a)_n]$. We say that an element

a of M codes an *ascending sequence of skies* if a codes an increasing sequence of a nonstandard length and for every $i < lh\, a$ and every function F definable without parameters in M, $M \models F((a)_i) < (a)_{i+1}$. If $a \in M$ codes an ascending sequence of skies, then we write $a \in ASS(M)$ and we let $M(\omega, a)$ denote $\bigcup_{n\in\omega} [0, (a)_n]$.

The following are two simple but fundamental facts.

Proposition 3.1 *Let M be recursively saturated.*
(i) *For every $b \in M$, there exists $a \in ASS(M)$ such that $(a)_0 > b$.*
(ii) *For every $a \in ASS(M)$, the model $M(\omega, a)$ is recursively saturated and $M(\omega, a) \prec_e M$.*

If $I \subseteq_e M$, $J \subseteq_e N$, and f is an isomorphism of I into J, then we say that it is an *(M, N)-isomorphism* if for every $A \subseteq I$ which is coded in M, $f * A$ is coded in N, and, conversely, for every subset $B \subseteq J$ which is coded in N, $f^{-1} * B$ is coded in M. Observe that if $M \subseteq_e M_1$ and $N \subseteq_e N_1$, then f is an (M, N)-isomorphism of I onto J if and only if it is an (M_1, N_1)-isomorphism of I onto J.

Also, it is easy to see that if g is an isomorphism of M onto N and $g * I = J$, then g restricted to I is an (M, N)-isomorphism of I onto J.

The next theorem, based on the results of [9], was crucial for the proof of the positive result of [3] mentioned in the introduction.

Theorem 3.2 *Let M and N be countable recursively saturated, similar models. Let $M_0 \prec_e M$, $N_0 \prec_e N$ have cofinality ω in M and N, respectively. Let $a \in ASS(M)$, $b \in ASS(N)$ be such that $M_0 < (a)_0$ and $N_0 < (b)_0$. Then every (M, N)-isomorphism of M_0 onto N_0 can be extended to an isomorphism of M onto N such that $f * M(\omega, a) = N(\omega, b)$.*

Corollary 3.3 *Let M and N be as above. For $n \in \omega$, let $a_n \in ASS(M)$, $b_n \in ASS(N)$ be such that $M(\omega, a_n) < (a_{n+1})_0$, $N(\omega, b_n) < (b_{n+1})_0$ and $M = \bigcup_{n\in\omega} M(\omega, a_n)$, $N = \bigcup_{n\in\omega} N(\omega, b_n)$. Then the structures $(M, \{M(\omega, a_n)\}_{n\in\omega})$, $(N, \{N(\omega, b_n)\}_{n\in\omega})$ are isomorphic.*

If M_0 is a countable recursively saturated model, then by the basic isomorphism theorem, there is a countable recursively saturated model M_1 such that $M_0 \prec_e M_1$, and for some $a_0 \in ASS(M_1)$, $M_0 = M_1(\omega, a_1)$. We may iterate this procedure ω_1 times, taking unions at limit stages. So after ω_1 steps, we obtain an ω_1-like, recursively saturated model $M = \bigcup_{\alpha\in\omega_1} M_\alpha$, such that for every $\alpha \in \omega_1$, there is an $a_\alpha \in ASS(M_{\alpha+1})$ such that $M_\alpha = M_{\alpha+1}(\omega, a_\alpha)$. Now by the (finite case of) Corollary 3.3 and Theorem 1.5, we have that all models of the above form are finitely determinate, and if they are similar, then they are also $L(aa)$-equivalent.

Next, we will show how to construct $2^{\aleph_1}$ such models which are mutually nonembeddable.

In the proof of Theorem 2.4 we constructed nonisomorphic similar models M and N by producing suitable filtrations $\{M_\alpha\}_{\alpha\in\omega_1}$, $\{N_\alpha\}_{\alpha\in\omega_1}$ such that for sufficiently many α, no isomorphism of M_α onto N_α could be extended to an isomorphism of $M_{\alpha+1}$ onto $N_{\alpha+1}$. This obviously cannot be done in the case of finitely determinate structures constructed from ascending sequences of skies. We cannot produce $M_{\alpha+1}$ and $N_{\alpha+1}$ to "block" all isomorphisms from M_α to N_α. But we can block some of them, and in the presence of the $\Diamond$ principle this is just enough.

The tool for blocking isomorphisms will be the following lemma.

Lemma 3.4 *Let M be a countable recursively saturated model, and let $\{X_n\}_{n\in\omega}$ be a family of subsets of M of order type ω which are cofinal in M. Then there is a recursively saturated model N such that for some $a \in ASS(N)$, $M = N(\omega, a)$ and none of the X_n's is coded in N.*

Proof: Let N be a recursively saturated countable model such that for some $a \in ASS(N)$, $M = N(\omega, a)$. It is enough to show that there is an automorphism f: $M \to M$ such that for every $n \in \omega$, $f * X_n$ is not coded in N. Assume that every set occurs infinitely many times in the sequence $\{X_n\}_{n\in\omega}$, and let $\{Y_n\}_{n\in\omega}$ be the family of subsets of M of order type ω which are coded in N.

We will construct an automorphism f by the usual back and forth procedure with every back and forth step followed by an adjustment described below.

Suppose we have a partial automorphism given by two finite sequences $\bar{a}$ and $\bar{b}$ such that for every formula ϕ, $M \models \phi(\bar{a}) \leftrightarrow \phi(\bar{b})$. Then take the first k such that Y_k has not been used in the construction yet. Since M is recursively saturated and X_k is unbounded in M, we can find an element x of X_k which is greater than all the elements definable from $\bar{a}$ in M. Now consider the following type:

$$t(\bar{b}, v) = \{\phi(\bar{b}, v) \colon M \models \phi(\bar{a}, x)\} \ .$$

It is not difficult to see that $t(\bar{b}, v)$ is realized arbitrarily high in M (see [9], Lemma 2.5). But also if M' is a recursively saturated model such that $\bar{b} \in M' \prec_e M$, then $t(\bar{b}, v)$ is realized arbitrarily high in M'. Now observe that the set $M' \cap \bigcup_{n\le k} Y_n$ is finite. Thus, we can find x' realizing $t(\bar{b}, v)$ in M' such that $x' \notin \bigcup_{n\le k} Y_n$, and we can prolong our partial automorphism by putting $f(x) = x'$.

If f is an automorphism constructed according to the above procedure then for every n, $k \in \omega$ we have $f * X_n \neq Y_k$, which finishes the proof.

We will use the following version of the $\Diamond$ principle (which is an easy consequence of the usual one). There is a sequence $\{f_\alpha \colon \alpha \in \omega_1\}$ of functions $f_\alpha \colon \alpha \to \alpha$ such that for every $f \colon \omega_1 \to \omega_1$ the set $\{\alpha \in \omega_1 \colon f \restriction \alpha = f_\alpha\}$ is stationary in ω_1.

A binary tree T is called an ω_1-*Kurepa* tree if T has at least $\aleph_2$ branches of length ω_1, and for every $\alpha \in \omega_1$, the set T_α of elements of T of rank α is countable. The existence of ω_1-Kurepa trees follows from $V = L$ and is independent from *ZFC*. (See [2], Theorem 55).

Theorem 3.5 ($V = L$)[1] *For every countable recursively saturated model M_0 there is a family $\mathfrak{A}$ of finitely determinate, recursively saturated, ω_1-like models similar to M_0, such that $\mathfrak{A}$ has cardinality $\aleph_2$, and for all distinct elements M and N of $\mathfrak{A}$, M is $L(aa)$-equivalent to N but M cannot be elementarily embedded into N.*

Proof: For the sake of clarity, let us construct two mutually nonembeddable $L(aa)$-equivalent models first. Let $h: \omega_1 \to \omega_1$ be a continuous enumeration of the limit ordinal numbers smaller than ω_1. We will construct two chains of models $\{M_\alpha\}_{\alpha\in\omega_1}$, $\{N_\alpha\}_{\alpha\in\omega_1}$, where $M_0 = N_0$ is a given countable recursively saturated model and for every α the ordinal $h(\alpha)$ is the universe of M_α and N_α. If λ is a limit ordinal, then, as usual, $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ and $N_\lambda = \bigcup_{\alpha<\lambda} N_\lambda$. For the successor step assume that we already have M_α and N_α recursively saturated models with universes $h(\alpha)$. Let $F = f_{h(\alpha)}$, given by the $\Diamond$ principle. If F is neither a cofinal embedding of M_α into N_α nor a cofinal embedding of N_α into M_α, then we simply take $M_{\alpha+1}$ and $N_{\alpha+1}$ to be any countable recursively saturated models such that for some $a_\alpha \in ASS(M_{\alpha+1})$, $b_\alpha \in ASS(N_{\alpha+1})$, we have $M_\alpha = M_{\alpha+1}(\omega, a_\alpha)$ and $N_\alpha = N_{\alpha+1}(\omega, b_\alpha)$. If F happens to be a cofinal elementary embedding of M_α into N_α or N_α into M_α or both, then we proceed as follows (assume the third and worst possibility). First we take any $M_{\alpha+1}$ as above, and then we produce $N_{\alpha+1}$ such that for some $b_\alpha \in ASS(N_{\alpha+1})$, $N_\alpha = N_{\alpha+1}(\omega, b_\alpha)$ and none of the following subsets of N_α is coded in $N_{\alpha+1}$:

1. $F * X_\alpha$ for some $X_\alpha \subseteq M_\alpha$ of order type ω coded in $M_{\alpha+1}$
2. $F^{-1} * Y$ for all $Y \subseteq F * N_\alpha$ of order type ω coded in $\mathrm{M}_{\alpha+1}$.

So $N_{\alpha+1}$ does not code the image under F of a certain subset of M_α coded in $M_{\alpha+1}$. Also, the image under F of a certain subset of N_α coded in $N_{\alpha+1}$ is not coded in $M_{\alpha+1}$.

Let $M = \bigcup_{\alpha\in\omega_1} M_\alpha$, $N = \bigcup_{\alpha\in\omega_1} N_\alpha$. Suppose that $f: M \to N$ is an elementary embedding. Then for some $\alpha \in \omega_1$, we have $f \restriction M_\alpha = f_{h(\alpha)}$ and $f * M_\alpha \prec_{cof} N_\alpha$, where $\prec_{cof}$ means cofinal extension.

Now it is easy to verify that the image under f of any subset of M_α which has order type ω and is coded in $M_{\alpha+1}$, must be coded in $N_{\alpha+1}$. In particular, $f_{h(\alpha)} * X_\alpha$ is coded in $N_{\alpha+1}$, which is impossible by the construction. Similarly, we show that N cannot be elementary embedded in M.

Now we will describe a way of constructing $\aleph_2$ models with the above properties. Let T be an ω_1-Kurepa tree. For every $s \in T$ we will construct a recursively saturated model M_s by induction on the rank of s in T. Let $M_\phi = M_0$ and for s whose rank is a limit ordinal, let $M_s = \bigcup_{s'\subseteq s} M_{s'}$. To describe the successor step let us assume that for every $s \in T_\alpha$, we have a recursively saturated model M_s with the universe $h(\alpha)$. Let $\{M_n\}_{n\in\omega}$ be an enumeration of $\{M_s: s \in T_\alpha\}$. For every $n \in \omega$, we will construct recursively saturated countable models M_n^0, M_n^1 such that for some $a_n^i \in ASS(M_n^i)$, $M_n = M_n^i(\omega, a_n^i)$ for $i = 0, 1$. Let M_0^0 and M_0^1 be any models as above and suppose we already have M_k^i for $k < n$, $i = 0, 1$. Then we take $F = f_{h(\alpha)}$. By Lemma 3.4 we find M_n^i, $i = 0, 1$ satisfying the above conditions and such that the following sets are not coded in M_n^i for $i = 0, 1$:

1. $F * X_k^i$, for some $X_k^i \subseteq M_k$ which has order type ω and is coded in M_k^i, for every k such that $F * M_k \prec_{cof} M_n$, $i = 0, 1$.
2. $F^{-1} * Y$, for all $Y \subseteq F * M_n$ which have order type ω and are coded in M_k^0 or M_k^1, for every k such that $F * M_n \prec_{cof} M_k$.

So now if $s \in T_{\alpha+1}$, then we define M_s to be M_n^i, where $s' \frown i = s$ and $M_n = M_{s'}$.

For every branch B of T let $M(B) = \bigcup_{s \in B} M_s$. The same arguments as before show that if $B_1 \neq B_2$ then $M(B_1)$ is not elementarily embeddable in $M(B_2)$.

The obvious question connected with Theorem 3.5 is whether any set theoretical assumptions of the kind we used are really needed for the proof.

A second question concerns the variety of finitely determinate structures. Is every finitely determinate recursively saturated ω_1-like model expressible as the union of a continuous chain of models $\{M_\alpha\}_{\alpha \in \omega_1}$ such that for every $\alpha \in \omega_1$, $M_\alpha = M_{\alpha+1}(\omega, a)$ for some $a \in ASS(M_{\alpha+1})$? We believe that the answer to this question must be negative. In fact we would have a large family of finitely determinate, recursively saturated ω_1-like models without the above property if the answer to the following question was positive.

Suppose $M_0 \prec_e, \ldots, \prec_e M_n$, $N_0 \prec_e, \ldots, \prec_e N_n$ are such that M_n and N_n are similar and countable. Suppose that for every $i < n$, $(M_{i+1}, M_i) \equiv (N_{i+1}, N_i)$, both pairs are recursively saturated and M_i and N_i are semiregular in M_{i+1} and N_{i+1}, respectively. Is it true that then $(M_n, \ldots, M_0) \cong (N_n, \ldots, N_0)$? Recall that if $I \subseteq_e M$, then the *cofinality* of I in M, $cf(I)$, is the smallest cut $J \subseteq_e M$ such that there exists an increasing function coded in M such that $I = \bigcup_{i \in J} [0, f(i)]$. A cut I is called *semiregular* in I if $cf(I) = I$. Observe that the assumption about semiregularity of cuts in our question is essential since in the other case, we could have for some $i, j < n$, $cf(M_j) = M_i$ while $cf(N_j) \neq N_i$.

4 Nonisomorphic $L_{\infty\omega_1}(aa)$-equivalent models The ω_1-filtrations constructed in the proof of Theorem 3.5 have, in fact, stronger properties than those needed for the Eklof-Mekler characterization of elementary equivalence of finitely determinate structures. It follows from Theorem 3.2 and the next lemma that models built from such filtrations are also $L_{\infty\omega_1}(aa)$-equivalent.

Lemma 4.1 (Shelah, unpublished) *Let $\{M_\alpha\}_{\alpha\in\omega_1}$, $\{N_\alpha\}_{\alpha\in\omega_1}$ be ω_1-filtrations of models M and N, respectively. Suppose that for any $\alpha, \beta < \omega_1$, $H_{\alpha,\beta}$ is a nonempty set of isomorphisms from M_α onto N_β, and for all $\alpha < \alpha_1$, $\beta < \beta_1$, every isomorphism in $H_{\alpha,\beta}$ extends to an isomorphism in H_{α_1,β_1}. Then M and N are $L_{\infty\omega_1}(aa)$-equivalent.*

Corollary 4.2 ($V = L$) *For every countable recursively saturated model M_0, there is a family $\mathfrak{A}$ of recursively saturated ω_1-like models similar to M_0 such that $\mathfrak{A}$ has cardinality $\aleph_2$ and for all distinct M and N in $\mathfrak{A}$, M is $L_{\infty\omega_1}(aa)$-equivalent to N but M cannot be elementarily embedded into N.*

Remarks: The proof of Lemma 4.1 goes by a straightforward induction on the depth of $L_{\infty\omega_1}(aa)$ formulas. One shows that for all $\phi(x_0, \ldots, x_m, P_0, \ldots, P_n)$, where $P_0, \ldots, P_n$ are unary predicates for all $a_0, \ldots, a_m \in M_\alpha$, $M_{\alpha_0}, \ldots, M_{\alpha_n} \subseteq M$ and $h \in H_{\alpha,\beta}$ $\alpha, \beta < \omega_1$ we have:

$$M \models \phi(a_0, \ldots, a_m, M_{\alpha_0}, \ldots, M_{\alpha_n})$$
$$\text{iff } N \models \phi(h(a_0), \ldots, h(a_m), h * M_{\alpha_0}, \ldots, h * M_{\alpha_n})$$

We have simplified the statement of the lemma to the form needed for Corollary 4.2. The assumptions in a more general form can be written as follows. For any regressive functions f, g: $\omega_1 \to \omega_1$, there is a family of sets of partial isomorphisms $(H_{\alpha,\beta})_{\alpha,\beta<\omega_1}$ such that if $f(\alpha) = g(\beta)$ then $H_{\alpha,\beta}$ maps M_α isomorphically onto N_β and if $f(\alpha) = g(\beta)$ and $f(\alpha_1) = g(\beta_1)$, then every isomorphism in $H_{\alpha,\beta}$ extends to one in H_{α_1,β_1}.

The lemma in this formulation could be used for a proof of Corollary 4.2 without the assumption $V = L$, provided we had a positive answer to the question stated at the end of Section 3 and the answer was given by a theorem similar to Theorem 3.2.[2]

NOTE

1. In fact, the assumption of $\diamond$ is sufficient for the proof. Once we have $\diamond$ we have also CH, and as can be easily seen the construction that we give can be carried out along any ω_1-tree if all the levels T_α have cardinality at most $\aleph_1$.
2. The $V = L$ assumption can be deleted from the proof of Corollary 4.2 by a slightly different method. The answer to the question from the end of Section 3 is still unknown.

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