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Intension, Designation, and Extension

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It is not our business to set up prohibitions,
but to arrive at conventions.
Carnap[1]

Introduction This paper presents a semantical theory for free S_5 in all finite types. That is, the system of logic determined by the theory is a modal logic which allows quantification over individuals, propositions, properties of individuals, properties of propositions, properties of properties of individuals and propositions, and so on, without limit. Because the logic is free, it is not presupposed that any individuals exist and singular terms may be used even if they do not stand for anything.

The following features of the theory are worthy of note: (1) it permits one to formalize Descartes' version of the ontological argument; (2) it does not commit one to asserting that A and B express the same proposition as soon as one asserts $\Box(A \equiv B)$ (the box should be glossed as "necessarily"); (3) the theory

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Errors and infelicities are, of course, my responsibility.

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asserts that there is exactly one proposition which is its own negation, but, nevertheless, the theory is consistent.

In outline, the developments of the paper are as follows:

Section 2: Features (1)–(3) are discussed informally, and the theory presented here is compared with those of Frege and Montague.

Section 3: The formal language to be studied is officially introduced, and some fundamental semantical concepts are formally defined. It turns out that the unusual features of the theory being presented stem from the thoroughgoing use of partial functions to model semantical concepts. Why things are arranged this way is explained.

Section 4: Building on what has been done in Section 3, a semiformal treatment of features (1)–(3) is given. Sections 5–7 provide the definitions necessary in order to turn the semiformal arguments of Section 4 into proofs.

Section 5: The propositional calculus—the fragment of the theory which deals with formulas built up from propositional constants and propositional variables by means of connectives—is presented.

Because of the use of partial functions noted above, a system of defaults is employed to ensure that, ultimately, formulas will get truth values. Validity of formulas is then defined in such a way as to be independent of any particular choice of defaults. But before giving the definition of validity, it must be explained how the defaults operate. The required explanation is complex, so in presenting it attention is at first restricted to the propositional calculus.

Sections 6 and 7: The definitions of Section 5 are extended to cover the full system.

Section 8: Possible ways of developing the theory are discussed, and an application to set theory is given.

2 Three semantical theories Table 2.1 compares the theory presented here with those of Frege[2] and Montague.[3] The motive which led to the construction of the theory was to obtain "Yes" as an answer to the question: "Is there a theory which enables one to formalize Descartes' ontological argument?". Reasons for wanting to construct such a theory are given in Section 2.3. How this motive led to the construction of a theory having the other features to be discussed is explained in Section 3.4.

As the top section of Table 2.1 shows, the theory presented in this paper has three semantical categories: intension, designation, and extension. The other theories, of course, have two. In the theory being presented, a formula designates a proposition (if anything). Thus, the category called "designation" here is similar to what Frege called "sense" and what Montague called "intension".

The second section of the table shows that all three theories are *resolutely* two valued—there are no truth values besides the True and the False.[4] On the other hand, the theory presented here differs radically from the other two in its account of how the two truth values are assigned to formulas. A formula may not designate a proposition at all, and, even if it does, the proposition may not

Table 2.1. Box Score

	Frege's Theory	Montague's Theory	Theory Being Presented
Semantical categories	Sense, reference.	Intension, extension.	Intension, designation, extension.
Treatment of formulas	A has a sense. Sense of A determines reference of A. Reference of A is T or F.	A has an intension and an extension. Intension of A determines extension of A. Extension of A is T or F.	A has an intension. If A has a designation, intension of A determines it. Designation of A or a default determines extension of A. Extension of A is T or F.
Propositionist?	Yes	Yes	No
Formally consistent?	No	Yes	Yes
Extensionist?	No	Yes	No
Formalizes ontological argument?	No	No	Yes

determine a truth value. If either of these situations arises, a system of defaults assigns a truth value to the formula in the way to be explained below.[5]

2.1 Propositionism and the Russell proposition Propositionism is the view that the truth value of A is determined by the proposition A expresses. It is evident from the preceding remarks that propositionism is rejected by the theory of this paper. No doubt rejecting propositionism will seem a strange thing to do. Certainly, it is unusual. Section 3.4 argues that it is sensible.

The Russell proposition is the proposition which is its own negation. According to the theory presented here, the Russell proposition exists. Very likely, asserting this will seem horrible, rather than merely strange. Won't asserting it lead to inconsistency?

No. Asserting that the Russell proposition exists does not, by itself, lead to inconsistency. Inconsistency arises if one combines this assertion with propositionism, as the following argument shows.

Let '**R**' express the Russell proposition.

1. '**R**' expresses the same proposition as '~**R**.'
2. Hence, '**R**' has the same truth value as '~**R**.'
3. But either the truth value of '**R**' is T and the truth value of '~**R**' is F, or the truth value of '**R**' is F and the truth value of '~**R**' is T.
4. Hence, '**R**' does not have the same truth value as '~**R**.'
5. Therefore, '**R**' has the same truth value as '~**R**,' and '**R**' does not have the same truth value as '~**R**.'

Evidently, the argument fails if the step leading from 1 to 2 fails. But what, if not propositionism, licenses this inference? The developments which follow show that the answer is "Nothing".

"Russell proposition" is a suggestive piece of terminology. It is *not* intended to suggest that a new foundation for mathematics is being presented here—nothing of the sort is going on. It *is* intended to suggest that the following story might be true.

> What happened back in 1902 was that Russell discovered the proposition which is its own negation and communicated it to Frege. Because, on the one hand, that proposition was expressible in Frege's system and, on the other, Frege had built the system so that every formula would have a proposition ('*gedanke*' in his terminology) as its sense and every proposition expressed by a formula would determine the truth value of the formula expressing it, Frege's system was inconsistent.

More will be added to the story in Section 4.1, and a way of using the system to discuss set abstraction and the Russell paradox will be indicated in Section 8.2. But it will be clear that the scheme laid out in Section 8.2 provides a new way of looking at a known foundation for mathematics (the cumulative type hierarchy), not a new foundation for mathematics. The disclaimer issued above is to be taken seriously.

2.2 Extensionism Extensionism is the view that if $\Box(A \equiv B)$ is true, then A and B express the same proposition. Extensionist semantical theories rule out the possibility of construing belief as a relation between individuals and propositions, because doing so would, for example, lead to the conclusion that the following argument is valid:

1. Diophantus believed that 2 is even.
2. $\Box$(2 is even $\equiv$ every real-valued function continuous on a closed interval in the reals is uniformly continuous on that interval).
3. Hence, Diophantus believed that every real-valued function continuous on a closed interval in the reals is uniformly continuous on that interval.

Clearly, the argument is invalid. Diophantus was the greatest number theorist of antiquity and the right half of premise 2 is a famous old theorem of classical analysis, so the premises are true. On the other hand, the Greeks were so bad dealing with real numbers that the conclusion is certainly false.

It may be that construing belief as a relation between persons and propositions is not the right thing to do, but *logic* shouldn't rule out the possibility of constructing theories in which belief is such a relation. In order to construct such theories, a system of logic which rejects extensionism is required. The theory presented here does reject it—the example given in Section 4.2 shows that the theory is *inconsistent* with extensionism.[6]

2.3 The ontological argument As was remarked above, the motive which led to the construction of the theory being presented was a desire to devise a system of logic which would allow one to formalize Descartes' version of the ontological argument. It will now be argued that the philosophical importance of that argument is sufficient to justify the project. (The argument is also venerable and fun to think about, but those are separate matters.)

Assessing the claims of natural theology is a philosophically important task. The strongest argument for a negative assessment is the central argument of [8], which runs as follows: The existence of God cannot be proved a priori. Hence, if God's existence is to be proved, it must be proved by means of an inductive argument. But, given the evidence available to us and given that the principle of total evidence must be applied, it is clear that no such argument can succeed. Therefore, the existence of God cannot be proved.[7]

If one is convinced that the first premise is true, the argument is convincing. But Hume's argument for that premise ([8], p. 189) is problematic—if it were correct, it would show that that the number two might not exist, for example. Consequently, if one is to figure out what to make of Hume's argument, one must make a survey of the available a priori arguments and try to decide whether they make their point. The ontological argument is the best such argument, so the problem reduces to evaluating it.

The foregoing considerations led to the work which resulted in [17]. The following version of Anselm's argument[8] is analyzed there.

1. God is the being than which none greater can be conceived.
2. If God doesn't exist, then a being greater than God is conceivable.
3. Hence, God exists.

The analysis is satisfying for the following reasons. First, it turned out that there was a weakest way of expressing "a being greater than God is conceivable" in free T, but the resulting argument was invalid in free S_5. Thus, using the strongest possible construal of "God is the being than which none greater can be conceived" and the other premise and using the strongest logic, it turned out that the argument was invalid. On the other hand, adding the premise that God's existence is possible (Leibniz's premise[9]) produced an argument which was valid in free S_5 but not in the weaker systems. Since it seems impossible to decide whether the extra premise is true, [17] provides quite strong evidence that Hume was right, despite the problematic character of his argument to the effect that the existence of God cannot be proved a priori.

The unsatisfying thing about the analysis given in [17] is that it cannot be applied to Descartes's version of the argument, which runs as follows:

1. God is the being which has all perfections.
2. Existence is a perfection.
3. Therefore, God exists.

As soon as one says "God is the being which has all perfections", one is stuck with second-order quantification. The work reported here makes it possible to formalize Descartes' argument, and things turn out as they did with Anselm's argument.

3 Language, metalanguage, and basic semantical concepts This section lays out the basics of the formal apparatus to be considered and motivates the characteristics which lead to the features discussed in Sections 2.1 and 2.2. Because the concepts are unusual, considerable commentary is included along with the definitions.

3.1 The language A version of the simple theory of types is employed. To begin with, type symbols are introduced. The type structure these allow is, in fact, more general than the remarks of the introduction suggest. Besides individuals and propositions, properties of individuals and propositions, properties of such properties, etc., the structure includes n-nary relations where the terms of such a relation need not all be of the same type.

A version of the simple theory of types is employed.

Type symbols TS, the set of type symbols, is the smallest set satisfying the following conditions:

1. $\iota \in TS$.
2. If $\tau_1, \ldots, \tau_n \in TS$ $(n \geq 0)$, then $[\tau_1 \ldots \tau_n] \in TS$.

ι is the individual type symbol, and $[\tau_1 \ldots \tau_n]$ is a relation type symbol.

According to the specifications of Section 3.3, properties will be degenerate relations. For example, $[\iota]$ will be the type symbol for properties of individuals. Also, propositions will be degenerate properties, so $[\,]$ will be the type symbol for propositions.

Logical constants

$\cong^{[\tau\tau]}$, $E^{[\tau]}$
$\cong^{[\tau\tau]}$ is the identity constant with type symbol $[\tau\tau]$
$E^{[\tau]}$ is the existence constant with type symbol $[\tau]$.

Logical operators

$\sim$, &, $\Box$, $\forall$, !
! will be used as the description operator.

Variables and nonlogical constants Denumerably many variables with type symbol τ and arbitrarily many nonlogical constants with type symbol τ, for each τ.

Terms

1. If t is a constant or variable, τ is the type symbol of t, and $\tau \neq [\,]$, then t is a *term* with type symbol τ.
2. If $t_1, \ldots, t_n$ $(n \geq 0)$ are terms with type symbols $\tau_1, \ldots, \tau_n$, respectively, and either $n = 0$ and t is a constant or variable with type symbol $[\,]$ or $n \geq 1$ and t is a term with type symbol $[\tau_1 \ldots \tau_n]$, then $tt_1 \ldots t_n$ is a *term* with type symbol $[\,]$.
3. If t and u are terms with type symbol $[\,]$ and χ is a variable, then $\sim t$, $\& tu$, $\Box t$, and $\forall \chi t$ are *terms* with type symbol $[\,]$.
4. If t is a term with type symbol $[\,]$ and χ is a variable with type symbol τ, then $!\chi t$ is a *term* with type symbol τ.

The reader may wonder why propositional constants and propositional variables (i.e., those with type symbol $[\,]$) are included in case 2 of the foregoing definition, rather than being lumped together with the constants and variables of other types in case 1. The reason is that this way of arranging the grammar of terms gives a better fit between the rules of the grammar and the forms of the recursions of Sections 5–7 than the alternative way of proceeding would allow.

Formulas The terms which have type symbol $[\,]$ are formulas. Thus, formulas are those terms which (may) designate propositions.

Henceforth, A, B, C, $A_1, \ldots$ are to be formulas. It will become clear in the sequel that understanding the treatment of the formulas produced by case 2 of the definition of terms (the *predication case*) is crucial for understanding the theory. Where $tt_1 \ldots t_n$ is such a formula, t is the *head term of* the formula and $t_1, \ldots, t_n$ are the *tail terms of* the formula. It will also become clear that the semantical theory treats head terms and tail terms *very* differently.

3.2 The metalanguage The metalanguage is Gödel-Bernays with individuals. That is, the metalanguage countenances both classes and objects which are not classes, and a distinction is drawn between *sets* (those classes which are members of classes) and *proper classes* (which are classes, but are not members of any class.) The existence of the empty set, (unordered) pair sets, powersets, unions, and an infinite set is assumed. It is postulated that the membership relation is well-founded, and that the image of a set under a function (which may be a proper class) is a set. Also, class abstraction is permitted, so long as the abstraction condition does not involve unrestricted, bound variables taking proper classes as values. The axiom of choice is not included.

The following notational conventions are employed:

1. $\langle\langle x, y\rangle\rangle = \{\{x\}, \{x, y\}\}$. That is, $\langle\langle x, y\rangle\rangle$ is the ordinary ordered pair having x and y, respectively, as its elements.
2. $\langle\langle x\rangle\rangle = x$.
3. For $n \geq 1$, $\langle\langle x_1, \ldots, x_{n+1}\rangle\rangle = \langle\langle\langle\langle x_1, \ldots, x_n\rangle\rangle, x_{n+1}\rangle\rangle$. Thus, $\langle\langle x_1, \ldots, x_{n+1}\rangle\rangle$ is the ordinary $n + 1$-tuple which has $x_1, \ldots, x_{n+1}$, respectively, as its elements.

4. $\langle x_0, \ldots, x_{n-1} \rangle$ is the finite sequence with elements $x_0, \ldots, x_{n-1}$, respectively. So $\langle x_0, \ldots, x_{n-1} \rangle$ is a function which has $\{0, \ldots, n-1\}$ as its domain and has x_i as its value for the argument i $(0 \leq i \leq n-1)$.

Most of the semantical definitions, below, proceed by means of finite sequences, rather than ordered pairs and n-tuples. But, because functions are construed as sets of ordered pairs, pairs are needed in a couple of places to handle function abstraction. That is why notations for both pairs and sequences are included.

5. Where $x_0, \ldots, x_{n-1}$ are sets, $Seq\langle x_0, \ldots, x_{n-1} \rangle$ is the set of all finite sequences y of length n such that for all i $(0 \leq i \leq n-1)$, the ith element of y is a member of x_i.
6. Where f is a function, $\underline{D}f$ is the domain of f and $\underline{R}f$ is the range of f.
7. Where x and y are sets, $x \Rightarrow y$ is the set of partial mappings from x to y (i.e., the set of functions f such that $\underline{D}f$ is a subset of x and $\underline{R}f$ is a subset of y).
8. An element f of $x \Rightarrow y$ is *total* (with respect to $x \Rightarrow y$) if, and only if, $\underline{D}f = x$.
9. Where x and y are sets, $y^x = \{f \in x \Rightarrow y : f \text{ is total}\}$.
10. $f(w)$ is the result of applying f to w, if $w \in \underline{D}f$. $f(w)$ is undefined, otherwise. This remark applies *everywhere* in what follows.
11. '=' is used in such a way that if the value of one of the expressions flanking it is undefined and the value of the other is defined, the equation is false. If the values of both expressions are undefined, the equation is true.[10]

Let x and y be sets, let $\underline{u}$ be an object which is not an element of y, and let $y_1 = y \cup \{\underline{u}\}$. it is easy to see that $x \Rightarrow y$ has the same number of elements as y_1^x. This provides a simple way of figuring out how many functions are in $x \Rightarrow y$.

3.3 Basic semantical concepts The semantical theory employs possible worlds, but, as will become apparent in what follows, it employs them in unusual ways.

Basic ontologies

$\mathbf{O} = \langle D, \# \rangle$, where D is a set-valued function and $\# \in \underline{D}D$.

1. $W_{\mathbf{O}}$ = the set of *worlds of* $\mathbf{O} = \underline{D}D$.
2. $\#$ is the *actual world of* $\mathbf{O}$.
3. $D(w)$ = the set of *individuals existing in* w.

All interpretations will be built up from basic ontologies. Each such ontology has at least one world (the actual world of that ontology), but worlds need not have any individuals existing in them ($D(w)$ may be empty).

In what follows, part a of the definition glosses in metalinguistic prose the notation being defined, and part b does the actual defining.

Designation types

1. a. $DT(\mathbf{O}, \iota)$ = the individual designation type of **O**.
   b. $DT(\mathbf{O}, \iota)$ = the union over $w \in W_{\mathbf{O}}$ of $D(w)$.

Quantification over individuals will operate on the individual designation type of **O**, rather than being restricted to some $D(w)$. It will be possible to restrict the quantifier to individuals existing in a world by using the individual existence predicate appropriately.[11] Similar remarks apply to descriptions.

2. $DT(\mathbf{O}, [\tau_1 \ldots \tau_{n-1}])$
   a. = the $[\tau_1 \ldots \tau_{n-1}]$ *relation designation type of* **O**
   b. = $Seq\langle W_{\mathbf{O}}, DT(\mathbf{O}, \tau_1), \ldots, DT(\mathbf{O}, \tau_{n-1})\rangle \Rightarrow \{T, F\}$.

That is, an element of $DT(\mathbf{O}, [\tau_1 \ldots \tau_{n-1}])$ is a function which, when applied to a sequence $\langle w, x_1, \ldots, x_{n-1}\rangle$ consisting of a world and objects of appropriate types, may yield a truth value. Of course, it may yield nothing.

3. a. $PR(\mathbf{O})$ = the *proposition type of* **O**.
   b. $PR(\mathbf{O}) = DT(\mathbf{O}, [\,])$.

Clearly, propositions are completely degenerate relations, as promised in Section 3.1. Consequently, propositions take singleton sequences of worlds, rather than worlds, as arguments. A terminological dodge will smooth out talk about such matters. Consider a proposition f and a world w. f is *true at w* iff $f(\langle w\rangle) = T$. f is *false at w* iff $f(\langle w\rangle) = F$. f is *undefined at w* iff $f(\langle w\rangle)$ is undefined.

Intension types

1. $I(\mathbf{O}, \tau)$
   a. = the τ *intension type of* **O**
   b. = $W_{\mathbf{O}} \Rightarrow DT(\mathbf{O}, \tau)$.

Thus, a τ intension is a function which, when applied to a world, may yield a τ designation. It may yield nothing, and it may yield different designations when applied to different worlds.

2. $RI(\mathbf{O}, \tau)$
   a. = the τ *rigid intension type of* **O**
   b1. = $\{0\}$, if $DT(\mathbf{O}, \tau) = 0$
   b2. = the set of total, constant members of $I(\mathbf{O}, \tau)$, otherwise.

Remark: If b1 applies, then $\tau = \iota$.

Rigid intensions will be used to interpret variables. Clause b1 ensures that it will always be possible to interpret individual variables, even if there are no individuals.

Intensional interpretations

$\dagger = \langle \mathbf{O}, \underline{I}\rangle$, where $\underline{I}$ assigns intensions to the nonlogical constants.

Sections 5 and 7 show, for each intensional interpretation $\dagger$, how to define a function $\underline{I}^{\dagger}$ which assigns intensions to all terms.

The intension of a term may be thought of as the most basic part of its meaning. Allowing that a single intension may determine different designations in different worlds is a way of taking into account the fact that different assertions may be made by using the same sentence in different semantical situations.

Classical interpretations

> $* = \langle \dagger, \langle \Delta, \nabla \rangle \rangle$, where Δ and ∇ are defaults used, if necessary, to ensure that every formula has a truth value as its extension. Δ is the *head default* of the interpretation, and ∇ is the *tail default.*

The preceding paragraph exhibits the form of classical interpretations but does not really define them, because definitions of "head default" and "tail default" have not yet been given. Formal definitions of these terms will be deferred until Sections 5.2 (for head default) and 7.2 (for tail default). The intervening semiformal discussion should help the reader understand what is going on when he encounters the full, technical explanation of what the defaults are and how they operate.

Although every formula is assigned an intension in every intensional interpretation, the intension of a formula may not determine a designation at a world, and, even if a designation is determined by the intension, the designation may not determine a truth value. It is now possible to give a simple illustration of these points.

A basic ontology $\mathbf{O} = \langle D, \# \rangle$ is *bleak* iff $\underline{D}D = \{\#\}$ and $D(\#) = 0$. In a bleak ontology there are three propositions: the proposition which is true at #, the proposition which is false at #, and the proposition which is undefined at # (also known as the Russell proposition). There are four propositional intensions: three rigid intensions having the three propositions as values, and the undefined intension.

Consider a bleak ontology and a propositional constant P. If P is assigned the undefined intension, P will have no designation at #. If P is assigned the intension which has the Russell proposition as its value, P will have a designation at #, but the designation will not determine a truth value. In the other two cases, the intension will determine a designation and the designation will determine a truth value.

In a classical interpretation built up from a bleak ontology, if P is assigned either the undefined intension or the one which has the Russell proposition as its value, the head default will be applied to the intension in question.[12] It will yield one of the other two propositional intensions as its value, and the truth value of P will be determined via that intension.

The foregoing remarks provide the simplest possible illustration of what the head default does. Section 5 gives a general account of how it works for formulas involving neither bound variables nor predication. In Section 7 the tail default is added and all terms are treated.

3.4 Why things are partial As the preceding example suggests, the theory being presented rejects propositionism and asserts the existence of the Russell proposition because of the uniform use of partial functions at higher types. It is now time to motivate this.

Consider the following sentences:

1. Scott is a logician.
2. Godzilla is a monster.
3. God is a deity.

Clearly, when one uses any of them assertively, one speaks truly. But how does one manage to do this?

The case of "Scott is a logician" is unproblematic. "Scott" refers to Scott (i.e., Professor Dana Scott), "is a logician" expresses the property being a logician, and when one predicates "is a logician" of "Scott", one attributes the property in question to Scott. Since he has the property, a true proposition is expressed.

Things are otherwise with "Godzilla is a monster". To begin with, Godzilla doesn't exist. Does one speak truly when the sentence is used assertively because Godzilla exists in some other world and one is, somehow, talking about him and attributing to him our property monstrousness? Or is one's success merely a result of the way we regard certain activities of Japanese film makers?

Evidently, even nastier questions can be asked about "God is a deity". The metaphysician and the philosopher of language are, quite properly, concerned with such questions. But if the logician must answer them before formalizing arguments in which these sentences occur, he will never be able to get started.

These considerations suggest that the logician had better figure out a way of doing logic which avoids questions of these sorts if he is going to formalize the ontological argument and many other philosophically interesting arguments. This was a major motivation for the development of first-order free logic.[13] As long as attention is restricted to first-order logic, the main problem is how to handle sentences like 1-3, above, which involve singular terms which do not or may not denote. But moving to a higher-order system raises the question of what to do about sentences like "Scott is even". Here, there are no nondenoting singular terms, but it is hardly clear that being even is a property which can be attributed to Scott. Yet one can speak truly by using assertively the sentence "If Scott is an even prime, then Scott is even", so the problem of providing a semantical account of sentences like "Scott is even" cannot be avoided.

The theory of this paper resulted from a persistent attempt to extend the treatment of nondenoting singular terms provided by first-order systems of free logic to cover examples like the one given in the preceding paragraph. Allowing that attribution may fail, even if predication is successful, led naturally to the introduction of partial propositions. Also, there seemed to be no good reason to require that a formula should express the same proposition in every world and fairly good reasons for not doing this. Notoriously, the same sentence can be used to express different propositions in different situations, and the things being called "possible worlds" here are really just basic semantical situations in which formulas are to be assigned truth values. Taking these points into account led to the introduction of intensions of the sort considered here. And, what with everything being partial, there had to be defaults to ensure that, ultimately, every formula would get a truth value.

4 Reprise Although it is not yet possible to prove anything, it is possible to indicate how things will be proved eventually. That will now be done.

Officially speaking, the language of the system is written in prefix notation (operators are written before operands and no parentheses are used). Also, the logical constants carry type symbols along with them, in order to provide a formal way of distinguishing the various identity relations and kinds of existence involved in the type structure.

This way of arranging the notation was devised with the developments of Sections 5-8 in view, and it is fine for those purposes. However, in order to allow the reader to concentrate on the content of the semiformal explanations of this section, without having to worry about unfamiliar ways of arranging notation, the official notation of the system will be abused here. Infix notation will be employed (i.e., binary operators will be written between their operands and parentheses will be used), and '=' will be employed in a typically ambiguous way as an object language symbol for identity.

The reader will have to rely on his intuitions in order to interpret the variable binders and will have to take the author's word for how the defaults operate (or don't). The author warrants that the reader will not, thereby, be misled.

4.1 The Russell proposition (revisited) Where f is a proposition of the basic ontology $\mathbf{O}$, the negation of f is the proposition of $\mathbf{O}$ which is true at those worlds where f is false, false at those worlds where f is true, and undefined at those worlds where f is undefined. Evidently, there is exactly one proposition which is its own negation: the empty function (also known as the empty set, also known as the Russell proposition). The following definition provides a piece of notation which designates it at every world in every interpretation.

$$\mathbf{R} =_{df} !p(p = \sim p).$$

Where f and g are propositions of $\mathbf{O}$, their conjunction is the proposition which is true at worlds where both are true, false at worlds where at least one of them is false, and undefined at worlds where neither of these conditions holds. $A \rightarrow B$ is defined to be $\sim(A \,\&\, \sim B)$, and $A \equiv B$ is defined to be $(A \rightarrow B) \,\&\, (B \rightarrow A)$.

A is *closed* iff no variable is free in A. Let A be a closed formula. A is *intensionally valid* iff in every intensional interpretation †, at the actual world of † A designates a proposition which is true at that world. A is *classically valid* iff in every classical interpretation *, A has T as is extension at the actual world of *.

In the following, 1 is intensionally valid, 2 and 3 are classically valid:

1. $\mathbf{R} = \sim\mathbf{R}$.
2. $\mathbf{R} \equiv \mathbf{R}$.
3. $\sim(\mathbf{R} \equiv \sim\mathbf{R})$.

Since $\mathbf{R}$ designates the Russell proposition at every world in every interpretation and the Russell proposition is its own negation, it follows (via the definitions to be given below) that $\sim\mathbf{R}$ also designates the Russell proposition at every world in every interpretation. This suffices for the intensional validity of 1.

Because **R** designates the Russell proposition at every world in every interpretation and the Russell proposition is undefined at every world in every interpretation, **R** gets one of the two truth values as its extension *by default* at every world in every classical interpretation. Then truth tables apply. It follows that 2 and 3 are classically valid.

It should be clear from the preceding two paragraphs why one cannot infer $\mathbf{R} \equiv {\sim}\mathbf{R}$ from 1 and 2—although **R** and ~**R** designate the same proposition, that tells one nothing about their truth values. Consequently, it is fallacious to replace the right-hand occurrence of **R** in 2 by an occurrence of ~**R**. This provides an excellent example of the difference in the way head terms and tail terms are treated in the theory being presented—**R** and ~**R** are tail terms in 1, but they are head terms in 2 and 3. Evidently, the difference is important.

This brings us back around to the story about Frege and Russell which was begun in Section 2.1. One thing which needs to be done in order to turn it into more than a romance is to introduce the notations $\{x: x \notin x\} \in \{x: x \notin x\}$ and $\{x: x \notin x\} \notin \{x: x \notin x\}$ and get things to work out for them as they do for **R** and ~**R**. A way of doing this will be sketched in Section 8.2. The other thing which needs to be done is to show that the historical content fits the facts.

There is reason to think the historical content does fit the facts, at least as far as Frege's end of the affair is concerned. Frege was, in his way, a great fan of defaults. We find him, for example, telling us that we must explain what it means to add 1 to the Sun ([4], p. 33) and trying to explain how to negate 2 ([4], p. 35). Also, it is clear from [2] that he believed that some propositions (*gedanken*) expressible in natural languages determine no truth value. In [5] we can follow his painstaking efforts to make sure that this could not happen in the language of his formal system. But he adhered rigidly to the view that the reference of a formula is determined by its sense and tried to do all the necessary defaults by extending the domains of functions to totality, which will not work.[14]

4.2 Counterexample to extensionism For each basic ontology **O**, besides the Russell proposition, there are always at least two other **O** propositions: the *totally true proposition*, which is true at every world, and the *totally false proposition*, which is false at every world. The following definitions provide notations for these.

$$\mathbf{T} =_{df} \forall p(p = p).$$
$$\mathbf{F} =_{df} {\sim}\mathbf{T}.$$

It is now possible to explain why the theory being presented is inconsistent with extensionism. In the following, 1 is classically valid and 2, 3, and 4 are intensionally valid.

1. $\Box(\mathbf{R} \,\&\, {\sim}\mathbf{R} \equiv \mathbf{T} \,\&\, {\sim}\mathbf{T})$.
2. $(\mathbf{R} \,\&\, {\sim}\mathbf{R}) = \mathbf{R}$.
3. $(\mathbf{T} \,\&\, {\sim}\mathbf{T}) = \mathbf{F}$.
4. ${\sim}((\mathbf{R} \,\&\, {\sim}\mathbf{R}) = (\mathbf{T} \,\&\, {\sim}\mathbf{T}))$.

That 2, 3, and 4 are intensionally valid should be plausible from the definitions of **T** and **F** and what was said about negation and conjunction in Section 4.1. It should also be plausible that $\mathbf{R} \mathbin{\&} {\sim}\mathbf{R} \equiv \mathbf{T} \mathbin{\&} {\sim}\mathbf{T}$ is true at every world in every classical interpretation. This suffices for the classical validity of 1.

The reader must not be misled by what has just been said into thinking that the customary Kripke condition for the truth of formulas beginning with $\Box$ holds for all formulas. It is *not* always true that $\Box A$ has T as its extension at a world in a classical interpretation iff A has T as its extension at every world of that interpretation. This may be seen as follows.

$\Box$ will be interpreted by means of an operation which sends the totally true proposition to itself, sends any proposition which is false at some world or other to the totally false proposition, and sends all other propositions to the Russell proposition. Consider a basic ontology which has two worlds: # and w_1. Let the intension of the propositional constant P have the totally true proposition as its value in #, and let the intension of P have the totally false proposition as its value in w_1. Then $\Box P$ will designate the totally true proposition in #, but P will designate the totally false proposition in w_1. This suffices to make T the extension of $\Box P$ in # and to make F the extension of P in w_1. (Because everything is total, the defaults have no effect.)

4.3 Invalid and valid ontological arguments Let π be a variable for properties of individuals, let Π symbolize "is a perfection", let $\underline{g}$ be an individual constant symbolizing "God", and let $\underline{x}$ be an individual variable. Descartes' ontological argument may be formalized as follows:

1. $\forall\pi(\Box((\Pi\pi) = \mathbf{T}) \vee \Box((\Pi\pi) = \mathbf{F}))$
2. $\Box(\underline{g} = (!\underline{x}\Box\forall\pi((\Pi\pi) \rightarrow \Box(\pi\underline{x}))))$
3. ΠE
4. Hence, $E\underline{g}$

Or, in something like English:

1. For every property π of individuals, either it is necessary that the proposition that π is a perfection is the totally true proposition or it is necessary that the proposition that π is a perfection is the totally false proposition.
2. Necessarily, God is the being $\underline{x}$ such that it is necessary that for every property π of individuals, if π is a perfection, then necessarily $\underline{x}$ has π.
3. Existence is a perfection.
4. Hence, God exists.

Premise 2 is a strengthened version of Descartes' first premise, and 3 and 4 are straightforward formalizations of his second premise and conclusion. Premise 1 is a stability assumption—it says that being a perfection is an all or nothing affair which has the same upshot in every world. Although it's hard to say what being a perfection is like, it seems reasonable to say at least this.

Even given the added premise, the argument is invalid. To see this, consider a bleak ontology. Since there is only one world, the boxes can be ignored. Also, in a bleak ontology, there is only one property of individuals, 0, and this property is also individual existence. (In every interpretation built up from a basic ontology **O**, individual existence is the property f of individuals such that for every world w of **O** and every individual $\underset{\sim}{o}$ of **O**, $f(\langle w, \underset{\sim}{o}\rangle) = T$ if $\underset{\sim}{o} \in D(w)$, and $f(\langle w, \underset{\sim}{o}\rangle) = F$ if $\underset{\sim}{o} \notin D(w)$.) Let the intension of Π have as its value at # the function which has $\{\langle \#, 0\rangle\}$ as its domain and has T as its value. This will make $\forall\pi((\Pi\pi) = \mathbf{T})$ true. It follows that 1 and 3 will be true. The individual constant in 2 must be assigned an individual intension, and the description will have an intension by virtue of the recursive definition of $\underset{\sim}{I}^{\dagger}$ to be given below. But there is only one individual intension in a bleak ontology: 0. Hence, both the constant and the description will have 0 as their intension. Their having the same intension is sufficient for the truth of 2. Since the intension of the constant determines no individual at #, the tail default will produce an extension for 4. That extension will be F, because things are arranged so that no formula of the form Et is every made true by default.

On the other hand, adding as a premise that God's existence is possible yields a valid argument with the conclusion that God's existence is necessary. That is, for every classical interpretation *, if 1–4, below, are true at the actual world of *, then 5 is also true at the actual world of *.

1. $\forall\pi(\Box((\Pi\pi) = \mathbf{T}) \vee \Box((\Pi\pi) = \mathbf{F}))$
2. $\Box(\underset{\sim}{g} = (!x\Box\forall\pi((\Pi\pi) \rightarrow \Box(\pi\underset{\sim}{x}))))$
3. ΠE
4. $\sim\Box\sim E\underset{\sim}{g}$
5. Hence, $\Box E\underset{\sim}{g}$.

Premise 1 and the form of the description suffice to ensure that the intension of the description is constant. By premises 2 and 4, that intension is not degenerate. It follows that it is rigid and, via 2 again, that $\underset{\sim}{g}$ and the description have the same intension. Given the way the rules for the description operator will be arranged, it follows that $\Box\forall\pi((\Pi\pi) \rightarrow \Box(\pi\underset{\sim}{x}))$ will be true, if this intension is assigned to $\underset{\sim}{x}$. This and premise 3 imply that $\Box E\underset{\sim}{x}$ is true for this way of evaluating $\underset{\sim}{x}$. It follows that $\Box E\underset{\sim}{g}$ is true.

Before settling down to give the definitions necessary to turn the proof sketches of this section into proofs, it will be well to illustrate how this valid version of Descartes' ontological argument fares in some interpretations built on very simple ontologies.

A basic ontology is *Spinozistic* iff the actual world is both the only world of that ontology and the only individual in that world. Individual existence is the only candidate for being a perfection in such an ontology, and if one makes it such and assigns $\underset{\sim}{g}$ the only nondegenerate intension available, the premises and conclusion will be true.

Next, consider a basic ontology with two worlds, # and w_1, where Scott is the only individual in # and Godzilla is the only individual in w_1. If $\underset{\sim}{g}$ is assigned either the rigid intension having Scott as its value or the rigid intension having Godzilla as its value, the conclusion will be false. Given such an assignment of an intension to $\underset{\sim}{g}$, any way of making 1 and 3 true will make 2 false, because the description will not designate an individual and the tail default will always make $t = u$ false if t has a designation but u does not. On the other hand, it is possible to make $\Box E\underset{\sim}{g}$ true by assigning $\underset{\sim}{g}$ the Scottzilla intension—the individual intension which has Scott as its value in # and has Godzilla as its value in the other world.[15] Premise 2 will be false for this way of interpreting g. One other individual intension is worth considering before passing on to the next example—the Godott[16] intension, which has Godzilla as its value in # and has Scott as its value in w_1. If the Godott intension is assigned to $\underset{\sim}{g}$, $\Box{\sim}E\underset{\sim}{g}$ will be true.

As a final example, consider a basic ontology which has two worlds and has the same individuals in both, namely Professor Scott and Professor Peter Andrews. Here, both Scott and Andrews exist necessarily, so if either the intension having the former as its value in both worlds or the intension having the latter as its value in both worlds is assigned to $\underset{\sim}{g}$, the conclusion will be true and premise 4 will be true. If one makes individual existence the only perfection and does it carefully, premises 1 and 3 will also be true, but 2 will be false, because, as was remarked above, the description will fail to designate an individual and the tail default will be arranged so that $t = u$ is false if the intension of t determines a designation and the intension of u does not. If one allows that, besides individual existence, being Scott is the only other perfection, Scott undergoes apotheosis. A symmetric move would ensure the divinity of Andrews.

By the remark at the end of Section 3.2, there are $3^{3^4} = 3^{81}$ properties of properties of individuals in the ontology considered in the preceding paragraph, so it seems many other stories could be told about that ontology. However, enough stories have been told. It is time to get on with the formalism.

5 Propositional calculus The formulas of the *propositional calculus* are those which can be built up from variables and constants with type symbol [ ] by means of $\sim$, &, and $\Box$. In this section attention is restricted to formulas of the propositional calculus, which makes it possible to give a precise illustration of what the head default does without worrying about the tail default. From now on, prefix notation is employed in describing and writing object language expressions. Definitions which apply to the full system will be marked "General", and those which must be extended later will be marked "PC".

Designations (general)

a. $DE(\mathbf{O})$ = the set of $\mathbf{O}$ designations.
b. $DE(\mathbf{O})$ = the union over $\tau \in TS$ of $DT(\mathbf{O}, \tau)$.

That is, the set of **O** designations is the union of the **O** designation types.

Intensions (general)

a. $I(\mathbf{O})$ = the set of $\mathbf{O}$ intensions.
b. $I(\mathbf{O})$ = the union over $\tau \in TS$ of $I(\mathbf{O}, \tau)$.

And the set of **O** intensions is the union of the **O** intension types.

Determination of designations by intensions (general) For $f \in I(\mathbf{O})$ and $w \in W_{\mathbf{O}}$:

a. $DE(\mathbf{O}, w, f)$ = the $\mathbf{O}$ designation (if any) determined by f in w.
b. $DE(\mathbf{O}, w, f) = f(w)$.

Thus, the **O** designation (if any) determined by the **O** intension f is obtained by applying f to w.

5.1 Intensional interpretations (propositional calculus) Recall that an intensional interpretation consists of an ontology together with an assignment of intensions of that ontology to nonlogical constants. It will now be shown how to extend such an assignment to one which assigns intensions to complex formulas of the propositional calculus.

Determination of extensions (PC)

For $f \in PR(\mathbf{Q})$ and $w \in W_{\mathbf{O}}$:

a. $EX(\mathbf{O}, x, f)$ = the **O** extension (if any) determined by f in w.

b. $EX(\mathbf{O}, w, f) = f(\langle w \rangle)$.

Using the terminology introduced in Subsection 3.3, this amounts to saying that the **O** extension determined by the proposition f in w is T if f is true at w, F if f is false at w, and undefined, otherwise.

The logical operations (PC)

1. $N(\mathbf{O})$
   a. = the *negation operation of* **O**
   b. = the $f \in PR(\mathbf{O}) \Rightarrow PR(\mathbf{O})$ such that f is total, and for all $g \in PR(\mathbf{O})$ and all $w \in W_{\mathbf{O}}$, $\underline{D}f(g) = \underline{D}g$ and $f(g)(\langle w \rangle) = T$ iff $g(\langle w \rangle) = F$.

So the negation of the proposition g is the proposition which yields truth values at exactly the worlds where g does and yields the opposite truth value to the one produced by g at worlds where both produce truth values.

2. $K(\mathbf{O})$
   a. = the *conjunction operation of* **O**
   b. = the $f \in Seq\langle PR(\mathbf{O}), PR(\mathbf{O})\rangle \Rightarrow PR(\mathbf{O})$ such that f is total, and for all g, $h \in PR(\mathbf{O})$ and $w \in W_{\mathbf{O}}$:
      (1) $f(\langle g, h \rangle)(\langle w \rangle) = T$, if $g(\langle w \rangle) = T$ and $h(\langle w \rangle) = T$
      (2) $f(\langle g, h \rangle)(\langle w \rangle) = F$, if $g(\langle w \rangle) = F$ or $h(\langle w \rangle) = F$
      (3) $f(\langle g, h \rangle)(\langle w \rangle)$ is undefined, otherwise.

That is, the conjunction of the propositions g and h is the proposition which is true at worlds where both are true, false at worlds where at least one of g and h is false, and undefined, otherwise.

3. $L(\mathbf{O})$
   a. = the *necessity operation of* **O**
   b. = the $f \in PR(\mathbf{O}) \Rightarrow PR(\mathbf{O})$ such that f is total and for all $g \in PR(\mathbf{O})$ and $w \in W_{\mathbf{O}}$
      (1) $f(g)(\langle w \rangle) = T$, if for all $w_1 \in W_{\mathbf{O}}$, $g(\langle w_1 \rangle) = T$
      (2) $f(g)(\langle w \rangle) = F$, if for some $w_1 \in W_{\mathbf{O}}$, $g(\langle w_1 \rangle) = F$
      (3) $f(g)(\langle w \rangle)$ is undefined, otherwise.

Let us call the result of applying the necessity operation of $\mathbf{O}$ to the proposition g the *necessitation* of g. Then the necessitation of g is true at a world w if g is true at every world of $\mathbf{O}$, the necessitation of g is false at w if g is false at some world of $\mathbf{O}$, and the necessitation of g is undefined at w, otherwise.

Intensional interpretations (general)

$\dagger = \langle \mathbf{O}, \underline{I} \rangle$, where $\underline{I}$ is a function such that $D\underline{I}$ is the set of nonlogical constants and for all $t \in D\underline{I}$, $\underline{I}(t) \in I(\mathbf{O}, \tau)$, where τ is the type symbol of t.

Where V is the set of variables, $V^\dagger$ is $\{\psi$: ψ is a function, $\underline{D}\psi = V$, and for all $\chi \in \underline{D}\psi$, $\psi(\chi) \in RI(\mathbf{O}, \tau)$, where τ is the type symbol of $\chi\}$.

That is, $V^\dagger$ is the set of assignments of rigid intensions to variables.

Intension, designation, and extension (PC) For $\psi \in V^\dagger$, $w \in W_{\mathbf{O}}$, and t a formula of the propositional calculus, define $\underline{I}^\dagger(\psi, t)$, $\underline{DE}^\dagger(\psi, w, t)$, and $\underline{EX}^\dagger(\psi, w, t)$, respectively, the *intension of t in* $\dagger$ *under* ψ, the *designation* (if any) *of t in* $\dagger$ *at w under* ψ, and the *extension* (if any) *of t in* $\dagger$ *at w under* ψ, as follows:

Case $2^\dagger$*(PC):* t is a variable or constant.

If t is a variable, then $\underline{I}^\dagger(\psi, t) = \psi(t)$.

Or, put less formally, the intension of a variable in an intensional interpretation under an assignment of rigid intensions to variables is the intension obtained by applying the assignment to the variable in question.

If t is a constant, then $\underline{I}^\dagger(\psi, t) = \underline{I}(t)$.

In other words, the intension of a constant is the intension assigned to that constant by the intensional interpretation under consideration.

$\underline{DE}^\dagger(\psi, w, t) = DE(\mathbf{O}, w, \underline{I}^\dagger(\psi, t))$.

Thus, the designation of a propositional variable or constant in an intensional interpretation at a world under an assignment of rigid intensions to variables is the designation (if any) determined by the intension of that variable or constant in the ontology of the interpretation.

$\underline{EX}^\dagger(\psi, w, t) = EX(\mathbf{O}, w, \underline{I}^\dagger(\psi, t))$.

And in an intensional interpretation the extension of a propositional variable or constant is also determined by the intension of that variable or constant.

Case $3^\dagger$*(General):* t is $\sim A$.

$\underline{DE}^\dagger(\psi, w, t) = N(\mathbf{O})(\underline{DE}^\dagger(\psi, w, A))$.

So the designation of the negation of a formula is found by applying the negation operation of the interpretation to the designation of the formula. In the other two cases, below, the conjunction and necessity operations are used to produce designations for conjunctions and formulas beginning with box from the designations of the subformulas.

$\underline{I}^{\dagger}(\psi, t) = \{\langle\langle w_1, \underline{DE}^{\dagger}(\psi, w_1, t)\rangle\rangle : w_1 \in W_{\mathbf{O}}\}$.

That is, the intension of t in † under ψ is defined to be the function which, for every world $w_1 \in W_{\mathbf{O}}$, has the designation (if any) of t in † at w_1 under ψ as its value.

$\underline{EX}^{\dagger}(\psi, w, t) = EX(\mathbf{O}, w, \underline{I}^{\dagger}(\psi, t))$.

In the remaining cases, $\underline{I}^{\dagger}(\psi, t)$ and $\underline{EX}^{\dagger}(\psi, w, t)$ are always defined from $\underline{DE}^{\dagger}(\psi, w, t)$ in the preceding manner.

It follows that in an intensional interpretation of the propositional calculus, the extension (if any) of a formula is determined by its intension.

Case 4†(General): t is $\&AB$.

$\underline{DE}^{\dagger}(\psi, w, t) = K(\mathbf{O})(\langle\underline{DE}^{\dagger}(\psi, w, A), \underline{DE}^{\dagger}(\psi, w, B)\rangle)$.

Case 5†(General): t is $\Box A$.

$\underline{DE}^{\dagger}(\psi, w, t) = L(\mathbf{O})(\underline{DE}^{\dagger}(\psi, w, A))$.

Where A is a formula of the propositional calculus, A is *intensionally valid* iff for every assignment of values to the variables of A in every intensional interpretation †, at the actual world of † A designates a proposition which is true at that world.

No formula of the propositional calculus is intensionally valid. To see this, consider an arbitrary basic ontology $\mathbf{O}$, let $\underline{I}$ assign the rigid intension having the Russell proposition as its value to every propositional constant, and let ψ do the same for every propositional variable. Every formula of the propositional calculus will designate the Russell proposition at every world of the resulting interpretation.

5.2 Classical interpretations (propositional calculus) Inspection of the definitions given in Section 5.1 show that $\underline{I}^{\dagger}$ is defined for every $\psi \in V^{\dagger}$ and every formula of the propositional calculus. But $\underline{DE}^{\dagger}$ and $\underline{EX}^{\dagger}$ may be undefined for some triples of arguments. Further scrutiny of the definitions in question shows that this situation will arise only if the intension of some propositional constant or propositional variable fails to determine an extension at some world. Consequently, if each intension of a propositional variable or constant is defaulted to a total intension determining total designations, the other clauses of the definition can be used to assign default intensions, designations, and extensions to all formulas of the propositional calculus. The head default is used to carry out this scheme (and for more general purposes in Section 7.2).

Head defaults (general) Consider a basic ontology $\mathbf{O} = \langle D, \#\rangle$. Δ is a *head default for* $\mathbf{O}$ iff Δ is a function such that:

a. $\underline{D}\Delta$ = the union over $\tau \in TS - \{\iota\}$ of $Seq\langle\{\tau\}, I(\mathbf{O}, \tau)\rangle$
b. for all τ and f such that $\langle\tau, f\rangle \in \underline{D}\Delta$, $\Delta\langle\tau, f\rangle \in I(\mathbf{O},\tau)$ and, where $\Delta\langle\tau, f\rangle = f'$
c. for all $w \in W_0$, $f'(w)$ is a total element of $DT(\mathbf{O}, \tau)$ and either $w \notin \underline{D}f$ or $f'(w) \cap f(w) = f(w)$.

Clause a of the foregoing definition provides the generality which will be required in Section 7.2. Clauses b and c assert that the default yields a total intension which determines total designations agreeing with the designations determined by the intension with which one began.

> $\ddagger = \langle \dagger, \Delta \rangle$ is a *classical interpretation for* the propositional calculus iff $\dagger = \langle \mathbf{O}, \underline{I} \rangle$ is an intensional interpretation and Δ is a head default for $\mathbf{O}$.

Consider a classical interpretation for the propositional calculus, $\ddagger = \langle \dagger, \Delta \rangle$, and define $\underline{I}'(\psi, t)$, $\underline{DE}'(\psi, w, t)$, and $\underline{EX}'(\psi, w, t)$, respectively, as the *default intension, default designation,* and *default extension of t*, as follows, where t is a formula of the propositional calculus.

Case 2′(PC): t is a variable or constant.

> Let f be $\underline{I}^{\dagger}(\psi, t)$, and let $f' = \Delta\langle [\,], f \rangle$.
> $\underline{DE}'(\psi, w, t) = f'(w)$.
> $\underline{I}'(\psi, t) = f'$.
> $\underline{EX}'(\psi, w, t) = EX(\mathbf{O}, w, f')$.

Cases 3′–5′(General): Replace '†' by ''' everywhere in the specifications of cases $3^{\dagger}$–$5^{\dagger}$.

> $\underline{I}^{\ddagger}(\psi, t) = \underline{I}^{\dagger}(\psi, t)$, and $\underline{DE}^{\ddagger}(\psi, w, t) = \underline{DE}^{\dagger}(\psi, w, t)$.

Note that the intension and designation of t in a classical interpretation for the propositional calculus are the same as t's intension and designation in the intensional interpretation which is a part of the classical interpretation in question.

> $\underline{EX}^{\ddagger}(\psi, w, t) = \underline{EX}'(\psi, w, t)$.

But t's extension in a classical interpretation for the propositional calculus is its default extension.

6 *Basic semantical concepts (continued)* In preparation for the developments of Section 7, the definitions of basic semantical concepts will now be completed.

Determination of extensions (continued)

> For $\underline{o} \in DT(\mathbf{O},\iota)$ and $w \in W_{\mathbf{O}}$:
> a. $EX(\mathbf{O}, w, \underline{o})$ = the $\mathbf{O}$ *extension* (if any) *determined by* $\underline{o}$ *in w*.
> b. $EX(\mathbf{O}, w, \underline{o})$
> b1. $= \underline{o}$, if $\underline{o} \in D(w)$
> b2. = undefined, otherwise.

Thus, $\underline{o}$ determines an extension in w iff $\underline{o}$ exists in w, and that extension is $\underline{o}$ if $\underline{o}$ exists in w.

Consider $f \in DT(\mathbf{O}, [\tau_1 \ldots \tau_n])$ $(n \geq 1)$ and $w \in W_{\mathbf{O}}$. For all $i(1 \leq i \leq n)$, let $x_i = DT(\mathbf{O}, \tau_i)$, if $\tau_i \neq \iota$, and let $x_i = D(w) \cap DT(\mathbf{O}, \tau_i)$, if $\tau_i = \iota$. Then:

a. $\mathrm{EX}(\mathbf{O}, w, f)$ = the *extension of f in w.*
b. $EX(\mathbf{O}, w, f) = \{\langle\langle y_1, \ldots, y_n\rangle\rangle : \langle y_1, \ldots, y_n\rangle \in Seq\langle x_1, \ldots, x_n\rangle,$ and $f(\langle w, y_1, \ldots, y_n\rangle) = T\}$.

That is, the extension of the property or relation f in the world w is determined by fixing w, restricting attention to individuals existing in w, and picking off the appropriate set of n-tuples. This makes the extension of a relation the sort of object which is usually called a "relation" when one is doing set theory.

The preceding definition is an idle wheel as far as the language currently under discussion is concerned, because notations for talking about extensions of properties and relations are not available. Evidently, introducing them would be no problem.

For $w \in W_{\mathbf{O}}$:
a. $EX(\mathbf{O}, w)$ = the set of *w extensions of* $\mathbf{O}$.
b. $EX(\mathbf{O}, w) = \{EX(\mathbf{O}, w, x)$: for some $\tau \in TS$, $x \in DT(\mathbf{O}, \tau)\}$.

Thus, the set of w extensions of $\mathbf{O}$ is the set of extensions determined by $\mathbf{O}$ designations in w.

And:
a. $EX(\mathbf{O})$ = the set of $\mathbf{O}$ *extensions.*
b. $EX(\mathbf{O})$ = the union over $w \in W_{\mathbf{O}}$ of $EX(\mathbf{O}, w)$.

For $f \in I(\mathbf{O})$ and $w \in W_{\mathbf{O}}$:
a. $EX(\mathbf{O}, w, f)$ = the $\mathbf{O}$ *extension* (if any) *determined by f in w.*
b. $EX(\mathbf{O}, w, f) = EX(\mathbf{O}, w, f(w))$.

So the extension determined by the intension f in w is the extension determined by the designation determined by f in w.

Attribution Consider $[\tau_1 \ldots \tau_n]$ $(n \geq 0)$, $w \in W_{\mathbf{O}}, f \in I(\mathbf{O}, [\tau_1 \ldots \tau_n])$, and $g_1 \in I(\mathbf{O}, \tau_1), \ldots, g_n \in I(\mathbf{O}, \tau_n)$.

a. $PR(\mathbf{O}, w, \langle f, g_1, \ldots, g_n\rangle)$ = the $\mathbf{O}$ *proposition* (if any) *determined by f and* $g_1, \ldots, g_n$ *in w.*
b. $PR(\mathbf{O}, w, \langle f, g_1, \ldots, g_n\rangle)$
  b1. is undefined, if $w \notin \underline{D}f$
  b2. $= \{\langle\langle\langle w_1\rangle, f(w)(\langle w_1, g_1(w_1), \ldots, g_n(w_1)\rangle)\rangle\rangle : w_1 \in W_{\mathbf{O}}\}$, otherwise.

Understanding the foregoing definition is crucial for understanding what goes on in Section 7. Note first that if $n = 0$, then $PR(\mathbf{O}, w, \langle f, g_1, \ldots, g_n\rangle) = f(w)$. This ensures that the definitions given in Section 5 really are special cases of the ones to be given in Section 7.

Suppose $n \geq 1$. $PR(\mathbf{O}, w, \langle f, g_1, \ldots, g_n\rangle)$ will be undefined iff $w \notin \underline{D}f$. Suppose $w \in \underline{D}f$. Then $PR(\mathbf{O}, w, \langle f, g_1, \ldots, g_n\rangle)$ will be the proposition which, for all $w_1 \in W_{\mathbf{O}}$, is true at w_1 if $f(w)(\langle w_1, g_1(w_1), \ldots, g_n(w_1)\rangle) = T$, is false at w_1 if $f(w)(\langle w_1, g_1(w_1), \ldots, g_n(w_1)\rangle) = F$, and is undefined at w_1 if $f(w)(\langle w_1, g_1(w_1), \ldots, g_n(w_1)\rangle)$ is undefined. $f(w)(\langle w_1, g_1(w_1), \ldots, g_n(w_1)\rangle)$ may be undefined either because there is an i such that $g_i(w_1)$ is undefined or because, although there is no such i, $\langle w_1, g_1(w_1), \ldots, g_n(w_1)\rangle \notin \underline{D}f(w)$. In

Section 7.2, the head default will be used to deal with the latter source of undefinedness and the tail default will take care of the former.

The logical relations and intensions Definitions 1 and 2 under this heading say that the identity constants really express identity.

1. $IDE(\mathbf{O}, [\tau\tau])$
   a. = the $[\tau\tau]$ *identity designation of* $\mathbf{O}$
   b. = the $f \in DT(\mathbf{O}, [\tau\tau])$ such that f is total, and for all w, x, and y such that $\langle w, x, y\rangle \in \underline{D}f$, $f(\langle w, x, y\rangle) = T$ iff $x = y$.
2. $II(\mathbf{Q}, [\tau\tau])$
   a. = the $[\tau\tau]$ *identity intension of* $\mathbf{O}$
   b. = the element of $RI(\mathbf{O}, [\tau\tau])$ which has $IDE(\mathbf{O}, [\tau\tau])$ as its value.
3. $EDE(\mathbf{O}, [\tau])$
   a. = the $[\tau]$ *existence designation of* $\mathbf{O}$
   b. = the $f \in DT(\mathbf{O}, [\tau])$ such that f is total, $\tau = \iota$ only if for all w and x such that $\langle w, x\rangle \in \underline{D}f$, $f(\langle w, x\rangle) = T$ iff $x \in D(w)$, and $\tau \neq \iota$ only if for all w and x such that $\langle w, x\rangle \in \underline{D}f$, $f(\langle w, x\rangle) = T$.

Thus, existence at higher types is boring—every relation, property, and proposition exists in every world.

4. $EI(\mathbf{O}, [\tau])$
   a. = the $[\tau]$ *existence intension of* $\mathbf{O}$
   b. = the element of $RI(\mathbf{O}, [\tau])$ which has $EDE(\mathbf{O}, \tau)$ as its value.

The logical operations (continued)

5. $\prod(\mathbf{O}, \tau)$
   a. = the τ *universal quantification operation of* $\mathbf{O}$
   b. = the function f such that
      (1) $f \in Seq\langle W_{\mathbf{O}}, DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})\rangle \Rightarrow \{T, F\}$
      (2) if $DT(\mathbf{O},\tau) = 0$, then for all $w \in W_{\mathbf{O}}$, $f(\langle w, 0\rangle) = T$
      (3) if $DT(\mathbf{O},\tau) \neq 0$, $g \in DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})$, and g is not total, then f is undefined for $\langle w, g\rangle$
      (4) if $DT(\mathbf{O}, \tau) \neq 0$, $g \in DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})$, and g is total, then for all $w \in W_{\mathbf{O}}$:
         (a) $f(\langle w, g\rangle) = T$, if for all $x \in DT(\mathbf{O}, \tau)$, $g(x)(\langle w\rangle) = T$
         (b) $f(\langle w, g\rangle) = F$, if for some $x \in DT(\mathbf{O}, \tau)$, $g(x)(\langle w\rangle) = F$
         (c) $f(\langle w, g\rangle)$ is undefined, otherwise.

This definition paves the way for interpreting universal quantification as a variety of infinite conjunction. To see how this will work, consider a formula $\forall\chi A$, and suppose the designation of A is defined at every world, for every assignment of values to variables. Where τ is the type symbol of χ, a function $g \in DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})$ will be defined by letting the value of χ vary over $RI(\mathbf{O}, \tau)$. By fixing this function and considering $f(\langle w, g\rangle)$ as w varies over $W_{\mathbf{O}}$, a designation for $\forall\chi A$ will be obtained. Clause (2) allows for the possibility that there may be no individuals. Clause (3) causes undefinedness, if the indexing of propositions by elements of $DT(\mathbf{O}, \tau)$ fails anywhere.

6. $\downarrow(\mathbf{O}, \tau)$
   a. = the τ *description operation* of $\mathbf{O}$
   b. = the function f such that
      (1) $f \in Seq\langle W_{\mathbf{O}}, DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})\rangle \Rightarrow DT(\mathbf{O}, \tau)$
      (2) if $DT(\mathbf{O}, \tau) = 0$, then for all $w \in W_{\mathbf{O}}$, f is undefined for $\langle w, 0\rangle$
      (3) if $DT(\mathbf{O}, \tau) \neq 0$, $g \in DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})$, and $\underline{D}g \neq DE(\mathbf{O}, \tau)$, then for all $w \in W_{\mathbf{O}}$, f is undefined for $\langle w, g\rangle$
      (4) if $DT(\mathbf{O}, \tau) \neq 0$, $g \in DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})$, and $\underline{D}g = DT(\mathbf{O}, \tau)$, then
         (a) if there is an $x \in DT(\mathbf{O}, \tau)$ such that $g(x)(\langle w\rangle) = T$ and for all $y \in DT(\mathbf{O}, \tau)$, $x \neq y$ only if $g(y)(\langle w\rangle) = F$, then $f(\langle w, g\rangle) = x$
         (b) $f(\langle w, g\rangle)$ is undefined, otherwise.

Given what was said, above, about the quantifier, it should be clear that this is the natural way to interpret the description operator. Clauses (a) and (b) will ensure that a description has a designation iff the customary existence and uniqueness conditions hold intensionally.

7 The full system The definitions given in Section 5 will now be extended to cover all terms.

7.1 Intensional interpretations For $\psi \in V^\dagger$ and $w \in W_{\mathbf{O}}$, define $\underline{I}^\dagger(\psi, t)$, $\underline{DE}^\dagger(\psi, w, t)$, and $\underline{EX}^\dagger(\psi, w, t)$ as follows:

Case 1†*:* t is a constant or variable with type symbol τ and $\tau \neq [\,]$.

If t is a variable, then $\underline{I}^\dagger(\psi, t) = \psi(t)$.
If t is $\cong^{[\tau_1\tau_1]}$, *then* $\underline{I}^\dagger(\psi, t) = II(\mathbf{O}, [\tau_1\tau_1])$.
If t is $E^{[\tau_1]}$, *then* $\underline{I}^\dagger(\psi, t) = EI(\mathbf{O}, [\tau_1])$.
If t is a nonlogical constant, then $\underline{I}^\dagger(\psi, t) = \underline{I}(t)$.
$\underline{DE}^\dagger(\psi, w, t) = DE(\mathbf{O}, w, \underline{I}^\dagger(\psi\ t))$.
$\underline{EX}^\dagger(\psi, w, t) = EX(\mathbf{O}, w, \underline{I}^\dagger(\psi, t))$.

Case 2†*:* $t = uv_1 \ldots v_n$ $(n \geq 0)$, where either $n = 0$ and u is a variable or constant with type symbol $[\,]$ or $n \geq 1$, $v_1, \ldots, v_n$ are terms with type symbols $\tau_1, \ldots, \tau_n$, respectively, and u is a term with type symbol $[\tau_1 \ldots \tau_n]$.

If $n = 0$, $\underline{I}^\dagger(\psi, t)$ is determined as in Case 1.
Let $f, g_1, \ldots, g_n$ be $\underline{I}^\dagger(\psi, u), \underline{I}^\dagger(\psi, v_1), \ldots, \underline{I}^\dagger(\psi, v_n)$, respectively.
$\underline{DE}^\dagger(\psi, w, t) = PR(\mathbf{O}, w, \langle f, g_1, \ldots, g_n\rangle)$.
$\underline{I}^\dagger(\psi, t) = \{\langle\langle w_1, \underline{DE}^\dagger(\psi, w_1, t)\rangle\rangle : w_1 \in W_{\mathbf{O}}\}$.
$\underline{EX}^\dagger(\psi, w, t) = EX(\mathbf{O}, w, \underline{I}^\dagger(\psi, t))$.

In the remaining cases, $\underline{I}^\dagger(\psi, t)$ and $\underline{EX}^\dagger(\psi, w, t)$ are always defined from $\underline{DE}^\dagger(\psi, w, t)$ in the preceding manner.

Case 6†*:* t is $\forall\chi A$, where τ is the type symbol of χ.

If there is a $\psi_1 \in V^\dagger$ such that ψ_1 agrees with ψ off χ and $\underline{DE}^\dagger(\psi_1, w, A)$ is undefined, then $\underline{DE}^\dagger(\psi, w, t)$ is undefined.

Suppose there is no such ψ_1, and specify $f \in DT(\mathbf{O}, \tau) \Rightarrow PR(\mathbf{O})$ as follows. If $DT(\mathbf{O}, \tau) = 0$, then $f = 0$. Suppose $DT(\mathbf{O}, \tau) \neq 0$, and for each $x \in DT(\mathbf{O}, \tau)$, let ψ_x be the member of $V^\dagger$ such that ψ_x agrees with ψ off χ and $\psi_x(\chi)$ is the element of $RI(\mathbf{O}, \tau)$ which has x as its value. Then for all $x \in DT(\mathbf{O}, \tau)$, $f(x) = \underline{\underline{DE}}^\dagger(\psi_x, w, A)$.

$\underline{\underline{DE}}^\dagger(\psi, w, t) = \{\langle\langle\langle w_1\rangle, \prod(\mathbf{O}, \tau)(\langle w_1, f\rangle)\rangle\rangle: w_1 \in W_{\mathbf{O}}\}$.

Case 7†: t is $!\chi A$, where τ is the type symbol of ψ.

If there is a $\psi_1 \in V^\dagger$ such that ψ_1 agrees with ψ off χ and $\underline{\underline{DE}}^\dagger(\psi_1, w, A)$ is undefined, then $\underline{\underline{DE}}^\dagger(\psi, w, t)$ is undefined.

Suppose there is no such ψ_1, and let f be specified as in Case 6.

$\underline{\underline{DE}}^\dagger(\psi, w, t) = \downarrow(\mathbf{O}, \tau)(\langle w, f\rangle)$.

This concludes the definitions of $\underline{I}^\dagger$, $\underline{\underline{DE}}^\dagger$, and $\underline{\underline{EX}}^\dagger$. Observe that $\underline{I}^\dagger$ is defined for all $\psi \in V^\dagger$ and all t.

As usual, for closed t, $\underline{I}^\dagger(\psi, w, t)$, $\underline{\underline{DE}}^\dagger(\psi, w, t)$, and $\underline{\underline{EX}}^\dagger(\psi, w, t)$ do not depend on ψ. So $\underline{I}^\dagger(t)$, $\underline{\underline{DE}}^\dagger(w, t)$, and $\underline{\underline{EX}}^\dagger(w, t)$ can be defined by quantifying over $V^\dagger$ in the customary way. For such t, $\underline{\underline{DE}}^\dagger(t) = \underline{\underline{DE}}^\dagger(\#, t)$ and $\underline{\underline{EX}}^\dagger(t) = \underline{\underline{EX}}^\dagger(\#, t)$. If A is open, but does not contain free variables with type symbol ι, let A' be its universal closure, and define $\underline{I}^\dagger(A)$, $\underline{\underline{DE}}^\dagger(w, A)$, and $\underline{\underline{EX}}^\dagger(w, A)$ to be $\underline{I}^\dagger(A')$, $\underline{\underline{DE}}^\dagger(w, A')$, and $\underline{\underline{EX}}^\dagger(w, A')$, respectively.[17]

A is *intensionally valid* iff for every intensional interpretation $\dagger$, $\underline{\underline{EX}}^\dagger(A) = T$.

Let a variable χ with type symbol [] be chosen, and define:

$\mathbf{T} =_{df} \forall\chi \cong^{[[\,][\,]]} \chi\chi$.
$\mathbf{R} =_{df} !\chi \cong^{[[\,][\,]]} \chi \sim \chi$.
$\mathbf{F} =_{df} {\sim}\mathbf{T}$.

The foregoing definitions recast in official notation the definitions of **T**, **R**, and **F** given in Sections 4.1 and 4.2.

Theorem 7.1.1 *Consider an intensional interpretation* $\dagger = \langle \mathbf{O}, \underline{I}\rangle$.

a. $\underline{\underline{DE}}^\dagger(\mathbf{T}) =$ *the* $f \in PR(\mathbf{O})$ *such that* f *is total*, f *is constant, and* f *has* T *as its value.*
b. $\underline{\underline{DE}}^\dagger(\mathbf{R}) = 0$.
c. $\underline{\underline{DE}}^\dagger(\mathbf{F}) =$ *the* $f \in PR(\mathbf{O})$ *such that* f *is total, if is constant, and* f *has* F *as its value.*
d. $\underline{\underline{DE}}^\dagger(\&\mathbf{T} \sim \mathbf{T}) = \underline{\underline{DE}}^\dagger(\mathbf{F})$.
e. $\underline{\underline{DE}}^\dagger({\sim}\mathbf{R}) = \underline{\underline{DE}}^\dagger(\mathbf{R})$.
f. $\underline{\underline{DE}}^\dagger(\&\mathbf{R} \sim \mathbf{R}) = \underline{\underline{DE}}^\dagger(\mathbf{R})$.
g. $\underline{\underline{DE}}^\dagger(\cong^{[[\,][\,]]} \& \mathbf{R} \sim \mathbf{R} \& \mathbf{T} \sim \mathbf{T}) = \underline{\underline{DE}}^\dagger(\mathbf{F})$.
h. $\underline{\underline{DE}}^\dagger({\sim}\cong^{[[\,][\,]]} \& \mathbf{R} \sim \mathbf{R} \& \mathbf{T} \sim \mathbf{T}) = \underline{\underline{DE}}^\dagger(\mathbf{T})$.
i. $\underline{\underline{EX}}^\dagger(\mathbf{T}) = T$.
j. $\underline{\underline{EX}}^\dagger(\mathbf{R})$ *is undefined.*
k. $\underline{\underline{EX}}^\dagger(\mathbf{F}) = F$.

Proof: By inspection of the foregoing definitions.

Corollary 7.1.2

a. **T** *is intensionally valid.*

b. $\sim\cong^{[[\,][\,]]}$ & **R** $\sim$ **R** & **T** $\sim$ **T** *is intensionally valid.*

c. *If* A *is intensionally valid, then* $\cong^{[[\,][\,]]} A\mathbf{T}$ *is intensionally valid.*

Theorem 7.1.1d, e, and f and Corollary 7.1.2b substantiate the claims about intensional validity made in Sections 4.1 and 4.2.

7.2 Classical interpretations In providing default intensions, designations, and extensions for formulas of the full system, it suffices to blow everything up to totality in the predication case and in the case of descriptions with type symbol [ ]. Application of the head default will suffice in the latter case. In the former, it may be necessary to use the tail default, even after applying the head default to the intension of the head term. (Cf the remarks following the definition of attribution in Section 6.)

∇ is a *tail default for* **O** iff ∇ is a function such that:

a. $\underline{D}\nabla$ is the union over $\tau = [\tau_1 \ldots \tau_n] \in TS - \{\iota, [\,]\}$ of $\{\langle \tau, w, f, g_1, \ldots, g_n\rangle \in Seq\langle\{\tau\}, W_{\mathbf{O}}, I(\mathbf{O}, \tau), I(\mathbf{O}, \tau_1), \ldots, I(\mathbf{O}, \tau)\rangle : w \in \underline{D}f,$ $f(w)$ is a total element of $DT(\mathbf{O},\tau)$, and $w \notin (\underline{D}g_1) \cap \ldots \cap (\underline{D}g_n)\}$

b. for all $x \in \underline{D}\nabla$, $\nabla x \in \{T, F\}$

c. if $x = \langle[\tau_1\tau_1], w, II(\mathbf{O}, [\tau_1\tau_1]), g, g,\rangle$, then $\nabla x = T$

d. if $x = \langle[\tau_1\tau_1], w, II(\mathbf{O}, [\tau_1\tau_1]), g_1, g_2\rangle$ and $w \in (\underline{D}g_1) \cup (\underline{D}g_2)$, then $\nabla x = F$

e. if $x = \langle[\tau_1 \ldots \tau_i \ldots \tau_n], w, f, g_1, \ldots, g_i, \ldots, g_n\rangle \in \underline{D}\nabla$, $y = \langle[\tau_i\tau_i], w, II(\mathbf{O}, [\tau_i\tau_i]), g_i, h\rangle \in \underline{D}\nabla$, and $\nabla y = T$, then $x' = \langle[\tau_1 \ldots \tau_i \ldots \tau_n], w, f, g_1, \ldots, h, \ldots, g_n\rangle \in \underline{D}\nabla$ and $\nabla x = \nabla x'$

f. if $x = \langle[\tau_1], w, EI(\mathbf{O}, [\tau_1]), g\rangle \in \underline{D}\nabla$, then $\nabla x = F$.

Clause a says that the domain of ∇ is precisely what it needs to be in order to ensure that a truth value will be assigned in the predication case, even if application of the head default is insufficient to achieve this result. Clause b says that, in fact, application of the tail default will provide a truth value in this sort of situation. Clause c sees to it that $\cong^{[\tau\tau]} tt$ will always have T as its extension, and Clause d makes $\cong^{[\tau\tau]} tu$ false if one of t and u has a designation and the other does not. Clause e says that the overall effect of the tail default is compatible with its effect on identity, and Clause f says that $E^{[\tau]} t$ is never true by default.

An *extension default for* **O** is a sequence $\langle\Delta, \nabla\rangle$ such that Δ is a head default for **O** and ∇ is a tail default for **O**.

$* = \langle\dagger, \langle\Delta, \nabla\rangle\rangle$ is a *classical interpretation* iff $\dagger = \langle\mathbf{O}, \underline{I}\rangle$ is an intensional interpretation and $\langle\Delta, \nabla\rangle$ is an extension default for **O.**

Consider a classical interpretation $* = \langle\dagger, \langle\Delta, \nabla\rangle\rangle$, and define $\underline{I}'(\psi, t)$, $\underline{DE}'(\psi, w, t)$, and $\underline{EX}'(\psi, w, t)$ as follows.

Case 1′: t is a constant or variable with type symbol τ and $\tau \neq [\,]$.

$\underline{I}'(\psi, t) = \underline{I}^{\dagger}(\psi, t)$.
$\underline{DE}'(\psi, w, t) = \underline{DE}^{\dagger}(\psi, w, t)$.
$\underline{EX}'(\psi, w, t) = \underline{EX}^{\dagger}(\psi, w, t)$.

Case 2′: $t = uv_1 \dots v_n$ $(n \geq 0)$, where either $n = 0$ and u is a variable or constant with type symbol $[\,]$ or $n \geq 1$, $v_1, \dots, v_n$ are terms with type symbols $\tau_1, \dots, \tau_n$, respectively, and u is a term with type symbol $[\tau_1 \dots \tau_n]$.

Let $f, g_1, \dots, g_n$ be $\underline{I}^{\dagger}(\psi, u), \underline{I}^{\dagger}(\psi, v_1), \dots, \underline{I}^{\dagger}(\psi, v_n)$, respectively, let $f' = \Delta\langle[\tau_1 \dots \tau_n], f\rangle$, let $h = PR(\mathbf{O}, w, \langle f', g_1, \dots, g_n\rangle)$, and let $W' = Seq\langle W_{\mathbf{O}} - \{w_1 \colon \langle w_1\rangle \in \underline{D}h\}\rangle$.

Since f' was obtained by applying the head default to f and the appropriate type symbol, f' is a total intension which determines total designations in each world. But h may be partial, because one or more of the g's may not be total. The next order of business is to fix that by applying the tail default.

Define h' as follows. If $W' = 0$, then $h' = 0$. Suppose $W' \neq 0$, note that this implies $n \neq 0$, and consider $\langle w_1\rangle \in W'$. It follows that $w_1 \notin (\underline{D}g_1) \cap \dots \cap (\underline{D}g_n)$. $h'(\langle w_1\rangle) = \nabla\langle[\tau_1 \dots \tau_n], f', w_1, g_1, \dots, g_n\rangle$.

h' will provide a truth value at exactly those worlds where h fails to do so because one of the g's does not determine a designation. The default designation of t is obtained by combining h and h'.

$\underline{DE}'(\psi, w, t) = h \cup h'$.
$\underline{I}'(\psi, t) = \{\langle\langle w_1, \underline{DE}'(\psi, w_1, t)\rangle\rangle \colon w_1 \in W_{\mathbf{O}}\}$.
$\underline{EX}'(\psi, w, t) = EX(\mathbf{O}, w, \underline{I}'(\psi, t))$.

In the remaining cases, $\underline{I}'(\psi, t)$ and $\underline{EX}'(\psi, w, t)$ are always defined from $\underline{DE}'(\psi, w, t)$ in the preceding manner.

Case 6′: Replace '†' by ''' everywhere in the specifications of Case $6^{\dagger}$, excepting '$V^{\dagger}$'.

Case 7′: t is $!\chi A$, where τ is the type symbol of χ.

If $\tau \neq [\,]$, then $\underline{I}'(\psi, t)$, $\underline{DE}'(\psi, w, t)$, and $\underline{EX}'(\psi, w, t)$ are $\underline{I}^{\dagger}(\psi, t)$, $\underline{DE}^{\dagger}(\psi, w, t)$, and $\underline{EX}^{\dagger}(\psi, w, t)$, respectively.

If $\tau = [\,]$, let f be $\underline{I}^{\dagger}(\psi, t)$, and let $f' = \Delta\langle[\,], f\rangle$. $\underline{DE}'(\psi, w, t) = f'(w)$. $\underline{I}'(\psi, t) = f'$, and $\underline{EX}'(\psi, w, t) = EX(\mathbf{O}, w, f')$.

$\underline{I}^{*}(\psi, t) = \underline{I}^{\dagger}(\psi, t)$, and $\underline{DE}^{*}(\psi, w, t) = DE^{\dagger}(\psi, w, t)$.

Note that, as in the case of the propositional calculus, the defaults do not affect the intension and designation of a term in a classical interpretation.

If the type symbol of t is not $[\,]$, then $\underline{EX}^{*}(\psi, w, t) = \underline{EX}^{\dagger}(\psi, w, t)$. If the type symbol of t is $[\,]$, then $\underline{EX}^{*}(\psi, w, t) = \underline{EX}'(\psi, w, t)$.

And, in fact, if the type symbol of t is not $[\,]$, the defaults will not affect t's extension either.

For closed t, $\underline{I}^*(t)$, $\underline{DE}^*(w, t)$, and $\underline{EX}^*(w, t)$ are defined as usual, and $\underline{DE}^*(t)$, and $\underline{EX}^*(t)$ are $\underline{DE}^*(\#, t)$, and $\underline{EX}^*(\#, t)$, respectively. These notions are extended to open formulas not containing free individual variables via universal closures, and *classical validity* is defined in the obvious way.

Theorem 7.2.1 *Consider a classical interpretation* $* = \langle \dagger, \langle \Delta, \nabla \rangle \rangle$, *let* $\dagger = \langle \mathbf{O}, \underline{I} \rangle$, *let* $\mathbf{O} = \langle D, \# \rangle$, *and consider* $\psi \in V^\dagger$ *and* $w \in W_{\mathbf{O}}$.

a. $\underline{I}^*(\psi, t) = \underline{I}^\dagger(\psi, t)$.
b. $\underline{DE}^*(\psi, w, t) = \underline{DE}^\dagger(\psi, w, t)$.
c. *If the type symbol of* t *is not* [], *then* $\underline{EX}^*(\psi, w, t) = \underline{EX}^\dagger(\psi, w, t) = EX(\mathbf{O}, w, \underline{I}^\dagger(\psi, t))$.
d. *If* $\langle w \rangle$ *is in the domain of* $\underline{DE}^*(\psi, w, A)$, *then* $\underline{EX}^*(\psi, w, A) = \underline{EX}^*(\psi, w, A) = EX(\mathbf{O}, w, \underline{I}^\dagger(\psi, A))$.
e. $\underline{EX}^*(\psi, w, A) = T$ *or* $\underline{EX}^*(\psi, w, A) = F$.
f. $\underline{EX}^*(\psi, w, \cong^{[\tau\tau]} uv) = T$ *iff either* $\underline{DE}^\dagger(\psi, w, u)$ *and* $\underline{DE}^\dagger(\psi, w, v)$ *are defined and equal or* $\underline{DE}^\dagger(\psi, w, u)$ *and* $\underline{DE}^\dagger(\psi, w, v)$ *are undefined and* $\nabla \langle [\tau\tau], w, II(\mathbf{O}, [\tau\tau], \underline{I}^\dagger(\psi, u), \underline{I}^\dagger(\psi, v)\rangle = T$.
g. $\underline{EX}^*(\psi, w, \cong^{[\tau\tau]} uu) = T$.
h. *If* $\underline{EX}^*(\psi, w, \cong^{[\tau\tau]} v_i v] = T$, *then* $\underline{EX}^*(\psi, w, uv_1 \ldots v_i \ldots v_n) = \underline{EX}^*(\psi, w, uv_1 \ldots v \ldots v_n)$.
i. $\underline{EX}^*(\psi, w, E^{[\tau]} u) = T$ *iff either* $\tau = \iota$ *and* $\underline{EX}^\dagger(\psi, w, u)$ *is defined or* $\tau \neq \iota$ *and* $\underline{DE}^\dagger(\psi, w, u)$ *is defined.*
j. $\underline{EX}^*(\psi, w, \sim A) = T$ *iff* $\underline{EX}^*(\psi, w, A) \neq T$.
k. $\underline{EX}^*(\psi, w, \& AB) = T$ *iff* $\underline{EX}^*(\psi, w, A) = T$ *and* $\underline{EX}^*(\psi, w, B) = T$.
l. *Where* τ *is the type symbol of* χ, $EX^*(\psi, w, \forall\chi A) = T$ *iff either* $DT(\mathbf{O}, \tau) = 0$ *or* $DT(\mathbf{O}, \tau) \neq 0$ *and for all* $x \in DT(\mathbf{O}, \tau)$, $EX^*(\psi_x, w, A) = T$.

Proof: By induction on the complexity of terms.

Define:

$\rightarrow AB =_{df} \sim \& A \sim B$.
$\bigvee AB =_{df} \sim \& \sim A \sim B$.
$\equiv AB =_{df} \& \rightarrow AB \rightarrow BA$.

Given the proof sketches of Section 4 and what has been said since then, one can see that the following three theorems are true.

Theorem 7.2.2 *The following formulas are classically valid*:

a. $\equiv \mathbf{RR}$.
b. $\sim \equiv \mathbf{R} \sim \mathbf{R}$.
c. $\Box \equiv \& \mathbf{R} \sim \mathbf{R} \& \mathbf{T} \sim \mathbf{T}$.

Theorem 7.2.2 gives the official versions of the claims about classical validity made in Sections 4.1 and 4.2, and the next two theorems provide official formulations of the ontological arguments discussed in Section 4.3.

Let π be a variable with type symbol $[\iota]$, let Π be a nonlogical constant with type symbol $[[\iota]]$, let $\underline{x}$ be a variable with type symbol ι, and let $\underline{g}$ be a nonlogical constant with type symbol ι.

Theorem 7.2.3 *There is a classical interpretation * such that 1–3 below have T as their extension in * and 4 has F as its extension in *.*

1. $\forall\pi \vee \Box \cong^{[[\,][\,]]} \Pi\pi\mathbf{T}\Box \cong^{[[\,][\,]]} \Pi\pi\mathbf{F}$.
2. $\Box \cong^{[u]} g!\underline{x}\Box\forall\pi \rightarrow \Pi\pi\Box\pi\underline{x}$.
3. $\Pi E^{[\iota]}$.
4. $E^{[\iota]}\underline{g}$.

Theorem 7.2.4 *If 1-4 below have T as their extension in the classical interpretation *, then 5 has T as its extension in *.*
1. $\forall\pi \vee \Box \cong^{[[\,][\,]]} \Pi\pi\mathbf{T}\Box \cong^{[[\,][\,]]} \Pi\pi\mathbf{F}$
2. $\Box \cong^{[u]} g!\underline{x}\Box\forall\pi \rightarrow \Pi\pi\Box\pi\underline{x}$
3. $\Pi E^{[\iota]}$
4. $\sim\Box\sim E^{[\iota]}\underline{g}$
5. $\Box E^{[\iota]}\underline{g}$.

8 Concluding remarks Section 8.1 discusses the problem of getting the system of logic defined by the semantical theory presented here under control. Section 8.2 relates the semantical theory to set abstraction in the cumulative hierarchy.

8.1 Proof theory As yet, no proof theory for the system of logic defined by the semantical theory exists. Some needs to be created. The following remarks show that a sound and complete axiomatization of the set of formulas which are classically valid does not exist.

$$\exists\chi A =_{df} \sim\forall\chi \sim A \ .$$

Let Z symbolize "is a number", let S symbolize "is a successor of", and let 0 symbolize "0". **P** is to be the conjunction of the following formulas.

1. $\Box\forall\underline{x} \equiv Z\underline{x}\Box Z\underline{x}$
2. $\Box\forall\underline{x}\forall\underline{y} \equiv S\underline{x}\underline{y}\Box S\underline{x}\underline{y}$
3. $\exists\underline{x}\Box \cong^{[u]} \underline{x}0$
4. $Z0$
5. $\forall\underline{x} \rightarrow Z\underline{x}\exists\underline{y}\ \&\ Z\underline{y}S\underline{y}\underline{x}$
6. $\forall\underline{x}\forall\underline{y}\forall\underline{z} \rightarrow \&S\underline{z}\underline{x}S\underline{z}\underline{y} \cong^{[u]} \underline{x}\underline{y}$
7. $\forall\underline{x} \sim S\underline{x}0$
8. $\Box\forall\pi_\iota \rightarrow \&\pi 0\forall\underline{x} \rightarrow \pi\underline{x}\forall\underline{y} \rightarrow S\underline{y}\underline{x}\pi\underline{y}\forall\underline{x} \rightarrow Z\underline{x}\pi\underline{x}$.

Let A be a closed formula of ordinary, second-order arithmetic, and let A' be the natural translation of A into the language under consideration. Then A is true in the standard model of second-order arithmetic iff $\rightarrow\mathbf{P}A'$ is classically valid in the sense defined above. It follows via the first Incompleteness Theorem that the set of classically valid formulas is not recursively enumerable.

Given this, it will be necessary to introduce a more general class of interpretations in order to state and prove a completeness theorem. [10] is the source of this technique. The version of the technique developed in [9] would provide an appropriate starting point for an attempt to apply it to the theory presented here.

Before passing on to Section 8.2, it is worth pointing out that **P** *cannot* designate the totally true proposition in any interpretation—quantification over partial properties in the induction axiom will prevent this.

8.2 Set theory Although a set abstraction operator is not present in the language considered here, λ-abstraction can be defined. Using type symbol superscripts in the obvious way, consider χ^{τ} and A, and let $\chi^{[\tau]}$ be chosen from among the variables not free in A.

$$\lambda\chi^{\tau}A =_{df} !\chi^{[\tau]}\forall\chi^{\tau} \cong^{[[\,][\,]]}\chi^{[\tau]}\chi^{\tau}A \ .$$

What one can say about convertibility, given this definition, remains to be figured out. But it can be used to apply the theory presented here to set abstraction in the cumulative theory of types, as follows.

As usual, $\underline{R}_0 = 0$, $\underline{R}_{\alpha+1}$ is the power set of $\underline{R}_{\alpha}$, and, where α is a limit number, $\underline{R}_{\alpha}$ is the union over β less than α of $\underline{R}_{\beta}$. In order to make things as simple as possible, attention will be restricted to interpretations in which the actual world is the only world. This means that one can forget about $\Box$.

Consider an ontology, conforming to the condition just imposed, which has $\underline{R}_{\alpha}$ as its individual designation type, and let $\underline{P}_{\alpha}$ be $Seq\langle\{\#\}, \underline{R}_{\alpha}\rangle \Rightarrow \{T, F\}$. $\underline{P}_{\alpha}$ is the $[\iota]$ designation type of the ontology in question.[18] Let $\in$ have as its designation the total element of $[\iota\iota]$ which, for every member $\langle\#, x, y\rangle$ of $Seq\langle\{\#\}, \underline{R}_{\alpha}, \underline{R}_{\alpha}\rangle$, yields truth iff x is an element of y. Where $\underline{y}$ is chosen from among the variables distinct from $\underline{x}$ and not free in A, define:

$$\{\underline{x}\colon A\} =_{df} !\underline{y} \cong^{[[\iota][\iota]]}\lambda\underline{x}\in\underline{y}\lambda\underline{x}A \ .$$

Clearly, $\{\underline{x}\colon \notin \underline{x}\underline{x}\}$ will have no designation in this interpretation. It follows that both $\in\{\underline{x}\colon \notin \underline{x}\underline{x}\}\{\underline{x}\colon \notin \underline{x}\underline{x}\}$ and $\notin\{\underline{x}\colon \notin \underline{x}\underline{x}\}\{\underline{x}\colon \notin \underline{x}\underline{x}\}$ will designate the Russell proposition. Thus, things work out for these notations in such an interpretation as they did generally for **R** and ~**R**, above.

NOTES

1. From [1], p. 51.
2. For Frege's general semantical theory, see [4], [2], and [3]. For the fundamental part of the system of logic he developed on this basis, see [5].
3. See [15], [16], and [14]. For references to the sources from which Montague drew his ideas, see the introductory portions of [6].
4. ". . . we can keep our valuations bivalent—but we have to be clever in the way we use them," [20], p. 273.
5. The defaults used here are descendants of those used to handle nondenoting singular terms in the treatment of first-order free logic provided by [22], [23], and [24]. [10] contains a very clear exposition of the philosophical motivation of the free logic enterprise. For an excellent text covering this material, see [11].
6. Extensionism cannot be expressed in Frege's formal theory, because no modal operators are present in his system. It is clear, however, that his general semantical theory was not extensionist.

   > "2^2" and "2 + 2" do not have the same *sense*, nor do "$2^2 = 4$" and "2 + 2 = 4" have the same *sense*. ([5], p. 35)

   As regards extensionism and Montague's theory, see [15], p. 139. [21] provides a recent example of a theory which is propositionist, but not extensionist.

7. That Hume understood the principle of total evidence is clear from [9], Section X. For the argument just stated, see [8], pp. 189, 201-202, and *passim*.

8. Borrowed from [11], p. 204.

9. See [12], pp. 502–504 and 714–715.

10. Thus, free logic is seeping into the metalanguage. This makes it possible to write things down much more smoothly than would otherwise be possible (cf [19], pp. 181–182). However, the semantical rules for identity formulas in the object language are not so simple as the condition to which this note is appended—such a formula may be false, even if neither term flanking the identity sign designates anything. See Section 7.2.

11. This conforms to Montague's practice ([15], p. 126).

12. Actually, the head default will be applied to a sequence consisting of [ ] and the intension, in that order. See Section 5.2 for a precise account of head defaults.

13. See the references cited in Note 5. "... logic wherever possible ought not to wait upon philosophy, for logic wherever possible ought to be neutral between different philosophical analyses," [11], p. 213. Amen.

14. Evidently, much digging around in [18] and in sources, both primary and secondary, published more recently than those just cited would be required in order to nail this down.

15. This sort of situation can make true the premises and conclusion of the argument obtained via the analysis of [17], because there quantifiers are restricted to operate on the set of individuals existing in a world. Proceeding in the manner of the present paper would remedy this defect.

16. "Godott" is pronounced as if it contained only one occurrence of "t".

17. In systems of free logic, formulas containing free individual variables are not related to their universal closures in a pleasant way. For example, if there are no individuals, $\forall \underline{x} \sim \cong^{[u]} \underline{xx}$ will be intensionally true, though $\sim \cong^{[u]} \underline{xx}$ will never be true for any assignment of values to variables in any interpretation. The policy adopted here for dealing with such formulas is to avoid talking about them, insofar as this can be done. This conforms with the practice of [2]. It would also be possible to proceed in the manner of [13], where free individual variables are allowed and the usual relation between open formulas and their universal closures simply fails.

18. Observe that if x if a subset of y, then $x \Rightarrow z$ is a subset of $y \Rightarrow z$. Consequently, if $\alpha \leq \beta$, then $\underline{P}_\alpha$ is a subset of $\underline{P}_\beta$. This is pleasant. It would not be true if an undefined truth value were employed, instead of allowing properties to be undefined for some arguments.

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