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Classification Theory Over a Predicate I

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Introduction In this paper, the scene is set for the study of classification over a predicate. Let T be a complete first-order theory with among other things a unary predicate P. Instead of studying the structure and number of models of T we are now interested in the structure and number of models M of T *over* M^P (where M^P is the substructure of M with universe P^M). So for example we let $I_T(\lambda, \mu)$ be the greatest κ such that there are N of power μ and κ models $M \models T$, with $M^P = N$ and $|M| = \lambda$, which are pairwise non-N-isomorphic.

In Section 2 it is pointed out that, given N, those $M \models T$ with $M^P = N$ can be to some extent coded by L_2-reducts of *expansions* of N to T^* where $L(T^*) \supseteq L_2 \supseteq L(T)$, for suitable L_2, T^*. So in Section 1 the following is examined: given $L_1 \subseteq L_2 \subseteq L_3$ and T^* a theory in L_3, what are the possible numbers of expansions of N to L_2-reducts of models of T^* as N ranges over L_1-structures? This generalizes the context of the Chang-Makkai theorem (see [1]) and results in [6]. Some finer results are also obtained.

Such results are used to show that if $I_T(\lambda, \lambda)$ is not too big then for every $M \models T$, $\bar{a} \in M$, $tp(\bar{a}/P^M)$ is definable.

In Section 3 some stability-type notions are introduced. The general context here is: given $M \models T\, A \subseteq M$, $A \supseteq P^M$ one should study the space of those types p over A which can be realized in some $N \models T$ with $P^N = P^A$ $(= P^M)$. In future work fairly complete answers to spectrum problems (e.g., analogues of Morley's theorem) will be given by studying the space of such types for successively more "complicated" such A, using the techniques similar to [9]. Here we essentially consider only the case $A = M \models T$ and prove some non-structure theorems.[1]

1 Let us first establish notation for this section. $L_1 \subseteq L_2 \subseteq L_3$ are first-order languages (with equality), $L_2 - L_1 = \{P_i\colon i < \kappa\}$, and T is a theory in L_3 such that $T \models \exists x \exists y\, (x \neq y)$.

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For N an L_1-structure:

$D_1(N) = |\{M$: M is the L_2-reduct of an expansion of N to an L_3-structure which is a model of $T\}|$.

For N an L_3-structure which is a model of T:

$D_2(N) = |\{M$: M an L_2-structure, $M \upharpoonright L_1 = N \upharpoonright L_1$ and $M \cong N \upharpoonright L_2\}|$.

Now we define (for μ an infinite cardinal):

$D_1(\mu) = max\{D_1(N)$: $|N| = \mu$, N an L_1-structure$\}$ $(= D_1(\mu, T, L_2, L_1))$.
$D_2(\mu) = max$ $\{D_2(N)$: $N \models T$, $|N| = \mu\}$ $(= D_2(\mu, T, L_2L_1))$.

Remark: In case the maximum is not obtained we still use notation $D_i(\mu)$ as follows: D_i $(\mu) \geq \lambda$ means there is (suitable) N of cardinality μ with $D_i(N) \geq \lambda$.

Note that (if $N \models T$) $D_2(N) \leq D_1(N \upharpoonright L_1)$ and thus $D_2(\mu) \leq D_1(\mu)$ for all μ.

For λ an infinite cardinal define:

*Ded** $\lambda = max$ $\{\mu$: there is a tree with λ nodes and μ branches of the same height δ (for some δ)$\}$ (where use of "*max*" is as above).

The following essentially appears in [6] (Theorem 1).

Theorem 1.1 *Let $\kappa = 1$, $P_0 = P$, $\lambda \geq |T|$. The following are equivalent*:
(1) *For no $\phi \in L$ does $T \vdash \exists \bar{x} \forall \bar{y}$ $(P\bar{y} \leftrightarrow \phi(\bar{y}, \bar{x}))$*
(2) $D_1(\lambda) > \lambda$
(3) $D_1(\lambda) \geq Ded^* \lambda$
(4) $D_2(\lambda) > \lambda$
(5) $D_2(\lambda) \geq Ded^* \lambda$.

Let us continue for now with the case $\kappa = 1$, $P_0 = P$ and assume that $D_l(\lambda) \leq \lambda$ $(\forall \lambda \geq |T|)$. So we have a formula $\phi(\bar{y}, \bar{x})$ such that $T \vdash \exists \bar{x} \forall \bar{y}$ $(P\bar{y} \leftrightarrow \phi(\bar{y}, \bar{x}))$.

Let $E(\bar{x}_1, \bar{x}_2)$ be the formula

$$(\forall \bar{y})(\phi(\bar{y}, \bar{x}_1) \leftrightarrow \phi(\bar{y}, \bar{x}_2)) \ .$$

In $\mathfrak{C}^{eq}$ (see [7]) the equivalence classes of E become elements satisfying a predicate Q, and thus the choice of P is equivalent to the choice of a suitable element of Q.

It follows that if for some L_1-structure $M(|M| \geq |T|)$ $D_l(M)$ is infinite then for every $\mu \geq (T)$, $D_l(\mu) \geq \mu$.

On the other hand, if for every L_1-structure M, $D_l(M)$ is finite then by compactness this reduces to T proving a certain sentence which by abuse of notation we express by $T \vdash$ "Q is finite".

The following is then clear:

Theorem 1.2 *The following are equivalent* $(l = 1, 2)$ $\kappa = 1$, $P_0 = P$
(1) $T \vdash$ "*Q is finite*"
(2) $\forall \lambda$ $D_l(\lambda)$ *is finite*
(3) *for some* $\lambda \geq |T|$, $D_l(\lambda) < \lambda$
(4) $D_l(\lambda)(\lambda \geq |T|)$ *is constant.*

Corollary 1.3 ($\kappa = 1$, $P_0 = P$) *Exactly one of the following holds*
(1) $D_l(\lambda) \geq Ded^*\ \lambda$, $\forall\lambda \geq |T|$
(2) $D_l(\lambda) = \lambda$, $\forall\lambda \geq |T|$
(3) $D_l(\lambda)$ *is constant* (*and finite*) $\forall\lambda \geq |T|$.

The case in which κ is finite can be reduced to the case $\kappa = 1$ by considering the predicate

$$Q(\bar{y}_0\bar{y}_1\ldots\bar{y}_{\kappa-1}) = P_0(\bar{y}_0) \wedge \ldots \wedge P_{\kappa-1}(\bar{y}_{\kappa-1}) \ .$$

We now wish to describe the function $D_l(-)$ when $\kappa \geq \aleph_0$. So let us now assume that $\kappa \geq \aleph_0$.

For $i < \kappa$ let $L_2^i = L_1 \cup P_i$.

We clearly have $D_l(\mu, T, L_2^i, L_1) \leq D_l(\mu, T, L_2, L_1)$ for all $i < \kappa$, where by the above we have:

Fact 1.4 If for some $i < \kappa$ there is no $\phi(\bar{y}_i, x_i)$ such that $T \vdash \exists x_i \forall \bar{y}_i(P_i(\bar{y}_i) \leftrightarrow \phi(\bar{y}_i, \bar{x}_i))$, then

$$\forall\lambda \geq |T|,\ D_l(\lambda) \geq Ded^*(\lambda) \ .$$

So we now assume:

Assumption 1.5 $\forall i < \kappa$ there is $\phi_i(\bar{y}_i, \bar{x}_i)$ such that

$$\mathrm{T} \vdash \exists\bar{x}_i\forall\bar{y}_i(P_i(\bar{y}_i) \leftrightarrow \phi_i(\bar{y}_i, \bar{x}_i) \ .$$

As in Theorem 1.2 and with the same notation.

Theorem 1.6 *The following are equivalent*
(1) $\forall i$, $T \vdash$ "*Q_i is finite*"
(2) $\forall\lambda$, $D_l(\lambda) \leq 2^{|T|+\aleph_0}$
(3) *for some* $\lambda \geq |T|$, $D_l(\lambda) < \lambda$
(4) $D_l(\lambda)$ $(\lambda \geq |T|)$ *is constant.*

Remark 1.7: By Assumption 1.5, $D_1(\lambda) \leq \lambda^\kappa$ for all $\lambda \geq |T|$.

Let $M \models T$ and for each $i < \kappa$, let c_i^M be the element of M (actually of M^{eq}) which is $\bar{c}_i/E_i$, where $M \models P_i(\bar{y}_i) \leftrightarrow \phi_i(\bar{y}_i, \bar{c}_i)$. Note that if M_1, M_2 are distinct L_2-structures which are expansions of the same L_1-structure M, and are both reducts of models of T, then

$$\langle c_i^{M_1}\colon i < \kappa\rangle \neq \langle c_i^{M_2}\colon i < \kappa\rangle \ .$$

Definition 1.8 Let $S = \{\mu\colon \mu$ an infinite cardinal and $\exists\delta(cf\,\delta = \mu \vee \delta = \mu)$, $\exists M \models T\ \exists i(\gamma)$, $\gamma < \delta$ such that $c_{i(\gamma)}^M$ *is not algebraic in* $M \restriction L_1$ over $\{c_{i(\beta)}^M\colon \beta < \gamma\}$ but every c_i^M is algebraic in $M \restriction L_1$ over $\{c_{i(\gamma)}^M\colon \gamma < \delta\}$, and whenever $A \subseteq \{c_{i(\gamma)}\colon \gamma < \delta\}$, $|A| < \mu$ then some $c_{i(\gamma)}$ is not algebraic in $M \restriction L_1$ over $A\}$. Note that S is a (maybe empty) set of infinite cardinals $\leq\kappa$.

Theorem 1.9
(1) *If* $D_1(\lambda) > \lambda^{<\mu} + 2^{|T|}$ *then there is* $\chi \geq \mu$ *with* $\chi \in S$.
(2) *Let* $\lambda > |T|$, $D_1(\lambda) > \lambda$ *and* $(\forall\chi < \lambda)(\chi^\kappa < \lambda)$. *Then* $cf\,\lambda \in S$.

(3) *Let* $\mu > \lambda + |T|^\kappa$, μ *regular*, $D_1(\lambda) \geq \mu$. *Then*
(a) *if* λ *is regular then* $\exists\lambda^* < \lambda$, $D_1(\lambda^*) \geq \mu$
(b) *if* $L_3 = L_2$ *then* $\exists\lambda^* < \lambda$ $D_1(\lambda^*) \geq \mu$ *or* $cf\lambda \in S$.

Proof: (1) Let M be an L_1-structure of cardinality λ with $D_1(M_1) > \lambda^{<\mu} + 2^{|T|}$. Let $X =$ the set of sequences $\boldsymbol{c} = \langle c_i : i < \kappa\rangle$ in M corresponding to L_2-reducts of expansions of M to models of T. So $|X| > \lambda^{<\mu} + 2^{|T|}$. For each $\boldsymbol{c} \in X$, let $A_{\boldsymbol{c}} \subset \boldsymbol{c}$ be a set of minimal cardinality such that each $c_i \in \boldsymbol{c}$ is algebraic (in M) over $A_{\boldsymbol{c}}$. Note that (as $|A_{\boldsymbol{c}}| \leq |\boldsymbol{c}| \leq |T|$) for each $A_{\boldsymbol{c}}$ there are at most $2^{|T|}$ $\boldsymbol{c}' \in X$ with $A_{\boldsymbol{c}'} = A_{\boldsymbol{c}}$. So clearly there is $\boldsymbol{c}$ such that $|A_{\boldsymbol{c}}| = \chi \geq \mu$. By the minimality of $|A_{\boldsymbol{c}}|$ (with respect to $\boldsymbol{c}$) we can find $\langle c^j : j < \chi\rangle \subseteq A_{\boldsymbol{c}}$ such that c^j is not algebraic over $\{c^i : i < j\}$ and *every* $c \in A_{\boldsymbol{c}}$ is algebraic over $\{c^j : j < \chi\}$. Clearly $\chi \in S$.

(2) Let $D_1(\lambda) > \lambda$, $\lambda > |T|$, and $(\forall\chi < \lambda)(\chi^\kappa < \lambda)$.

Let M be as in (1) and put $M = \bigcup_{i<cf\lambda} M_i$ where $\langle M_i : i < cf\lambda\rangle$ is an elementary chain, $|M_i| < \lambda$ $\forall i < cf\lambda$. Clearly for each i there are $<\lambda$ different $\boldsymbol{c} \in X$ contained in M_i. Thus for some $\boldsymbol{c} \in X$, for unboundedly many $i < cf\lambda$, $\boldsymbol{c} \cap (M - M_i) \neq \varnothing$. Easily $cf\lambda \in S$.

(3) Again let N be an L_1-structure of cardinality λ with $D_1(N) \geq \mu$ (where $\mu > \lambda + |T|^\kappa$). Note $\lambda > |T|$.

(a) Let λ be regular and let $N = \bigcup_{i<\lambda} N_i$, where the N_i are a continuous increasing elementary chain of models of power $<\lambda$. Let $M^\alpha (\alpha < \mu)$ be expansions of N to models of T which are pairwise L_2-distinct.

For $i < \lambda$ let N_i^α be the L_3-structure whose universe is that of N_i and whose structure is induced from M^α.

By regularity of λ, for each $\alpha < \mu$ there is $i_\alpha < \lambda$ such that $N_{i_\alpha}^\alpha \prec M^\alpha$.

By regularity of μ there is i_* such that $Y = \{\alpha < \mu : i_\alpha = i^*\}$ has cardinality μ. It is then clear that $\{N_{i^*}^\alpha \restriction L_2 : \alpha \in Y\}$ are distinct, and so $D_1(N_{i^*}) \geq \mu$.

(b) Suppose $L_2 = L_3$. We may (by (a)) assume that λ is singular. Let $N = \bigcup_{i<cf\lambda} N_i$, $|N_i| < \lambda$, the N_i a continuous increasing elementary chain.

If for some expansion N' of N to a model of T, for all $i < cf\lambda$ there is j with $c_j^{N'} \notin N_i$ then as in the proof of (2) $cf\lambda \in S$.

If not then for every expansion N' of N to a model of T there is $i < cf\lambda$ such that all $c_j^{N'} \in N_i$.

As there are $\geq\mu$ such expansions N' and μ is regular $\mu > cf\lambda$, it follows that there are N^α, $\alpha < \mu$, distinct expansions of N to models of T, and $i_* < cf\lambda$ such that for all $\alpha < \mu$ $\forall i < \kappa$, $c_i^{N^\alpha} \in N_{i_*}$.

Remembering that $L_2 = L_3$ and that for each $i < \kappa$, $N^\alpha \models P_i(\bar{y}_i) \leftrightarrow \phi_i(\bar{y}_i, c_i)$ we see that $N_{i_*}^\alpha \prec N^\alpha$ and the $N_{i_*}^\alpha$ are pairwise distinct $(\alpha < \mu)$.

As $|N_{i_*}| < \lambda$, we see that $\exists\lambda^* < \lambda$, $D_1(\lambda^*) \geq \mu$, proving (3).

Lemma 1.10 *Let* $L_2 = L_3$, $\chi \in S$, $\lambda \geq |T|$. *Suppose there is a tree with* λ *nodes and* μ *branches of height* δ, *where* $\chi = cf\delta$. *Then* $D_1(\lambda)$, $D_2(\lambda) \geq \mu$.

Proof: Like [6].

Lemma 1.11 *Suppose* $\forall\mu < \lambda$, $\mu^{cf\lambda} \geq \lambda$. *Then there is a tree with* λ *nodes and* μ *branches of height* $cf\lambda$, *with* $\mu > \lambda$.

Proof: Our assumptions imply that $\lambda^{<cf\lambda} = \lambda$. So put $\mu = \lambda^{cf\lambda}$ and look at the tree ${}^{cf\lambda>}\lambda$.

Now, with no assumptions:

Theorem 1.12 *Let $L_2 = L_3$. Then exactly one of the following holds*
(1) $\forall\lambda \geq |T|$, $D_l(\lambda) \geq Ded^*\lambda$ $l = 1, 2$
(2) $\forall\lambda \geq |T|$, $D_l(\lambda) \geq \lambda$ *and there is* $S \subseteq \{\chi\colon \aleph_0 \leq \chi \leq |T|\}$ *such that*:
 (A) *if there is a tree with* λ *nodes and* μ *branches of height* δ, $cf\,\delta \in S$ *then* $D_l(\lambda) \geq \mu$, $l = 1, 2$
 (B) *if* $\forall\mu < \lambda$ $(\mu^\kappa \leq \lambda)$ *and* $\lambda \geq 2^{|T|}$ *then* $D_l(\lambda) > \lambda$ *iff* $cf\lambda \in S$
(3) $D_l(\lambda)$, $\lambda \geq |T|$, *is constant.*

Proof: If for some $\lambda \geq |T|$, $D_l(\lambda) < Ded^*\lambda$ then we can work with Assumption 1.5.

By Theorem 1.6, if $D_l(\lambda)$ is not constant, then $D_l(\lambda) \geq \lambda$ $\forall\lambda \geq |T|$.
Let S be as in Definition 1.8:

(2A) is Lemma 1.10.
(2B) Suppose $\mu < \lambda$, $\mu^* \leq \lambda$ and $\lambda \geq 2^{|T|}$. If $cf\lambda \in S$ then $cf\lambda \leq \kappa$. So the hypothesis of Lemma 1.11 holds, and thus by Lemma 1.10, $D_l(\lambda) > \lambda$.

If $D_1(\lambda) > \lambda$ then by Remark 1.7, $\lambda \neq \lambda^\kappa$ and thus using our assumptions, $\forall\mu < \lambda$ $\mu^\kappa < \lambda$. By Theorem 1.9(2) $cf\lambda \in S$.

Let us reassume Assumption 1.5 and seek some finer control over $D_l(\lambda)$.

Definition 1.13 $S_1 = \{\delta\colon \delta$ ordinal, $\exists M \models T$ and $\{c^M_{i(\gamma)}\colon \gamma < \delta\}$ such that: (1) $c^M_{i(\gamma)}$ is not definable (in $M \restriction L_1$) over $\{c^M_{i(\beta)}\colon \beta < \gamma\}$ and (2) every c^M_i is definable (in $M \restriction L_1$) over $\{c_{i(\gamma)}\colon \gamma < \delta\}\}$.

Lemma 1.14 *If* $\delta \in S_1$ *then* $\forall\lambda \geq |T|$, $D_l(\lambda) \geq min\ \{2^{|\delta|}, \lambda\}$.

Proof: Let $\delta \in S_1$. Clearly there is a $|T|^+$-saturated model N of T of cardinality $\geq\lambda$, $2^{|\delta|}$, such that $\langle c^N_{i(\gamma)}\colon \gamma < \delta\rangle$ witness $\delta \in S_1$.

We will drop the superscript N from c^N_i.

As for each $\gamma > \delta$, $c_{i(\gamma)}$ is not L_1-definable over $\{c_{i(\beta)}\colon \beta < \gamma\}$, we can find, inductively, in N elements d_τ for $\tau \in {}^{\delta>}2$ such that:

(1) for each $\tau \in {}^{\delta\geq}2$, $tp_{L_1}(\langle d_{\tau\restriction\gamma}\colon \gamma < lh(\tau)\rangle) = tp_{L_1}(\langle c_{i(\gamma)}\colon \gamma < lh(\tau)\rangle)$
(2) $d_{\tau\frown\langle 0\rangle} \neq d_{\tau\frown\langle 1\rangle}$ (when $l(\tau \frown \langle 0\rangle) < \delta$).

So there is an elementary extension N^* of N and for each $\eta \in {}^\delta 2$ an automorphism F_η of $N^* \restriction L_1$ taking $c_{i(\gamma)}$ to $d_{\eta\restriction\gamma}$ for all $\gamma < \delta$.

By adding predicates for these automorphisms and choosing an elementary substructure of power λ, we can obtain a model M of T of power λ with $D_2(M) \geq min(\lambda, 2^{(\delta)})$.

Lemma 1.15 *Let* $T \vdash$ "*Q_i is finite*" *for all* $i < \kappa$, *T complete. Then*
(1) $\delta_1, \delta_2 \in S_1 \Rightarrow |\delta_1| = |\delta_2|$ *and* $\delta \in S_1 \Rightarrow |\delta| \in S_1$
(2) *if* $L_2 = L_3$ *and* $\delta \in S_1$ *then* $D_1(\lambda) \geq 2^{|\delta|}$, $\forall\lambda \geq |T|$.

Proof: (1) First note that by the completeness of T if some $\delta_1 \in S_1$ is finite then $\forall\delta \in S_1$ $\delta = \delta_1$.

So suppose that $\delta_1, \delta_2 \in S_1$ with $\aleph_0 \leq |\delta_1| < |\delta_2|$. As T is complete there is a saturated model M of T witnessing both $\delta_1 \in S_1$ and $\delta_2 \in S_1$. Let $\langle c_{i_1(\gamma)}$: $\gamma < \delta_1\rangle$ witness $\delta_1 \in S_1$ and $\langle c^M_{i_2(\gamma)}$: $\gamma < \delta_2\rangle$ witness $\delta_2 \in S_1$.

We can find an infinite $U \subset \delta_2$ and finite $w \subset \delta_1$ such that $\forall\gamma \in U$, $c_{i_1(\gamma)}$ is definable (*in* $M \upharpoonright L_1$) over $\bar{c} = \langle c_{i_1(\alpha)}: \alpha \in w\rangle$.

As $T \vdash$ "Q_i is finite" $\forall i$, there are only finitely many $\bar{c}'$ in M with $tp_{L_1}(\bar{c}') = tp_{L_1}(\bar{c})$. So there are only finitely many possible sequences $\langle c^j$: $j \in U\rangle$ with the same L_1-type as $\langle c_{i_2(\gamma)}: \gamma \in U\rangle$. This contradicts the fact that for all $\gamma < \delta$ $c_{i_2}(\gamma)$ is not (L_1)-definable over $\{c_{i_2(\beta)}: \beta < \gamma\}$.

So $|\delta_1| = |\delta_2|$.

It now easily follows that $\delta \in S_1 \Rightarrow |\delta| \in S_1$.

(2) Working in a big model N of T as in the proof of Lemma 1.14 we find $\{d_\tau: \tau \in {}^{\delta>}2\}$ satisfying (1) and (2) of that proof.

Note that, as $T \vdash$ "Q_i is finite", $\forall i < \kappa$ $|\{d_\tau: \tau \in {}^{\delta>}2\}| = |\delta| \leq \kappa \leq |T| \leq \lambda$.

Let $N \prec M$ be of cardinality λ containing d_τ $\forall\tau \in {}^{\delta>}2$.

As $L_2 = L_3$, for each $\eta \in {}^{\delta}2$ $\langle d_{\eta\upharpoonright\gamma}: \gamma < \delta\rangle$ gives rise to a different expansion N^η of $N \upharpoonright L_1$ to a model of T.

So $D_1(\lambda) \geq 2^{|\delta|}$.

We now give some examples.

Example 1.16: Let α be a countable ordinal and $T_\alpha = Th(H(\beth_\alpha), \in)$.

Let $L_0 = \{\in\}$ $(= L(T_\alpha))$, $L_1 = L_0 \cup \{c_n: n < \omega\}$, $L_2 = L_1 \cup \{F\}$ where F is a unary predicate.

$T = T_\alpha \cup \{$"F is a function with domain ω, $\forall n < \omega(F(n)$ is a sequence of ordinals of length $n)$, $\forall n < m < \omega(F(m) \upharpoonright n = F(n))$"$\} \cup \{F(n) = c_n: n < \omega\}$.

Let $N \models T_\alpha$. Then if ω^N is standard, clearly $D_1(N) =$ the cardinality of the set of (really) countable sequences of ordinals in N.

On the other hand, if ω^N is nonstandard then let n_1 be a nonstandard member of ω^N and then for every expansion N' to a model of T

$$\forall n < \omega\ N' \models c_n = F(n') \upharpoonright n\ .$$

Thus $D_1(N) \leq |N|$.

Noting that if $M \models T_\alpha$ $|M| > \beth_\alpha$ then ω^M is nonstandard we see that $D_1(\lambda) > \lambda$ iff $\lambda \leq \beth_\alpha$ and $\lambda^{\aleph_0} > \lambda$.

Example 1.17: Let for each $l < \omega$, F_l be an l-place function from ω_1 to ω such that for any countably infinite $A \subseteq \omega$,

(*) $\{F_l(x_1, \ldots, x_l): x_1, \ldots, x_l \in A,\ l < \omega\} = \omega$.

Let $L_0 = \{G_l: l < \omega\}$, $L_1 = L_0 \cup \{c_n: n < \omega\}$, $L_2 = L_1 \cup \{d_i: i < \omega_1\}$, where G_l is an l-place function symbol.

Let $T = \{G_l(d_{i_1}, \ldots, d_{i_l}) = c_n: l < \omega,\ i_1, \ldots, i_l < \omega_1,\ n < \omega\ F_l(i_1, \ldots, i_l) = n\}$.

Let N be an L_0-structure and suppose N_1, N_2 are expansions of N to models of T such that $\{d_i^{N_1}: i < \omega_1\} \cap \{d_i^{N_2}: i < \omega_1\}$ is infinite. Then by (*) $N_1 \upharpoonright L_2 = N_2 \upharpoonright L_2$.

Thus, if $D_1(\lambda) > \lambda$ then there is a family of λ^+ subsets of λ each of power $\aleph_1$, the intersection of any two of which is finite.

2 Here we begin the study of classification over a predicate. T will be a complete theory in a relational language L (with equality) containing, among other things, a unary predicate P with $T \vdash$ "P is infinite".

If M is an L-structure then by M^P we mean the L-structure whose universe is P^M and whose structure is that induced by M. We are interested in the number and structure of models M of T *over their P-part*. So the strongest categoricity property is: for any $M \models T$, M is determined, up to isomorphism over M^P, *by* M^P. (Note that if $T \vdash \forall x \neg Px$ then T has this property iff T is the theory of a *finite* structure.) Nonstructure theorems will say, for example, that for some M_0 there are "many" $M \models T$ (maybe of a given cardinality) up to M_0-isomorphism, with $M^P = M_0$.

Definition Let N be an L-structure.

$I_T(\lambda, N)$ = the number of models M of T of cardinality λ with $M^P = N$, up to isomorphism over N.

$$I_T(N) = \sum_{\lambda} I_T(\lambda, N).$$

$$I_T(\lambda, \mu) = \max_{|N|=\mu} I_T(\lambda, N).$$

$$I_T(\mu) = \max_{|N|=\mu} I_T(N).$$

When T is clear from the context it will be omitted.

Lemma 2.1 *There is a theory T' in a language $L' \supseteq L$ with $|T'| = |T|$ and such that any L-structure N can be expanded to a model of T' if and only if $N = M^P$ for some $M \models T$.*

Proof: Let $L' = L \cup \{f\} \cup \{R^*: R$ a relation symbol of $L\}$ where f is a unary function symbol.

Let N' be an L'-structure. T' will say the following of N':

(i) f is a $1-1$ function from N into N
(ii) $P^* = Im\, f$
(iii) $\forall \bar{x}(R(\bar{x}) \leftrightarrow R^*(f(\bar{x})))$ for every symbol R of L
(iv) the structure $N'' = N' \upharpoonright \{R^*: R \in L\}$ is a model of T^*, where T^* is T with R^* replacing R.

Clearly T' works.

So note that $\{N: \exists M \models T, M^P = N\}$ is a PC_Δ-class.

We will be interested in the following possible properties of T.

Property (I) For every L-formula $\phi(\bar{x})$ there is an L-formula $\psi(\bar{x})$ such that for every $M \models T$ and $\bar{a} \in M^P$,

$$M \models \phi(\bar{a}) \text{ iff } M^P \models \psi(\bar{a}) .$$

Property (II) For any $M \models T$ and $\bar{a} \in M$, $tp(\bar{a}/P^M)$ is definable over P^M (i.e., for any $\phi(\bar{x}, \bar{y})$ there is $\psi_\phi(\bar{y}, \bar{c})$ $\bar{c} \in P^M$ such that $\forall \bar{d} \in P^M$ $M \models \phi(\bar{a}, \bar{d}) \leftrightarrow \psi_\phi(\bar{d}, \bar{c})$).

Remark 2.2: Note that if T satisfies (I) then for the purposes of our study we can assume that T is Morleyized; i.e., that for each $\phi(\bar{x}) \in L$ there is $R_\phi(\bar{x})$ a relation symbol of L with

$$T \vdash \phi(\bar{x}) \leftrightarrow R_\phi(\bar{x}) \ .$$

Also note that if T satisfies (II) then by compactness, for every $\phi(\bar{x}, \bar{y})$ there is $\psi_\phi(\bar{y}, \bar{z})$ such that for any $\bar{a} \in M \models T\ l(\bar{a}) = l(\bar{x})$, there is $\bar{c} \in P^M$ such that $tp_\phi(\bar{a}/P^M)$ is definable by $\psi_\phi(\bar{y}, \bar{c})$.

In this section we show, using material from Section 1, how we can assume T to satisfy Properties (I) and (II) above.

Lemma 2.3 *If for some $\lambda \geq |T|$, $I_T(\lambda, \lambda) = 1$ then T satisfies (I).*

Proof: If T does not satisfy (I) then we can easily obtain $M_1, M_2 \models T$ of cardinality λ with $M_1^P = M_2^P = N$, N of cardinality λ, and $\bar{a} \in N$ such that $(M_1, \bar{a}) \not\cong (M_2, \bar{a})$, contradicting our hypothesis.

Lemma 2.4 *Let T^* be $T \cup \{\forall\bar{x}(R_\phi(\bar{x}) \leftrightarrow \phi(\bar{x})\colon \phi \in L\}$ where the R_ϕ are new relation symbols. Let $T^+ = (T^*)'$ (from Lemma 2.1). Then*
(i) *for any L-structure N, $I_T(\lambda, N) = \sum\{I_{T^*}(\lambda, N^*)\colon N^*$ is an $L(T^*)$-expansion of $N\}$*
(ii) $I_T(\lambda, \lambda) \geq D_1(\lambda, T^+, L(T^*), L), I_{T^*}(\lambda, \lambda)$
(iii) $I_T(\lambda, \mu) \leq D_1(\mu, T^+, L(T^*), L)\ I_{T^*}(\lambda, \mu)$.

Proof: Immediate.

Corollary 2.5 *If $I_T(\lambda, \lambda) \geq \aleph_0$ then*

$$I_T(\lambda, \lambda) = D_1(\lambda, T^+, L(T^*), L)\ I_{T^*}(\lambda, \lambda) \ .$$

Bearing in mind what we know from Section 1 of the function D_1 and also 2.2, 2.3, 2.4, and 2.5 we now make:

Assumption 2.6 $T = T^*$ (and thus T satisfies (I)).

Definition 2.7 Let $M \models T$, $\bar{a} \in M$.

Let $L_2 = L \cup \{R_{\phi(\bar{x},\bar{a})}\colon \phi(\bar{x}, \bar{y}) \in L\}$ and we expand M to an L_2-structure $M_{\bar{a}}$ by putting $M_{\bar{a}} \models R_{\phi(\bar{x},\bar{a})}\bar{b}$ iff $M \models \phi(\bar{b}, \bar{a})$.

Let $T_{\bar{a}} = Th(M_{\bar{a}})$ and let $T_{\bar{a}}^+$ be $(T_{\bar{a}})'$ from Lemma 2.1. Let $D_{\bar{a}}(\lambda) = D_1(\lambda, T_{\bar{a}}^+, L_2, L)$ and similarly for $S_{\bar{a}}$.

Thus an L-structure N has an expansion to a model of $T_{\bar{a}}^+$ iff there is $M \models T$ and $\bar{a} \in M$ such that $M^P = N$ and $M_{\bar{a}} \models T_{\bar{a}}$.

Note that if N (L-structure) has two different expansions N^1, N^2 to L_2-reducts of models of $T_{\bar{a}}^+$ then there are $M_1, M_2 \models T\ \bar{a}_1 \in M_1\ \bar{a}_2 \in M_2$ with $M_1^P = M_2^P = N$ and $tp(\bar{a}_1/N) \neq tp\ (\bar{a}_2/N)$. (Moreover we can get $|M_1| = |M_2| = |N|$.)

Lemma 2.8 *Let $M \models T$, $\bar{a} \in M$.*
(i) *If $tp(\bar{a}/P^M)$ is not definable over P^M then $\forall\lambda \geq |T|$, $D_{\bar{a}}(\lambda) \geq Ded^*(\lambda)$.*
(ii) *If $tp(\bar{a}/P^M)$ is definable over P^M but not definable almost over some finite $\bar{c}$ in P^M, then $S_{\bar{a}} \neq \emptyset$.*

(iii) *If $tp(\bar{a}/P^M)$ is definable over P^M but not definable over some finite $\bar{c}$ in P^M, then $(S_1)_{\bar{a}} \neq \emptyset$.*

Proof: (i) Note that $tp(\bar{a}/P^M)$ being definable over P^M means for each $R_{\phi(\bar{x},\bar{a})}$ there is an L-formula $\psi(\bar{x}, \bar{z})$ such that

$$T_{\bar{a}}^{+} \vdash (\exists \bar{z})(\forall \bar{x} R_{\phi(\bar{x},\bar{a})}(\bar{x}) \leftrightarrow \psi(\bar{x}, \bar{z})) \ .$$

Now use Fact 1.4.

(ii) and (iii) are similar.

Theorem 2.9

(i) *If for some $\bar{a}$ and $\lambda \geq |T|$, $D_{\bar{a}}(\lambda) > \lambda$ then $I_T(\lambda, \lambda) \geq D_{\bar{a}}(\lambda)$.*

(ii) *If $|T| \leq \mu \leq \lambda$, $D_{\bar{a}}(\lambda) > \lambda$ for some $\bar{a}$, then $I_T(\mu, \mu) \geq \mu$.*

Proof: (i) Our hypotheses and the remarks following 2.7 give us $M_i \models T$ and $\bar{a}_i \in M_i$ for $i < D_{\bar{a}}(\lambda)$ such that $\forall i\ M_i^P = N$, $|M_i| = |N| = \lambda$, and the $tp(\bar{a}_i/N)$ are all different.

Clearly we can choose $X \subseteq D_{\bar{a}}(\lambda)$, $|X| = D_{\bar{a}}(\lambda)$ such that for all $i, j \in X$ with $i < j$, $tp(\bar{a}_i/N)$ is omitted in M_j.

So the M_i, $i \in X$ are pairwise nonisomorphic, whereby $I_T(\lambda, \lambda) \geq D_{\bar{a}}(\lambda)$.

(ii) Let, by (i), $M_i (i < \mu)$ and $\bar{a}_i \in M_i$ be such that $M_i^P = N\ \forall i < \mu$ and $tp(\bar{a}_i/N)$ is omitted in M_j for $j < i$ (also $|M_i| = |N| = \lambda$, $\forall i$).

Now we can easily define, for $\alpha < \mu$ and $i < \mu$, models $N_i^\alpha \prec M_i$ such that:

(a) $\bar{a}_i \in N_i^\alpha$ and $|N_i^\alpha| = \mu\ \forall i, \alpha$

(b) $\bigcup\limits_{\substack{\beta<\alpha \\ j<\mu}} P^{N_j^\beta} \subseteq N_i^\alpha\ \forall \alpha < \mu$

(c) $\forall j < i$, $\beta < \alpha$ if $\bar{a} \in N_j^\beta$ then $tp(\bar{a}/P^{N_j^\alpha} \cap P^{N_i^\alpha}) \neq tp(\bar{a}_i/P^{N_j^\alpha} \cap P^{N_i^\alpha})$.

If we put $N^i = \bigcup\limits_{\alpha<\mu} N_i^\alpha$ for $i < \mu$ we see that $(N^i)^P = (N^j)^P$ for $i, j < \mu$ and $tp(\bar{a}_i/(N_i)^P)$ is omitted in N^j for $j < i$.

So $I(\mu, \mu) \geq \mu$.

Corollary 2.10 *If for some $M \models T$ and $\bar{a} \in M$ $tp(\bar{a}/P^M)$ is not definable over P^M then*

$$\forall \lambda \geq |T|\ I_T(\lambda, \lambda) \geq Ded^*(\lambda) \ .$$

Proof: By Lemma 2.8 and Theorem 2.9.

Thus we can now make

Assumption 2.11 For every $M \models T$, $\bar{a} \in M$ $tp(\bar{a}/P^M)$ is definable over P^M.

For each $\phi(\bar{x}, \bar{y}) \in L$ we let $\psi_\phi(\bar{y}, \bar{z})$ be as in Remark 2.2.

3 We continue with Assumptions 2.6 and 2.11, introduce some suitable stability-theoretic notions, and prove essentially some "nonstructure" theorems.

We work inside a big saturated model $\mathfrak{C}$ of T.

Fact 3.1: Given $\phi(\bar{x}, \bar{y}) \in L$ let $\theta_\phi(\bar{x}, \bar{z})$ be the formula $\forall \bar{y} \in P\ (\phi(\bar{x}, \bar{y}) \leftrightarrow \psi_\phi(\bar{y}, \bar{z}))$. Then for each $M \models T$, $\bar{a} \in M$ there is $\bar{c} \in P^M$ such that $\models \theta_\phi(\bar{a}, \bar{c})$ and $\theta_\phi(\bar{x}, \bar{c}) \vdash tp_\phi(\bar{a}/P^M)$. Moreover, for every $N \succ M$, $\theta_\phi(\bar{x}, \bar{c}) \vdash tp_\phi(\bar{a}/P^N)$.

Proof: Easy.

Theorem 3.2 *If M is saturated and of power $\lambda > |T|$, then M is λ-primary over P^M (i.e., $M = \{a_i: i < \alpha\}$ where for each $j < \alpha$ $tp(a_j/P^M \cup \{a_i: i < j\})$ is λ-isolated).*

Proof: Let $M' = \{a_i: i < \lambda\}$. By Fact 3.1, for any $j < \lambda$ $tp(\langle a_i: i \le j\rangle/P^M)$ is isolated over a set of power $<\lambda$. Thus $tp(a_j/P^M \cup \{a_i: i < j\})$ is λ-*isolated.*

For A any subset of $\mathfrak{C}$ we let P^A denote the set of elements of A satisfying $P(x)$ and A^P the corresponding substructure of $\mathfrak{C}$.

Definition 3.3 $A \subset \mathfrak{C}$ is said to be *complete* if whenever $\models \exists x(\phi(x, \bar{a}) \wedge Px)$, $\bar{a} \in A$ then for some $b \in P^A \models \phi(b, \bar{a})$.

Note that by virtue of Assumption 2.6, the completeness of some $A \subset \mathfrak{C}$ is a function of the theory of A (as a structure in its own right).

Remark 3.4:
(1) If $A \subset \mathfrak{C}$ is complete then $A^P \prec \mathfrak{C}^P$.
(2) If $M \models T$, $A \subset M$, $A \supset P^M$ then A is complete.
(3) If A, T are countable then A is complete iff there is $M \models T$, $M \supset A$ with $P^M = P^A$ (by omitting types).
(4) If $A^P \prec \mathfrak{C}^P$ then A is complete if and only if for every $\bar{a} \in M$ and $\phi(\bar{x}, \bar{y}) \in L$ there is $\bar{c} \in P^A$ such that $\models \theta_\phi(\bar{a}, \bar{c})$.

Theorem 3.5 *Let $\lambda = \lambda^{<\lambda} \ge |T|$. Let $|A| = \lambda$, A complete and A^P λ-compact. Then there is a λ-compact $M \supseteq A$ ($M \models T$) with $M^P = A^P$. (Similarly if we replace λ-compact by λ-saturated.)*

Proof: We first make the following claim:

Claim *Let $p(\bar{y})$ be a consistent type over A, $p(\bar{y}) \vdash \bar{y} \in P$, $|p| < \lambda$. Then p is realized by some $\bar{c} \in A$.*

Proof of claim: For each $\phi(\bar{a}, \bar{y}) \in p(\bar{y})$ choose (by 3.4(4)) $\bar{c}_{\phi,\bar{a}} \in A^P$ such that $\models \theta_\phi(\bar{a}, \bar{c}_{\phi,\bar{a}})$.

It is clear that $p(\bar{y})$ is equivalent to

$$q(\bar{y}) = \{\psi_\phi(\bar{y}, \bar{c}_{\phi,\bar{a}}): \phi \in L, \bar{a} \in A, \phi(\bar{a}, \bar{y}) \in p(\bar{y})\} .$$

By the λ-compactness of A^P (and Assumption 2.6), q is realized in A^P. So the claim is proved.

The theorem now follows by a standard Henkin construction.

So the property of A being *complete* "formalizes" the property that there is $M \supseteq A$ $M^P = A^P$ (and, as above, corresponds to it for suitable A). Similarly we wish to "formalize" the notion of a type p over a complete A being realized in such an M, and stability comes in as the study of the space of such types for suitable A.

Definition 3.6 Let A be complete. $S_*(A) = \{p(\bar{x}) \in S(A): p(\bar{x}) \vdash \bigwedge_i \neg P(x_i)$ and for some (any) realization $\bar{a}$ of p, $A \cup \bar{a}$ is complete$\}$.

We develop the stable/unstable dichotomy. In fact for complete A we will define the notion "A is stable" and define suitable ranks on types over A. This will be done in a context rather more general than that exploited in this paper.

Definition 3.7 Let A be complete, $m < \omega$, $\bar{x}$ a fixed m-tuple of variables, Δ_1 a set of L-formulas of the form $\phi(\bar{x}, \bar{y})$ (any $\bar{y}$), Δ_2 a set of L-formulas of the form $\phi(\bar{x}, \bar{y}, \bar{z})$ (any $\bar{y}$, $\bar{z}$), and $p(\bar{x})$ an m-type (not necessarily complete) over A.

We define $R^n_A(p, \Delta_1, \Delta_2, 2)$ (which we shall call here $R_A(p)$ as Δ_1, Δ_2, m are fixed).

So by induction $R_A(p) \geq \alpha$ is defined as follows:

(i) $R_A(p) \geq 0$ if $p(\bar{x}) \cup \bigwedge_{i=1}^{m} \neg Px_i$ is consistent.
(ii) For δ limit, $R_A(p) \geq \delta$ if $R_A(p) \geq \alpha$ $\forall \alpha < \delta$.
(iii) For α *even*, $R_A(p) \geq \alpha + 1$ if for every finite $q \subset p$ there are $r_0(\bar{x})$, $r_1(\bar{x})$ explicitly contradictory Δ_1-types over A such that

$$R_A(q \cup r_i) \geq \alpha \text{ for } i < 2 \ .$$

(iv) For α *odd*, $R_A(p) \geq \alpha + 1$ if for every finite $q \subseteq p$, $\phi(\bar{x}, \bar{y}, \bar{z}) \in \Delta_2$ and $\bar{b} \in A$ there is $\bar{d} \in P^A$ such that

$$R_A(q \cup \{\forall \bar{z} \in P(\phi(\bar{x}, \bar{b}, \bar{z}) \leftrightarrow \psi_\phi(\bar{z}, \bar{d})\}) \geq \alpha \ .$$

Explanation: We will be interested in $S_*(A)$ for complete A. In particular we want the existence of $p \in S(A)$ with $R(p) = \infty$ to give rise to many members of $S_*(A)$. This is where clause (iv) above comes in.

The following are trivial (for 3.9 keep in mind Assumption 2.6).

Lemma 3.8
(i) *Let A be complete, $p(\bar{x})$, $q(x) \in S(A)$ $p(\bar{x}) \subset q(\bar{x})$, $\Delta_1 \supseteq \Delta_1'$, $\Delta_2 \subseteq \Delta_2'$ then* $R_A(p, \Delta_1, \Delta_2, 2) \geq R_A(q, \Delta_1, \Delta_2, 2)$
(ii) *any $p(\bar{x})$ over (complete) A has a finite subtype q with $R_A(q, \Delta_1, \Delta_2, 2) = R_A(p, \Delta_1, \Delta_2, 2)$.*

Lemma 3.9 *For finite Δ_1, Δ_2, n and $\phi(\bar{x}, \bar{y}) \in L$ there is $\theta(\bar{y})$ such that for any complete A, $\bar{a} \in A$, $R_A(\phi(\bar{x}, \bar{a}), \Delta_1, \Delta_2, 2) \geq n$ iff $A \models \theta(\bar{a})$.*

Lemma 3.10 *Let A be complete, $p(\bar{x}) \in S_*(A)$, Δ_1, Δ_2 arbitrary, then*

$$R_A(p, \Delta_1, \Delta_2, 2) \textit{ is even or } \infty \ .$$

Proof: Suppose $R_A(p, \Delta_1, \Delta_2, 2) = \alpha < \infty$ and α is odd. Choose $\phi(\bar{x}, \bar{y}, \bar{z}) \in \Delta_2$, $\bar{b} \in A$. As $p(\bar{x}) \in S_*(A)$ there is $\bar{d} \in P^A$ such that $\forall \bar{z} \in P(\phi(\bar{x}, \bar{b}, \bar{z}) \leftrightarrow \psi_\phi(\bar{z}, \bar{d})) \in p(\bar{x})$. But then, by (iv) of Definition 3.7, clearly $R_A(p, \Delta_1, \Delta_2, 2) \geq \alpha + 1$, a contradiction. So α is even.

Lemma 3.11 *Let A be complete, $p(\bar{x}) \in S_*(A)$, $\Delta_1 = \{\phi(\bar{x}, \bar{y})\}$ and $R_A(p, \Delta_1, \Delta_2, 2) < \omega$ for some finite Δ_2. Then $p \restriction \phi$ is definable over A in the* *sense that there is* $\chi(\bar{y}, \bar{c})$, $\bar{c} \in A$ *such that for any* $\bar{b} \in A$, $\phi(\bar{x}, \bar{b}) \in p$ *iff* $A \models \chi(\bar{b}, \bar{c})$.

Proof: Let $R_A(p, \Delta_1, \Delta_2, 2) = n < \omega$ (so n is even). Let $p_0 \subset p$ be finite such that

$$R_A(p_0, \Delta_1, \Delta_2, 2) = n$$

Let $\bar{b} \in A$. Clearly $\phi(\bar{x}, \bar{b}) \in p$ iff $R_A(p_0 \cup \phi(\bar{x}, \bar{b}), \Delta_1, \Delta_2, 2) > n$.
Now use Lemma 3.9.

Definition 3.12
(i) Let $p(\bar{x})$ be over A, A complete. p *is* Δ_1*-big for* A if $R_A(p, \Delta_1, \Delta_2, 2) \geq \omega$ for all finite Δ_2.
(ii) Let A be complete. A is *unstable* if for *some* finite Δ_1 $\{\bar{x} = \bar{x}\}$ is Δ_1-big for A.

Remark: For a fixed complete A, the interesting property is rather $-A'$ is stable for every $A' \equiv A$.

Lemma 3.13 *Let A be complete and stable. Then*

$$|S_*(A)| \leq |A|^{|T|} .$$

Proof: Let $p(\bar{x}) \in S_*(A)$. For each $\phi(\bar{x}, \bar{y}) \in L$, let Δ_2 be finite such that $R_A(p, \{\phi(\bar{x}, \bar{y})\}, \Delta_2, 2) < \omega$.

By 3.11, $p \restriction \phi$ is "definable over A". As ϕ is arbitrary p is "definable over A". Clearly the proposition follows.

Lemma 3.14 *Let A be complete,* λ*-compact and unstable with* $|A| = \lambda \geq |T|$. *Then* $|S_*(A)| = 2^\lambda$. *Moreover there are* 2^λ *types in* $S_*(A)$ *which are* Δ_1*-contradictory for some finite* Δ_1.

Proof: Let Δ_1 be finite such that $\bar{x} = \bar{x}$ is Δ_1-big for A. So $R_A(\bar{x} = \bar{x}, \Delta_1, \Delta_2, 2) \geq \omega$ for all finite Δ_2. Without loss of generality Δ_1 is one formula $\phi(\bar{x}, \bar{y})$.

List all pairs $\langle \phi(\bar{x}, \bar{y}', \bar{z}'), \bar{b} \rangle$, $\phi' \in L$, $\bar{b} \in A$ as

$$\{\langle \phi_\alpha(\bar{x}, \bar{y}_\alpha, \bar{z}_\alpha), \bar{b}_\alpha \rangle : \alpha < \lambda\} .$$

We define, for $\alpha < \lambda$ and $\eta \in {}^\alpha 2$ types $p_\eta(\bar{x})$ over A, as follows:

(1) $p_{\langle\rangle} = $ "$\bar{x} = \bar{x}$"
(2) for α limit, $\eta \in {}^\alpha 2$, $p_\eta = \bigcup_{\beta<\alpha} p_{\eta \restriction \beta}$
(3) given p_η, $\eta \in {}^\alpha 2$, then $p_\eta^0 = p_\eta \cup \{\phi(\bar{x}, \bar{c})\}$ $p_\eta^1 = p_n \cup \{\neg\phi(\bar{x}, \bar{c})\}$ for some $\bar{c} \in A$, and $p_{\eta^\frown\langle i \rangle} = p_\eta^i \cup \{\forall \bar{z} \in P(\phi_\alpha(\bar{x}, \bar{b}_\alpha, \bar{z}_\alpha) \leftrightarrow \psi_{\phi_\alpha}(\bar{z}_\alpha, \bar{d}_\alpha^i))\}$ $i < 2$, for some $\bar{d}_\alpha^i \in P^A$ $i = 0, 1$; *and* $R_A(p_{\eta^\frown\langle i \rangle}, \{\phi(\bar{x}, \bar{y}\}, \Delta_2, 2) \geq \omega$ for $i = 0, 1$, all finite Δ_2.

Clearly if this can be done, then for each $\eta \in {}^\lambda 2$ let $p_\eta = \bigcup_{\alpha<\lambda} p_{\eta \restriction \alpha}$, and complete each p_η to some $p_\eta^* \in S(A)$. By (3) each $p_\eta^* \in S_*(A)$ and, $\eta_1 \neq \eta_2$ implies $p_{\eta_1}^* \neq p_{\eta_2}^*$.

So we must just check that (3) can be accomplished. So let $p_\eta(\eta \in {}^\alpha 2)$ be given. By induction, $|p_\eta| < \aleph_0 + |l(\eta)|^+ \leq \lambda$ and $R_A(p_\eta, \{\phi(\bar{x}, \bar{y})\}, \Delta_2, 2) \geq \omega$ for all finite Δ_2.

By λ-compactness of A and Lemmas 3.8 and 3.9 we can find $\bar{c}$ such that

$$R_A(p_\eta \cup \{\phi(\bar{x}, \bar{c})\}, \{\phi(\bar{x}, \bar{y})\}, \Delta_2) \geq \omega \text{ for all finite } \Delta_2$$

and

$$R_A(p_\eta \cup \{\neg\phi(\bar{x}, \bar{c})\}, \{\phi(\bar{x}, \bar{y})\}, \Delta_2) \geq \omega \text{ for all finite } \Delta_2.$$

Similarly, by considering those Δ_2 containing $\phi_\alpha(\bar{x}, \bar{y}_\alpha, \bar{z}_\alpha)$ we can find $p_{\eta^\frown\langle 0\rangle}, p_{\eta^\frown\langle 1\rangle}$ as required.

Corollary 3.15 *Suppose $\lambda = \lambda^{<\lambda} \geq |T|$ and that some model of T is unstable. Then there are $M \models T$, $M \prec N$ with $P^M = P^N$, $|M| = |P^M| = \lambda$, $|N| = \lambda^+$.*

Proof: Clearly if some model is unstable, then so is every model (by Lemma 3.9). As $\lambda = \lambda^{<\lambda}$ let $M \models T$ be λ-compact of power λ. Clearly M^P is also λ-compact of power λ. By 3.14 $|S_*(M)| = 2^\lambda$ so there is $p(\bar{x}) \in S_*(M)$ not realized in M. Let $\bar{a}$ realize $p(\bar{x})$. So $M \cup \bar{a}$ is complete and $(M \cup \bar{a})^P = M^P$ is λ-compact. So by Theorem 3.5 there is $M_1 \supset M \cup \bar{a}$, $M_1^P = M^P$, M_1 λ-compact of cardinality λ. By continuing this way and using Theorem 3.5 (also at limit stages) we get $N \succ M$ as required. (Note N will also be λ-compact by our construction.)

Theorem 3.16 *Suppose some model of T is unstable, and that $\lambda = \lambda^{<\lambda} \geq |T|$. Then there is $M \models T$, M λ-compact, $|M| = \lambda^+$, $|P^M| = \lambda$ such that one of the following holds:*

(1) *there is $N \prec M$, $|N| = \lambda$, $P^M \subset N$ and finite Δ_1 such that M realizes λ^+ pairwise distinct Δ_1-types over N.*

(2) *M can be written as $M = \bigcup_{\alpha<\lambda^+} M_\alpha$ where $M_\alpha^P = M^P$, $\forall\alpha$.*

M_α is a strictly increasing elementary chain, for $cf\,\alpha = \lambda$ $M_\alpha = \bigcup_{i<\alpha} M_i$, and there is a formula $\phi(\bar{x}, \bar{y}) \in L$ and for $\alpha < \beta < \lambda^+$ a set I_α^β of sequences of the form $\bar{a}^\frown\bar{b}^\frown\bar{c}(l(\bar{a}) = l(\bar{x}), l(\bar{b}) = l(\bar{c}) = l(\bar{y}))$, $I_\alpha^\beta \subseteq M_\beta - M_\alpha$ such that for every $\alpha < \lambda^+$ with $cf\,\alpha = \lambda$, and $\bar{d} \in I_\alpha^{\alpha+1}$ there are β_i $i < \lambda$ with $\bigcup_i \beta_i = \alpha$ and for each $i < \lambda$ there are $\bar{d}_i = \bar{a}_i{}^\frown\bar{b}_i{}^\frown\bar{c}_i \in I_{\beta_i}^{\beta_i+1}$ such that for each $i, j \leq \lambda$

$$i \leq j \text{ iff } \models \psi(\bar{d}_i, \bar{d}_j)$$

where $\psi(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{x}_2, \bar{y}_2, \bar{z}_2)$ is $\neg(\phi(\bar{x}_2, \bar{y}_1) \leftrightarrow \phi(\bar{x}_2, \bar{z}_1))$.

Proof: Let us suppose that there is no suitable M satisfying (1) and we show that (2) holds.

Let us fix $\phi(\bar{x}, \bar{y}) \in L$ such that $\Delta_1 = \{\phi(\bar{x}, \bar{y})\}$ witnesses the instability of some (every) model.

Let $\theta(\bar{y}, \bar{x})$ be $\phi(\bar{x}, \bar{y})$ and we shall say that $p(\bar{x})$ (ϕ, θ) splits over B if there are $\bar{d}_0\bar{d}_1$, with $tp_\theta(\bar{d}_0/B) = tp_\theta(\bar{d}_1/B)$ but $\phi(x, \bar{d}_0) \in p$, $\neg\phi(x, \bar{d}_1) \in p$.

As in Corollary 3.15 we start with λ-compact M_0 of power λ and define inductively $M_\alpha\ \alpha < \lambda^+$, $\bar{a}_\alpha \in M_{\alpha+1} - M_\alpha$ with $M_\alpha^P = M_0^P$, each M_α λ-compact of power λ, and also satisfying

(i) if $cf\,\alpha = \lambda$ then $M_\alpha = \bigcup_{i<\alpha} M_i$

(ii) for all $j < \lambda^+$, if $P^M \subseteq M_{j+1} \prec M$ then every ϕ-type or θ-type over M_j realized in M is realized in M_{j+1}.

(iii) If $cf\,\alpha = \lambda$ then $tp(\bar{a}_\alpha/M_\alpha)$ (ϕ, θ)-splits over M_j, $\forall j < \alpha$.

We should just check that (ii) and (iii) are obtainable. So we start with (ii). Suppose M_j is given. If M_{j+1} as in (ii) cannot be found then we can clearly define λ-compact $M_{j,\xi}$ of cardinality λ for $\xi < \lambda^+$ and $\bar{a}_\xi \in M_{j,\xi+1}$ such that $P^{M_{j,\xi}} = P^{M_j}$ $\forall\xi$, the $M_{j,\xi}$ are increasing with ξ and either $tp_\phi(\bar{a}_\xi/M_j)$ or $tp_\theta(\bar{a}_\xi/M_j)$ is not realized in $M_{j,\xi}$. Then clearly $\bigcup_{\xi<\lambda^+} M_{j,\xi}$ satisfies (1) of the theorem. So M_{j+1} can be found.

Now for (iii). Let in fact $\alpha < \lambda^+$ be limit and let $j < \alpha$, and $p(\bar{x}) \in S_*(M_\alpha)$ not (ϕ, θ)-split over M_j.

As, by (ii), every θ-type over M_j realized in M_α is realized in M_{j+1} it follows that $p \restriction \phi$ is determined by $(p \restriction \phi) \restriction M_{j+1}$, which in turn is by (ii) realized in M_{j+2}. Thus, there are at most λ ϕ-types of such p. On the other hand, by 3.14 there are 2^λ $p(\bar{x}) \in S_*(M_\alpha)$ which are ϕ-distinct. Thus we can find $p(\bar{x}) \in S_*(M_\alpha)$ which (ϕ, θ)-splits over M_j $\forall j < \alpha$. Let $\bar{a}_\alpha$ realize p.

Now, for $\alpha < \beta < \lambda^+$ let $I_\alpha^\beta = \{\bar{a} \frown \bar{b} \frown \bar{c}$: $\bar{a}, \bar{b}, \bar{c} \in M_\beta - M_\alpha$, $tp(\bar{a}/M_\alpha)$ (ϕ, θ) splits over M_j $\forall j < \alpha$ and $tp_\theta(\bar{b}/M_\alpha) = tp_\theta(\bar{c}/M_\alpha)\}$. Note that if $cf\,\alpha = \lambda$ then $I_\alpha^{\alpha+1} \neq \emptyset$ ($\bar{a}_\alpha \frown \bar{b} \frown \bar{c} \in I_\alpha^{\alpha+1}$ for suitable $\bar{b}$, $\bar{c}$).

Now we show that the conclusion of (2) holds for the I_α^β. Let $cf\,\alpha = \lambda$. Put $\bar{d}_\lambda = \bar{a}_\alpha \frown \bar{b} \frown \bar{c} \in I_\alpha^{\alpha+1}$. Now $tp(\bar{a}_\alpha/M_\alpha)$ (ϕ, θ)-splits over M_j $\forall j < \alpha$. Thus $\left(\text{as } M_\alpha = \bigcup_{j<\alpha} M_j\right)$ we can define inductively $\beta_i < \alpha$, $\forall i < \lambda$, and $\bar{d}_i = \bar{a}_i \frown \bar{b}_i \frown \bar{c}_i \in I_{\beta_i}^{\beta_i+1}$ such that $\bar{b}_i$, $\bar{c}_i$ witness the (ϕ, θ)-splitting of $tp(\bar{a}_\alpha/M_\alpha)$ over M_{β_i} and $tp_\phi(\bar{a}_i/M_{\beta_i} \cup \{\bar{b}_i, \bar{c}_i\}) = tp_\phi(\bar{a}_\alpha/M_{\beta_i} \cup \{\bar{b}_i, \bar{c}_i\})$ (*). Now note that if $i \leq j \leq \lambda$ then

$$\models \neg(\phi(\bar{a}_j, \bar{b}_i) \leftrightarrow \phi(\bar{a}_j, \bar{c}_i)) \qquad \text{(by (*) and induction)}$$

and if $j < i \leq \lambda$ then $\models \phi(\bar{a}_j, \bar{b}_i) \leftrightarrow \phi(\bar{a}_j, \bar{c}_i)$, as $tp_\theta(\bar{b}_i/M_{\beta_{j+1}}) = tp_\theta(\bar{c}_i/M_{\beta_{j+1}})$.

So the theorem is proved.

Theorem 3.17 *Suppose that (2) of Theorem 3.16 holds. Let $\mu \geq |T|$, $\mu^{<\aleph_\alpha} = \mu$ and $\exists\lambda \geq \mu(\lambda = \lambda^{<\lambda} > |T|)$. Then $I(\mu, \mu) \geq |\alpha|$.*

Proof: For the given λ let M and M_α, $\alpha < \lambda^+$ be as in (2) of Theorem 3.16. Let us fix $\alpha < \lambda^+$ with $cf\,\alpha = \lambda$ and fix $\beta_i\ i < \lambda$ as in Theorem 3.16.

Let $\aleph_\beta < \aleph_\alpha$ be regular, and let $M^\beta = \bigcup_{i<\aleph_\beta} M_{\beta_i}$.

Claim I *M^β is not $\aleph_\beta^+$-saturated, but is $\aleph_\beta$-saturated.*

Proof: Note that M^β *omits* the (consistent) set $\{\psi(\bar{d}_i, \bar{w})$: $i < \aleph_\beta\}$, so is *not* $\aleph_\beta^+$-saturated.

On the other hand, each M_{β_i} $i < \aleph_\beta$ is λ-saturated, so $\aleph_\beta$-saturated, so as $\aleph_\beta$ is regular M^β is $\aleph_\beta$-saturated.

Claim II *For each regular* $\aleph_\beta < \aleph_\alpha$ *there is* $N^\beta \prec M^\beta$ *such that* $|N^\beta| = |P^{N^\beta}| = \mu$. N^β *is* $\aleph_\beta$*-saturated, but not* $\aleph_\beta^+$*-saturated.*

Proof: For any $i < \aleph_\beta$ and $X \subset M_{\beta_i}$ with $|X| \leq \mu$ there is $N_{\beta_i} \prec M_{\alpha_i}$, $N_{\beta_i} \supseteq X$ with $|N_{\beta_i}| = |P^{N_{\beta_i}}| = \mu$ and N_{β_i} is $\aleph_\beta$-saturated. This is because M_{β_i} is λ-saturated, $\aleph_\beta \leq \lambda$ and $\mu^{\aleph_\beta} = \mu$.

Now we can obtain an increasing chain N_{β_i} $i < \aleph_\beta$ and put $N^\beta = \bigcup_{i<\aleph_\beta} N_{\beta_i}$ and we can also specify that $\bar{d}_i \in N^\beta$ $\forall i < \aleph_\beta$. N^β clearly satisfies Claim II.

Finally, we can easily choose the N^β, $\beta < \alpha$ such that $|N^\beta| = |N^\gamma|$ $\forall \beta, \gamma$.

Clearly then $I(\mu, \mu) \geq |\alpha|$.

Theorem 3.18 *Suppose (1) of Theorem 3.16 holds for* $\lambda = \aleph_0$, (*so T is countable*), *and also* $2^{\aleph_0} < 2^{\aleph_1}$. *Then T has* $2^{\aleph_1}$ *models of power* $\aleph_1$ *with the same P-part.*

Proof: Add constants for N to get a language L'. Let ψ be the $L'_{\omega_1\omega}$ sentence saying $\bigwedge Th(M, a)_{a\in N} \wedge (\forall x)\left(Px \leftrightarrow \bigvee_{a\in P^M} x = a\right)$. So ψ has a model realizing uncountably many types (namely M). By Keisler [4], ψ has 2^{ω_1} nonisomorphic models of power ω_1. These models clearly have the same P-part. As $2^{\aleph_0} < 2^{\aleph_1}$, $2^{\aleph_1}$ of their L-reducts are not isomorphic.

The methods of [8] can be adapted to prove

Theorem 3.19 *Suppose (1) of Theorem 3.16 holds for* $\lambda = \lambda^{<\lambda} > |T|$ *and assume* $\diamondsuit_\lambda$, $2^\lambda < 2^{\lambda^+}$. *Then there are* 2^{λ^+} *nonisomorphic models of power* λ^+ *of T with the same P-part.*

NOTE

1. Some remarks are in order concerning the history of this paper. In late 1975 the second author sent, on request, to the first author handwritten notes on the subject-matter of this paper. In 1979 the second author had these notes typed up into a paper which was circulated and also submitted to this Journal. In 1983 this paper found its way back into the hands of the first author who then rewrote the paper into its present form.

   Some earlier work on the same general topic was done by Gaifman (e.g., [2]) who characterized the property that every model of T is unique and rigid over its P-part. Other work was done by Hodges [3] who studied categoricity over a predicate in the case of Abelian groups with a distinguished subgroup, and Pillay [5] who characterized syntactically $\aleph_0$-categoricity over a predicate.

REFERENCES

[1] Chang, C. C. and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[2] Gaifman, H., "Characterisations of uniqueness and rigidity properties," preprint.

[3] Hodges, W. A., "Relative categoricity in Abelian groups," preprint.

[4] Keisler, H. J., *Model Theory for Infinitary Logic*, North-Holland, Amsterdam, 1971.

[5] Pillay, Anand, "$\aleph_0$-categoricity over a predicate," *Notre Dame Journal of Formal Logic*, vol. 24 (1983), pp. 527-536.

[6] Shelah, Saharon, "Remark to 'Local definability theory' of Reyes," *Annals of Mathematical Logic*, vol. 2 (1970), pp. 441-447.

[7] Shelah, Saharon, *Classification Theory and the Number of Non-Isomorphic Models*, North-Holland, Amsterdam, 1978.

[8] Shelah, Saharon, "Models with second order properties III. Omitting types in $L(Q)$," *Archiv für Mathematische Logic und Grundlagenforschung*, vol. 21 (1980), pp. 1-11.

[9] Shelah, Saharon, "Classification theory for non-elementary classes I. The number of uncountable models of $\psi \in L_{\omega_1 \omega}$. Parts A, B," *Israel Journal of Mathematics*, vol. 46 (1983), pp. 212-273.

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