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An Exposition of Shelah's 'Main Gap': Counting Uncountable Models of ω-Stable and Superstable Theories

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Introduction Throughout this introduction, we let T be a countable complete theory in (finitary) first-order logic having infinite models.

Two of the outstanding conjectures of model theory concern the number $I(T,\kappa)$ of isomorphism types of models of T having a fixed cardinality κ. Morley's conjecture says that $I(T,\kappa)$ is a monotonically increasing function of κ, for uncountable cardinals κ: κ uncountable, and $\kappa < \lambda$ imply $I(T,\kappa) \leq I(T,\lambda)$. The other is Vaught's conjecture: $I(T,\aleph_0) \leq \aleph_0$, or $I(T,\aleph_0) = 2^{\aleph_0}$.

Saharon Shelah's deep and extensive work in the exploration and classification of all possible complete theories can be seen as motivated to a large extent by Morley's conjecture. The results of this work point toward the possibility that Morley's conjecture will eventually be proved by giving more or less explicitly all possible spectrum-functions $\kappa(\geq \aleph_1) \mapsto I(T,\kappa)$, with each possibility (hopefully) conforming to Morley's requirement.

In [4], Shelah proved that $I(T,\kappa) = 2^\kappa$ ($\kappa \geq \aleph_1$) for all unstable T. One of the main results of [5] is that the same holds for all T that are not superstable. At the end of [5], some partial results are given for very special totally

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transcendental and superstable theories (so-called nonmultidimensional ones; see Theorems 2.3 and 2.4 in Chapter IX). Although these results are special, the tools used in proving them have turned out to be very useful, and in particular, those tools, orthogonality and regular types (dealt with in Chapter V of [5]), are the chief ones for the results presented in this paper.

Let us mention that, independently of Shelah, Alistair Lachlan also proved somewhat related theorems on the spectrum function $I(T, \kappa)$ for T totally transcendental and $\kappa \geq \aleph_1$ (see [1] and [2]).

The present paper is an exposition of the contents of Shelah's papers [6], [7]; all results stated and proved in this paper are due to Shelah.

In these papers, Shelah proves the Morley conjecture for totally transcendental theories almost completely. He introduces a key property of the theory T, which is actually quite simply stated: for T totally transcendental, T has *NDOP* (the negation of the "dimensional order property") if over certain sets A one has a minimal $\aleph_0$-saturated model M (i.e., such that there is no $\aleph_0$-saturated N with $A \subset N \subsetneq M$); A here is any set of the form $M_1 \cup M_2$ with two $\aleph_0$-saturated models M_1, M_2, each extending a third one M_0 such that M_1 and M_2 are independent (in the nonforking sense) over M_0. For the benefit of the reader not well-versed in general stability theory, let us note that this independence means that for any finite tuple $\vec{a}$ of elements of M_2, the Morley rank of the type of $\vec{a}$ over M_1 equals that of the type of $\vec{a}$ over M_0. Let us note that the property *NDOP* is defined for an arbitrary stable theory T (see Definition 1.1 below). *NDOP* turns out to be a most important dividing line in the realm of totally transcendental, and more generally, superstable theories. With *DOP* signifying the negation of *NDOP*, Shelah proves that $I(T, \kappa) = 2^\kappa$ for T totally transcendental having *DOP* ($\kappa \geq \aleph_1$). Actually, the same equality is proved for an arbitrary superstable T having *DOP* for $\kappa \geq 2^{\aleph_0}$ (see Theorem 2.3 below). On the other hand, for theories having *NDOP* there is a structure theory developed that, in many cases, allows a precise determination of $I(T, \kappa)$. In particular, the depth of T, $d(T)$, is defined; $d(T)$ is an ordinal or ∞. If $d(T) = \infty$ and T is superstable, then again it turns out that $I(T, \kappa) = 2^\kappa$ for most κ (see 5.2 below). If, however, $d(T)$ is an ordinal, and, say, T is totally transcendental, then $d(T)$ is necessarily a countable ordinal, and, e.g., we obtain the upper estimate $I(T, \aleph_\alpha) \leq \beth_{d(T)+1}(|\omega + \alpha|)$ in case $d(T) \geq \omega$ (see 5.1 below). (Note that Shelah's $Dp(T)$ in [6] equals our $d(T) + 1$ if $d(T) \geq \omega$.)

The phrase "the main gap" refers to the following state of affairs.

Main Gap Theorem *Either the spectrum function $I(T, -)$ is the maximal one ($I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ for all $\alpha \geq 1$), or, if not, it necessarily has to satisfy the inequality*

$$I(T, \aleph_\alpha) < \beth_{\omega_1}(|\omega + \alpha|)$$

for all $\alpha \geq 1$.

The fact of the Main Gap for totally transcendental T is a consequence of the results just mentioned. It clearly holds for any unsuperstable T, by the earlier result of Shelah mentioned above. Until recently, it remained open for T superstable, not totally transcendental. Recently, Shelah has completed, among others, the proof of the full Main Gap Theorem for any T and that of the full Morley conjecture.[1]

The results for theories having *NDOP* are obtained through a structure theory concerning all models in the case where T is totally transcendental, and concerning models that are slightly-more-than-$\aleph_0$-saturated ($\aleph_\epsilon$-saturated models) in case T is only superstable. This structure theory introduces "representations" of models ($\aleph_\epsilon$-saturated models) in the form of certain trees of "small" models; in the case where T is totally transcendental, the model represented by the tree is simply the one that is prime over the union-set of the models in the tree. Naturally, there is an 'existence' part, and a 'uniqueness' part of this structure theory. The former asserts the existence of representations. This turns out to be a fairly easy consequence of *NDOP* (see Section 4 below). 'Uniqueness' holds only in a suitably weak sense; although it is a general result not using *NDOP* or even superstability, its proof is more elaborate (see Section 3 below).

The aim of the authors of the present paper was to expose the inherent elegance, power, and even sometimes the simplicity, of the mathematics involved in Shelah's work on the "main gap". This work should be seen to have two distinct levels. The first level consists of the "local theory", the fundamental general truths "in the small" that serve as the framework for all subsequent concepts and constructions. (Shelah named his Chapter III in [5] "Global theory"; we would have said "local theory" instead.) The first level itself has two layers: the first centers around the notion of independence (nonforking), the second, applicable mainly to superstable, or even, totally transcendental theories, is based, in addition, on the notions of orthogonality and regular types. It was found impossible to write a short and informative section of preliminaries to this paper. Instead, the paper [3] was prepared to serve as such a compendium of preliminaries. It lists the basic definitions and facts of the "first layer" mentioned above, mainly without proof, and it gives an essentially self-contained treatment of the "second layer". References of the form A.2, B.3, etc. refer to [3].

The second level of Shelah's work is the subject matter of this paper; it can be understood only on the basis of the "local theory".

Let us make a few remarks for the reader interested in comparing this paper to the originals, [6] and [7]. First of all, some key 'local' facts such as 1.1 in [6] and 1.1 in [7] were put into [3]. The construction of many models for theories having *DOP* given in Section 2 is different from the one outlined in [6]; it was found by the first author, with a view to applicability to Vaught's conjecture (see [8]). We should note that Shelah was aware of the possibility of such a proof as well. The quasi-uniqueness theorem of Section 3 does not as such appear in Shelah's work, but all the ingredients of its proof appear there; it was formulated in the form stated by the second author. The existence of representations is proved in a considerably shorter (in fact, a very short) way in Section 4; both authors had their shares in arriving at the proof. The fifth section containing the actual results on the spectrum function is very elementary. In fact, the possibility of a separation of the structural theory and the numerical computations was the main impetus for this presentation of the material. We should note that Section 5 contains a trick communicated to us by Shelah, which may not so appear in the corresponding computations in [7].

1 DOP The "dimensional order property" (*DOP* for short) is a "hidden order property" in the sense that if T has *DOP*, then it is possible to "code" (or "define") arbitrary orders in models of T. This will not be obvious from the definition (although at the beginning of Section 2 in [6] one reads that *DOP* 'clearly means there is a hidden order property'); rather, it will be shown in the next section. Actually, referring to 'order' here is not important, since we will be able to code essentially arbitrary relations. All in all, the expression "dimensional order property" does not seem to be a particularly happy choice. Also, its negation is the 'positive' property that makes a theory nice and simple. The negation of *DOP* will be written *NDOP*.

We consider a-models M_0, M_1, M_2, M satisfying

$$\begin{gathered} M_0 \subset M_1,\ M_0 \subset M_2 \\ M_1 \underset{M_0}{\perp} M_2 \\ M\ a\text{-prime over } M_1 \cup M_2. \end{gathered} \tag{1}$$

Definition 1.1 T has *DOP* if there are a-models M_0, M_1, M_2, M satisfying (1), and a nonalgebraic type $p \in S(M)$ such that $p \perp M_1$, $p \perp M_2$. Equivalently, T has *NDOP* if whenever we have a-models as in (1), any nonalgebraic type over M is nonorthogonal either to M_1 or to M_2.

Remark: The definition in [6] (2.1 Definition) is the following: T has *DOP* if there are a-models as in (1) such that M is not a-minimal over $M_1 \cup M_2$, i.e., there is an a-model M' with $M_1 \cup M_2 \subset M' \subsetneq M$. We will show below that *NDOP* in the sense of Definition 1.1 implies the a-minimality of M. For the equivalence (not used in the present paper) of the two definitions, we refer the reader to Section 2 of [6].

Proposition 1.2 *Let T be superstable. T has NDOP iff for a-models as in* (1), *and for any type p* (*over any set*)

$$p \not\perp M \Leftrightarrow p \not\perp M_1 \text{ or } p \not\perp M_2\ . \tag{2}$$

Proof: Since $p \in S(M)$ being nonalgebraic clearly implies $p \not\perp M$, condition (2) implies *NDOP*. Assume *NDOP*. In condition (2), the direction $\Leftarrow$ is automatic. Suppose $p \not\perp M$. By D.11(v), there is a regular $q \in S(M)$ such that $p \not\perp q$. By *NDOP*, $q \not\perp M_1$, say. By D.5′(ii), D.8, $p \not\perp M_1$ follows.

Proposition 1.3 *Let T be t.t. Then T has NDOP iff the following holds: whenever M_0, M_1, M_2, M are* (*ordinary*) *models satisfying*

$$\begin{gathered} M_0 \subset M_1,\ M_0 \subset M_2 \\ M_1 \underset{M_0}{\perp} M_2 \\ M \text{ prime over } M_1 \cup M_2 \end{gathered} \tag{1$'$}$$

then for any nonalgebraic $p \in S(M)$, $p \not\perp M_1$ or $p \not\perp M_2$.

Proof: '*only if*': Suppose we have models as in (1)′. Let us find a-models N_0, N_1, N_2 such that we have the independent diagram Figure 1 (see after A.13)

$$\begin{tikzcd}
N_1 & M_1 \arrow[l] & & M_2 \arrow[r] & N_2 \\
& & M_0 \arrow[ul] \arrow[ur] \arrow[d] & & \\
& & N_0 \arrow[uull] \arrow[uurr] & &
\end{tikzcd}$$

Figure 1.

(with the arrows being inclusions); this is possible by first adjoining N_0 'freely' to the independent diagram of the M's by A.12, and then extending $M_1 \cup N_0$ to N_1, $M_2 \cup N_0$ to N_2 'freely' again by A.12. In particular, we have $N_1 \underset{N_0}{\perp} N_2$. By the technical Lemma B.11, we have $M_1 \cup M_2 \underset{T-V}{<} N_1 \cup N_2$. Let M be any prime model over $M_1 \cup M_2$; let $\langle a_i : i < \alpha \rangle$ be a construction of M over $M_1 \cup M_2$; write $A_i = \{a_j : j < i\}$. We claim that M is atomic over $N_1 \cup N_2$, in fact, that $\langle a_i : i < \alpha \rangle$ is a construction of M over $N_1 \cup N_2$. Indeed, by B.12, we have $M_1 \cup M_2 \cup A_i \underset{T-V}{<} N_1 \cup N_2 \cup A_i$ (since M, hence A_i, is atomic over $M_1 \cup M_2$); hence, since a_i is isolated over $M_1 \cup M_2 \cup A_i$, by B.13 it is isolated over $N_1 \cup N_2 \cup A_i$, as required. It follows (see B.8) that there is an a-prime model N over $N_1 \cup N_2$ containing M. Now, suppose $p \in S(M)$ is nonalgebraic; consider $q = p|N$. By *NDOP*, $q \not\perp N_1$, say; hence $p \not\perp N_1$. By the independence of the diagram, $N_1 \underset{M_1}{\perp} M_2$, hence by C.12(i), $N_1 \underset{M_1}{\perp} M$; from $p \in S(M)$ and $p \not\perp N_1$ it follows that $p \not\perp M_1$ by C.8. This completes the proof.

'*if*': we show, using the new version of *NDOP* that if we have a-models as in (1), then in fact M is (ordinary) prime over $M_1 \cup M_2$. This will obviously make the original version of *NDOP* a consequence of the new version. In fact, we show that M is minimal over $M_1 \cup M_2$. Suppose $M_1 \cup M_2 \subset M' \subsetneqq M$. Let p be any type over M' realized in $M - M'$. If $p \not\perp M_0$, then by D.17 and D.19′ we would have some $a \in M - M'$ with $a \underset{M_0}{\perp} M'$. Since then $a \underset{M_0}{\perp} M_1 \cup M_2$, C.12(ii) (!) implies $a \underset{M_0}{\perp} M$, a contradiction to $a \in M - M_0$. Thus we have $p \perp M_0$. By the new version of *NDOP*, $p \not\perp M_1$, say. By D.17 there is an *SR* type $q \in S(M_1)$ with $p \not\perp q$; by D.19′ $q|M'$ is realized in $M - M'$, by b, say; also, $q \perp M_0$ (see D.5′(ii)). Since $M_1 \underset{M_0}{\perp} M_2$, we have $b \underset{M_1}{\perp} M_2$; by C.12(ii) again, $b \underset{M_1}{\perp} M$, a contradiction to $b \in M - M_1$.

Proposition 1.2′ *Let T be t.t. Then T has NDOP iff for models as in* (1)′, *and any p*, (2) *holds.*

Proof: By 1.3, as 1.2.

For the purposes of the next section we 'prepare' *DOP*; we show that if there are items witnessing *DOP*, there must be certain special ones too. We assume that T is superstable.

Proposition 1.4 *Let T be superstable. Assume that T has DOP. Then there is a system of witnesses for DOP as in 1.1 having the additional property that*

M_i is of the form $M_0[a_i]$, i.e., it is a-prime over $M_0 \cup \{a_i\}$, for some a_i regular over M_0 (i.e., such that $t(a_i/M_0)$ is regular) ($i = 1, 2$).

Proof: Temporarily, call M' special over M if $M' = M[a]$ for some a regular over M. To prove the contrapositive of the assertion, we assume the condition of *NDOP* for the situation with the additional condition of M_i being special over M ($i = 1, 2$). First we deduce from this the following version: whenever M_k is special over M_0 (all a-models) for $1 \le k \le n$, $\langle M_k \rangle_{1 \le k \le n}$ is independent over M_0, and M is the prime model over $A =_{df} \bigcup_{1 \le k \le n} M_k$, then $p \not\perp M$ implies $p \not\perp M_k$ for some k, $1 \le k \le n$. The proof is by induction on n; $n = 2$ is the assumption. The following is an easy *exercise*: if $\hat{M}_k$ is a-prime over $M_n \cup M_k$ ($1 \le k < n$), then the a-prime model over $\hat{A} = \bigcup_{1 \le k < n} \hat{M}_k$ is a-prime over A (hint: use C.12(ii)). Hence (by the uniqueness of a-prime models, B.8), we may assume that our M is a-prime over $\hat{A}$. Also, if $M_k = M_0[a_k]$ with a_k regular over M_0, then $\hat{M}_k = M_n[a_k]$, and we have that $\langle \hat{M}_k \rangle_{1 \le k < n}$ is independent over M_n. Suppose $p \not\perp M$; by induction hypothesis, $p \not\perp \hat{M}_k$ for some k, $1 \le k < n$. Hence, by the restricted *NDOP*, $p \not\perp M_k$ or $p \not\perp M_n$, as desired.

Secondly, we show *NDOP* for the case when M_i is finitely generated over M_0, $M_i = M_0[\vec{a}^i]$ ($i = 1, 2$). By the decomposition Theorem D.10, we may assume *wlog* that $\vec{a}^i = \langle a_0^i, \ldots, a_{k^i}^i \rangle$, with each a_j^i regular over M_0, with both $\vec{a}^1$, $\vec{a}^2$ being independent systems over M_0. Since $M_1 \underset{M_0}{\perp} M_2$, their concatenation is independent over M_0 too.

Let M_j^i be a copy of $M_0[a_j^i]$ ($i = 1, 2$; $j \le k^i$). The a-prime model over $B^i = \bigcup_{j \le k} M_j^i$ is isomorphic to M_i over $M_0 \cup \{\vec{a}^i\}$; hence, by a suitable choice of the copies M_j^i, M_i is a-prime over B^i. M being a-prime over $M_1 \cup M_2$, we see that it is a-prime over $\bigcup_{j \le k^1} M_j^1 \cup \bigcup_{j \le k^2} M_j^2$. Now, if $p \not\perp M$, then by the version of *NDOP* we already have, $p \not\perp M_j^i$ for some i and j, and $p \not\perp M^i$ for $i = 1$ or $i = 2$.

Finally, we consider the general case. Suppose $p \in S(M)$ is nonalgebraic and $p \perp M_1$, $p \perp M_2$, to derive a contradiction. Let $B \subset M$ be finite such that p is based on B, and let $B_i \subset M_i$ be finite ($i = 1, 2$) such that $st(\vec{B}/B_1 \cup B_2 \cup M_0) \vdash t(\vec{B}/M_1 \cup M_2)$; let $\hat{M}_i = M_0[B_i] \subset M_i$ ($i = 1, 2$). Since B is a-atomic over $\hat{M}_1 \cup \hat{M}_2$, there is an a-prime model $\hat{M}$ over $\hat{M}_1 \cup \hat{M}_2$ such that $B \subset \hat{M}$. Clearly, $p \not\perp \hat{M}$, $p \perp \hat{M}_1$, $p \perp \hat{M}_2$; also, $\hat{M}_1 \underset{M_0}{\perp} \hat{M}_2$. We have obtained a contradiction to the form of *NDOP* we proved above.

Corollary 1.5 *Suppose T is superstable and satisfies DOP. Then there are finite sets A, $D_1 \supset A$, $D_2 \supset A$ and $B \supset D_1 \cup D_2$ and we have a regular type $p \in S(B)$ such that D_1/A, D_2/A are of weight 1, $D_1 \underset{A}{\perp} D_2$, B is dominated by D_1D_2/A, by D_2/D_1 and by D_1/D_2 and $p \perp D_1$, $p \perp D_2$. Moreover, we can arrange that $d_1 \equiv d_2(A)$ or $d_1/A \perp d_2/A$ for d_i enumerating D_i.*

Proof: By 1.4 we have a-models as in (1) and in addition, $M_i = M_0[c_i]$ ($i = 1, 2$), with c_i regular over M_0; and we have $p \in S(M)$ nonalgebraic such that $p \perp M_i$ ($i = 1, 2$). By D.9, C.13″(iii), we may assume p to be regular. Define

$B \subset M$, $D_i \subset M_i$ $(i = 1, 2)$, $A \subset M_0$ to be finite sets such that the following are all satisfied:

$$c_i \in D_i,\ A \subset D_i,\ D_1 \cup D_2 \subset B\ ,$$
$$st(\vec{B}/D_1D_2) \vdash t(\vec{B}/M_1M_2)\ ,$$
$$p \text{ is based on } B,$$
$$B \underset{D_i}{\perp} M_i\ (i = 1, 2)\ , \tag{3}$$
$$B \underset{A}{\perp} M_0\ . \tag{4}$$

Such choices are clearly possible. B is a-atomic over $M_0 \cup (D_1 \cup D_2)$, as well as over $M_1 \cup D_2$ and $M_2 \cup D_1$; from these facts, and by (4) and (3) respectively, we conclude, using C.11(ii) and C.12(ii), that B is dominated by D_1D_2/A, as well as by D_2/D_1 and D_2/D_1. In case $c_1/M_0 \perp c_2/M_1$, we also have $d_1/M_0 \perp d_2/M_1$. Otherwise, M_1 and M_2 are M_0-isomorphic, in which case we can clearly achieve $d_1 \equiv d_2(A)$. c_i/M_0 is of weight 1 by D.8; d_i is dominated by c_i/M_0 by C.12(ii); hence d_i/M_0 is of weight ≤ 1 by D.2(ix); since $c_i \in D_i$, $w(d_i/M_0) = 1$. For the type p of the assertion, use $p|B$. It is clear now that all requirements are met.

We consider some examples of theories.

Let T_1 be the theory of the structure M of the similarity type $L = \{A, B, f_n\ (n < \omega)\}$ (A unary, B ternary relation symbols, f_n binary operation symbol) for which A^M is an infinite (countable) set; for $a \neq a'$ in A^M, $B_{aa'} =_{df} \{b \in |M|: M \models Bbaa'\}$ equals $\{f_n^M(a, a'): n < \omega\}$, $f_n^M(a, a') = f_n^M(a', a)$ $f_n^M(a, a') \neq f_m^M(a, a')$ for $n \neq m$ (and $a \neq a'$), each $B_{aa'}$ (for $a \neq a'$) is disjoint from A^M and all other $B_{a_1a_1'}$, and B and f_n are given as far as they are unspecified above in some trivial (irrelevant) way. Then an arbitrary model M of T_1 is like M above, except that A^M is not necessarily countable, and $B_{aa'}$ is not necessarily exhausted by the $f_n^M(a, a')$ $(n < \omega)$. It is easy to see that T_1 is ω-stable (t.t.). T_1 has *DOP*. First of all, note that the prime model M over $M_1 \cup M_2$ is the definable closure of $M_1 \cup M_2$; one has to add all $f_n(a, a')$ for $a \in A(M_1)$, $a' \in A(M_2)$. Taking M_0, M_1, M_2, with $M_1 \underset{M_0}{\perp} M_2$ and $A(M_0) \subsetneqq A(M_1)$, $A(M_0) \subsetneqq A(M_2)$, we choose $a \in A(M_1) - A(M_0)$, $a' \in A(M_2) - A(M_0)$; then for $b \in B_{aa'}(\mathfrak{C})$, with $b \neq f_n(a, a')$ $(n < \omega)$, we have that $t(b/M) \perp M_1$, $t(b/M) \perp M_2$. Note also that now M is minimal over $M_1 \cup M_2$, hence the version for ordinary models of the alternative definition of *DOP* mentioned after 1.1 would not be appropriate.

The isomorphism type of an arbitrary (symmetric, irreflexive) binary relation R can be coded by the isomorphism type of a model of T_1: take A^M to be the underlying set of R, and make sure that $B_{aa'}$ contains at least one additional element besides the $f_n(a, a')$ precisely when $R(a, a')$ holds. Clearly, in particular, if we denote the model so obtained by M_R, then $M_{R_1} \simeq M_{R_2}$ iff $R_1 \simeq R_2$. Hence, among others, T_1 has the maximal number of models in every infinite power. In the next section, we'll show that, to a large extent, any s.s. theory having *DOP* resembles T_1.

Next, let T_2 be the theory of the structure M over the similarity type $\{f\}$ (f a unary operation symbol) such that $M \models f^n(a) \neq a$ $(n \geq 1)$("there are no loops") and $M \models \forall a \exists$ infinitely many b such that $f(b) = a$. Again, T_2 is t.t. Now, the prime model over $M_1 \cup M_2$ is the union $M_1 \cup M_2$ itself. It is easy to see that T_2 has *NDOP*.

Any model of T_2 is given up to isomorphism, roughly speaking, by specifying the cardinality of the set of predecessors of each element in the model or by a "tree of cardinalities". Theorem 4.3′ below asserts a somewhat similar analysis for models of an arbitrary s.s. theory having *NDOP*. It is clear that T_2 has the maximal number of nonisomorphic models in every uncountable power. This is related to the fact that T_2 is *deep*, in the sense specified in Section 5; it is proved there that all deep theories share the mentioned property of T_2 (Theorem 5.2).

Finally, we mention a family of examples. For any ordinal α, consider the language $L_\alpha = \{E_\beta: \beta < \alpha\}$, E_β a binary relation symbol. T_3^α is the theory whose models are the L_α-structures M in which each E_β is an equivalence relation with infinitely many infinite equivalence classes, and for $\beta_1 < \beta_2 < \alpha$, each E_{β_2}-class is the union of infinitely many E_{β_1}-classes. It is not hard to see that T_3^α is t.t., and it has *NDOP*. A little analysis shows that T_3^α does not have the maximal number of nonisomorphic models in suitable uncountable powers. This is in accordance with the fact that T_3^α is *shallow* (opposite of 'deep', see Section 5). In fact, Theorem 5.1 (among others) asserts that every shallow, *NDOP* theory shares the last mentioned property of T_3^α. Also, the depth of T_3^α as introduced in Section 5 turns out to be $\alpha \pm \epsilon$.

2 DOP implies many models Throughout this section T is superstable.

Let λ_0 be the least cardinal $\geq |S(A)|$ for any set A of cardinality of $|T|$ (= the number of formulas of the language of T). It is easy to see that every set A of cardinality $\geq \lambda_0$ has an a-model extension of the same cardinality as A itself. If T is t.t. and countable, $\lambda_0 = \aleph_0$. If T is (superstable and) countable, $\lambda_0 \leq 2^{\aleph_0}$.

Definition 2.1 A stationary type q is called *trivial* if every *nf* extension q' (over A, say) of q satisfies the following: whenever I is a set of elements realizing q' such that any two-element subset of I is independent over A, then I is independent over A.

Remark: Since it is easy to see that the property of q' in the definition is inherited from q' to any q'' such that q' is an *nf* extension of q'', we have that triviality is a parallelism invariant.

Lemma 2.2 *Let A be a set, c a type of weight 1 over A, and p a regular type such that $p \perp A$ and $D = dom(p)$ is dominated by c/A, and $c \subset D$. Assume that c/A is nontrivial. Then for every uncountable $\lambda \geq \lambda_0$, T has 2^λ isomorphism types of a-models of power λ.*

Proof: Suppose p is a nontrivial stationary type. Let q' be an *nf* extension of q, $dom\ q' = B$, and I a set of elements realizing q' such that I is not independent over B but every two-element subset of it is; moreover, choose I such that I has the smallest possible (finite) cardinality. Let c_0, c_1, c_2 be three distinct elements of I, $I' = I - \{c_0, c_1, c_2\}$. By the minimality of I, $q'' = t(c_i/BI')$ is an *nf* extension of q $(i < 3)$, $\{c_0, c_1, c_2\}$ is not independent over $B \cup I'$, but every two-element subset of it is. Below, a *triangle over* a set $\hat{A}$ will mean a three-element set J (of tuples) such that J is not independent over $\hat{A}$, but every two-element subset of J is independent over $\hat{A}$.

Assume the hypotheses of the lemma. By the above argument, we may assume in addition that there is a triangle of elements realizing $t(c/A)$ (by suitably extending A to some A' such that, among others, $D \underset{A}{\perp} A'$; see also C.11(i)). We may also assume in addition that both A and D are finite (for this, note that if $\vec{c} = \{c_1, c_2, c_3\}$ is a triangle over A, and $c_1c_2c_3 \underset{A_0}{\perp} A$, then $\vec{c}$ remains a triangle over A_0; also use C.11(ii)). Let d^0 be a tuple enumerating D. Since d^0 is dominated by c/A, d^0 has weight 1 over A; also, as it is easily seen, the triangle $\vec{c}$ gives rise to a triangle $\vec{d} = \langle d_0^0, d_1^0, d_2^0\rangle$ of elements realizing $t(d^0/A)$. We may, moreover, assume that $A = \varnothing$, since having many nonisomorphic models with the distinguished finite subset A implies having many nonisomorphic models in the original language. We may now forget about c as well. To summarize, we have a weight-1 type p with $dom(p) = d^0$ such that d^0 has weight 1 (over $\varnothing$), $p \perp \varnothing$, and there exists a triangle (over $\varnothing$) $\langle d_0^0, d_1^0, d_2^0\rangle$ of elements realizing the same type as d^0.

Let $\mathcal{G}$ be a family of power λ of pairwise nonisomorphic binary relations (graphs) G satisfying the following: denoting the field of G by $|G|$, we have $\|G\| < \lambda$, G is symmetric, irreflexive, and connected; moreover for any $a \in |G|$ there are at least two distinct $b \in |G|$ such that $\langle a, b\rangle \in G$, and finally, there are no G-triangles, i.e., $\langle a, b\rangle$, $\langle b, c\rangle$, $\langle c, a\rangle$ cannot all be in G. Also, assume that the fields of the relations in $\mathcal{G}$ are pairwise disjoint. It is easy to construct such a family $\mathcal{G}$.

We construct a system $D_\mathcal{G}$ of elements indexed by the set $I = \bigcup_{G \in \mathcal{G}} |G| \cup \{\{i, j\}: \langle i, j\rangle \in G, G \in \mathcal{G}\}$; the element indexed by $i \in |G|$ is denoted by d^i, the element indexed by $\{i, j\}$ is denoted by d^{ij}. We consider on I the partial ordering $<$ in which $i < \{i, j\}$, $j < \{i, j\}$, and no other relations hold. We define the system $D_\mathcal{G}$ to be independent relative to $<$ and to satisfy $d^i d^j d^{ij} \equiv d_0^0 d_1^0 d_2^0$. By applying A.12 twice (first, to get the d^i, next, to get the d^{ij}) such a system $D_\mathcal{G}$ exists. $D_\mathcal{G}$ also denotes the corresponding set of elements; $D_G = \{d^i: i \in |G|\} \cup \{d^{ij}: \langle i, j\rangle \in G\}$, and for $X \subset \mathcal{G}$, $D_X = \bigcup_{G \in X} D_G$.

For any $X \subset \mathcal{G}$, we construct an a-model $M = M_X$ of power λ such that:

(i) For any $d \in D_X$, the type p_d has dimension $< \lambda$ in M (p_d is the type over d obtained by "replacing" d^0 by d in $p \in S(d^0)$. (See the beginning of Section A of [1].)

(ii) For any stationary type q with a finite domain included in M such that $q \perp p_d$ for all $d \in D_X$, we have $dim(q, M) = \lambda$.

We will show that such M_X exists, moreover that $M_X \simeq M_{X'}$ implies $X = X'$; this will prove the lemma.

Let $X \subset \mathcal{G}$, $G \in X$, $d \in D = D_G$, $p = p_d$, $D' = \bigcup_{G' \in X - \{G\}} D_G$. By the 'relative independence' built into $D_\mathcal{G}$, we have $D \perp D'$; since $p \perp \varnothing$, it follows that $p \perp D'/D$. By D.12″(iii), it follows that for M_0 the a-prime model over $D_X = D \cup D'$, we have $dim(p_d, M_0) \leq card(D) + \aleph_0 < \lambda$. Since $\lambda \geq \lambda_0$, we have $card(M_0) = \lambda$. Applying D.12′ to the family $\{p_d|M_0: d \in D_X\}$ of types, we conclude that M_X exists as required (see (i) and (ii) above).

We describe now how to "recover" the set $X \subset \mathcal{G}$ from $M = M_X$, proving the second assertion above.

By a *class* we mean an equivalence class $[d]$ of the forking relation $\not\perp$ (over $\emptyset$) among elements d of M realizing the type of d^0; by a *good class*, a class containing some d such that $dim(p_d, M) < \lambda$. A *triangle* $\langle \mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2 \rangle$ of classes is such that for some $d_i \in \mathfrak{D}_i$ $(i < 3)$, $\langle d_0, d_1, d_2 \rangle$ is a triangle. Since d^0 is of weight 1, $\langle \mathfrak{D}_0, \mathfrak{D}_1, \mathfrak{D}_2 \rangle$ is a triangle implies that $\langle d_0, d_1, d_2 \rangle$ is a triangle for any $d_k \in \mathfrak{D}_k$.

The construction has built some triangles of good classes (or *good triangles*) into M: for all $G \in X$, *all the triples* $\langle [d^i], [d^j], [d^{ij}] \rangle$ for $\langle i, j \rangle \in G$.

Claim *The latter are precisely all the good triangles in M.*

Proof of Claim: First of all, the good classes in M are precisely the $[d]$ for $d \in D_X$; these latter are good classes by (i); and if $\mathfrak{D} = [d']$ is not among these, then $d' \perp d$ for all $d \in D_X$, hence by $p_{d'} \perp \emptyset$, we have $p_{d'} \perp p_d$ for all $d \in D_X$, hence by (ii), $dim(p_{d'}, M) = \lambda$, as required. Secondly, we claim that if $d_k \in D_X$ $(k < 3)$, $\mathfrak{D}_k = [d_k]$, and the $\mathfrak{D}_k$ form a triangle, then there is $G \in \mathcal{G}$ and $\langle i, j \rangle \in G$ such that $\{d_0, d_1, d_2\} = \{d^i, d^j, d^{ij}\}$. It is clear that for a (unique) G we must have $\{d_0, d_1, d_2\} \subset D_G$; the rest of the proof is an elementary check using relative independence and the nonexistence of G-triangles. The details are as follows. Having fixed G, we call any $d^i \in D = D_G$ an element of the first kind, any $d^{ij} \in D$ one of the second kind. Every element d^{ij} of the second kind has two *roots* d^i, d^j of the first kind. Suppose $\{d_0, d_1, d_2\} \subset D$ is a triangle. If each d_k $(k < 3)$ is an element of the second kind, one of the d_k, say d_0, has a root which is not a root of either of the other elements of the triangle (otherwise there would be a G-triangle). If neither root of d_0 is a root of d_1 or d_2, 'relative independence' of D yields $d_0 \perp d_1 d_2$, a contradiction to $\{d_0, d_1, d_2\}$ being a triangle. Otherwise, by 'relative independence' of D, we have $d_0 \underset{d}{\perp} d_1 d_2$, where d is the other root of d_0; since $d_0 \perp d$, it follows that $d_0 \perp d_1 d_2$, a contradiction. If two of the d_k, say d_0 and d_1, are of the second kind, d_2 of the first kind, then either d_2 is the common root of d_0 and d_1, in which case we have $d_0 \underset{d_2}{\perp} d_1$, hence $d_0 \perp d_1 d_2$ (which is false), or d_2 is not a root of d_0 say, and then $d_0 \underset{C}{\perp} d_1 d_2$ for $C = \emptyset$ or $C = \{d\}$, with d the common root of d_0 and d_1, hence $d_0 \perp d_1 d_2$ again. It is clearly impossible to have all three of the d_k of the first kind. Hence one, say d_0, is of the second kind, $d_0 = d^{ij}$, and d_1, d_2 are of the first kind; it is easy to conclude that $\{d_1, d_2\} = \{d^i, d^j\}$ must be the case. This concludes the proof of the Claim.

Now it is quite clear that from the isomorphism type of $M = M_X$ we can recover X itself. Namely, using that each G is connected, we see that the sets $\{[d] : d \in D_G\}$ of good classes, for $G \in X$, are recovered as the connected components of the good classes with respect to the relation: $\mathfrak{D}$, $\mathfrak{D}'$ are related iff there is a triangle of good classes containing $\mathfrak{D}$ and $\mathfrak{D}'$. For a fixed $G \in X$, the classes $[d^i]$ of elements of the first kind are recovered as those that belong to at least two good triangles. Finally, the relation G itself is recovered by noting that $\langle i, j\rangle \in G$ iff there is a good triangle containing $[d^i]$ and $[d^j]$.

This concludes the proof of the lemma.

Theorem 2.3 *Suppose T has DOP. Then for every uncountable $\lambda \geq \lambda_0$, T has 2^λ isomorphism types of a-models of power λ.*

Proof: By the reduction 1.5, we have finite sets A, $D_0 \supset A$, $D_1 \supset A$ and $B \supset D_0 \cup D_1$, and we have a regular type $p \in S(B)$ such that D_0/A, D_1/A are of weight 1, $D_0 \underset{A}{\perp\!\!\!\!\perp} D_1$, B is dominated by D_0D_1/A, D_1/D_0 and D_0/D_1; $p \perp D_0$, $p \perp D_1$; moreover, either $d_0^0 \equiv d_1^0(A)$ or $d_0^0/A \perp d_1^0/A$ for d_i^0 enumerating D_i. *Wlog*, $A = \emptyset$. Let b^0 enumerate B. By Lemma 2.2, we may assume that $t(d_1^0/d_0^0)$ and $t(d_0^0/d_1^0)$ are trivial (since $p \perp d_0$, $p \perp d_1$, and B is dominated by d_1^0/d_0^0 and d_0^0/d_1^0). Hence (since $d_0^0 \perp\!\!\!\!\perp d_1^0$), $t(d_0^0)$ and $t(d_1^0)$ are trivial too.

A 0-*class* (1-*class*) is an equivalence class $[d]$ under $\not\perp\!\!\!\!\perp$ of an element $d \equiv d_0^0$ ($d \equiv d_1^0$) (in the universe, or later, in a fixed model). $\mathfrak{D}_k$, $\mathfrak{D}_k'$ etc., always denote a k-class. If $d_0^0 \equiv d_1^0$, then 0-classes and 1-classes coincide.

Whenever $\langle \mathfrak{D}^i: i \in I\rangle$ is a family of distinct classes (0-classes and 1-classes), then for any $d^i \in \mathfrak{D}^i$ ($i \in I$), $\langle d^i\rangle_{i\in I}$ is an independent system; this follows by the triviality of $t(d_0^0)$, and $t(d_1^0)$. (Consider the cases $d_0 \equiv d_1$, $d_0 \perp d_1$ separately.)

Any type of the form p_b with $bd_0d_1 \equiv b^0d_0^0d_1^0$ is said to be *over* (d_0, d_1). Given $\mathfrak{D}_0$, $\mathfrak{D}_1$, a type over (d_0, d_1) for some $d_0 \in \mathfrak{D}_0$, $d_1 \in \mathfrak{D}_1$ is said to be *over* $(\mathfrak{D}_0, \mathfrak{D}_i)$. If $\mathfrak{D}_0$, $\mathfrak{D}_0'$, $\mathfrak{D}_1$, $\mathfrak{D}_1'$ are distinct, p is over $(\mathfrak{D}_0, \mathfrak{D}_1)$, and p' is over $(\mathfrak{D}_0', \mathfrak{D}_1')$, then $p \perp p'$; to see this notice that for $d_k \in \mathfrak{D}_k$, $d_k' \in \mathfrak{D}_k'$ we have $d_0d_1 \perp\!\!\!\!\perp d_0'd_1'$, hence $b \perp\!\!\!\!\perp b'$ if $bd_0d_1 \equiv b'd_0'd_1' \equiv b^0d_0^0d_1^0$, since b^0 is dominated by $d_0^0d_1^0$; $p \perp p'$ follows from $p \perp \emptyset$ by C.8. In other words, if $\{\mathfrak{D}_0, \mathfrak{D}_0'\} \cap \{\mathfrak{D}_1, \mathfrak{D}_1'\} = \emptyset$, p and p' are as before, and $p \not\perp p'$, then either $\mathfrak{D}_0 = \mathfrak{D}_0'$ or $\mathfrak{D}_1 = \mathfrak{D}_1'$. We show that, in fact, 0 or 1 can be chosen uniformly.

Claim 1 *For one of $k = 0$ or $k = 1$, the following is true. Whenever $\{\mathfrak{D}_0, \mathfrak{D}_0'\} \cap \{\mathfrak{D}_1, \mathfrak{D}_1'\} = \emptyset$, p is over $(\mathfrak{D}_0, \mathfrak{D}_1)$, p' is over $(\mathfrak{D}_0', \mathfrak{D}_1')$ and $p \not\perp p'$, then $\mathfrak{D}_k = \mathfrak{D}_k'$.*

Proof of Claim 1: Suppose the claim fails to be true. Figure 2 contains the proof:

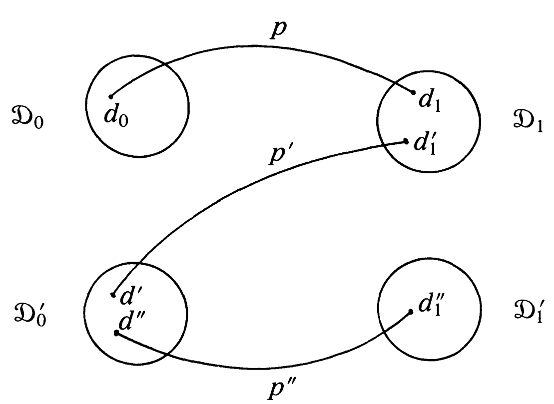


Figure 2.

In detail: there are distinct $\mathfrak{D}_0$, $\mathfrak{D}_0'$, and $\mathfrak{D}_1$, with p over $(\mathfrak{D}_0, \mathfrak{D}_1)$ and p' over $(\mathfrak{D}_0', \mathfrak{D}_1)$ such that $p \not\perp p'$ (by the assertion failing for $k = 0$). By applying a suitable isomorphism, the items exemplifying the failure of the assertion for $k = 1$ can be used to give a class $\mathfrak{D}_1'$ distinct from each of $\mathfrak{D}_0$, $\mathfrak{D}_0'$, $\mathfrak{D}_1$ and a type p'' over $(\mathfrak{D}_0', \mathfrak{D}_1')$ such that $p' \not\perp p''$. By D.5′(iii) we infer $p \not\perp p''$, which contradicts the assertion made before the Claim.

From now on, we assume that the assertion of Claim 1 holds for $k = 0$.

Our plan of constructing many models of power λ is as follows. We let G be any binary relation on a set $|G|$ of power λ such that $|G|$ is the disjoint union of sets I and J of power λ with the property that $\langle i, j\rangle \in G$ implies $i \in I$ and $j \in J$ (G is a bipartite graph); in addition, we assume that for every $i \in I$ ($j \in J$) there is at least one $j \in J$ ($i \in I$) such that $\langle i, j\rangle \in G$; finally, we assume that there is a unique $\hat{i} \in I$ such that for all $j \in J$ we have $\langle \hat{i}, j\rangle \in G$. Depending on G, we construct a model $M = M_G$. We let d^i and d^j for $i \in I$, $j \in J$ be elements such that $d^i \equiv d_0^0$, $d^j \equiv d_1^0$, and b^{ij} an element such that $d^i d^j b^{ij} \equiv d_0^0 d_1^0 b^0$ for every $\langle i, j\rangle \in G$; we stipulate furthermore that the system $C = \langle d^i\rangle_{i\in I} \frown \langle d^j\rangle_{j\in J} \frown \langle b^{ij}\rangle_{\langle i,j\rangle\in G}$ be independent with respect to the partial ordering in which $i < \langle i, j\rangle$, $j < \langle i, j\rangle$ and no other relations holds; C exists by A.12. C also denotes the set of all the d^i, d^j, b^{ij}. We will show that there is a model $M = M_G$ containing the set C such that (i) every type in the set $\mathcal{P} = \{p_{b^{ij}}: \langle i, j\rangle \in G\}$ has dimension $\le \omega$ in M; (ii) every stationary type $q \perp$ to all types in $\mathcal{P}$ and with a finite domain included in M has dimension λ in M.

We will not quite be able to recover G from M_G; however, if G^* is the binary relation on I defined by $\langle i, i'\rangle \in G^*$ iff there are infinitely many $j \in J$ such that $\langle i, j\rangle \in G$ and $\langle i', j\rangle \in G$, then we will recover G^* from $(M_G, d^{\hat{i}})$; i.e., $(M_{G_1}, d^{\hat{i}}) \simeq (M_{G_2}, d^{\hat{i}})$ will imply $G_1^* \simeq G_2^*$. Since it is easy to construct 2^λ G's such that the G^* are pairwise nonisomorphic, the proof of the theorem will be completed.

To prove the existence of $M = M_G$, using the notation introduced above we show

Claim 2 *For $b = b^{i_0 j_0}$ ($\langle i_0, j_0\rangle \in G$) we have*

$$p_b \perp C/b \ .$$

Proof of Claim 2: Let U be any set extending b such that $U \underset{b}{\perp} C$ and let $\bar{p}$ be $p|U$; we need to show

$$\bar{p} \overset{w}{\perp} C/U \ ,$$

i.e., that $\bar{p}$ has a unique extension to $U \cup C$. Clearly (see A.11), if we replace the item $b = b^{i_0 j_0}$ by U in the system C (so that $\langle i_0, j_0\rangle$ will index U instead of b), the system remains 'relatively independent'.

Let $d_0 = d^{i_0}$, $d_1 = d^{j_0}$, $C_0 = \{d^i: i \in I - \{i_0\}\}$, $C_1 = \{d^j: j \in J - \{j_0\}\}$, $B_0 = \{b^{i_0 j}: \langle i_0, j\rangle \in G \;\&\; j \neq j_0\}$, $B_1 = \{b^{ij}: \langle i, j\rangle \in G \;\&\; i \neq i_0\}$.

Consult Figure 3. By relative independence, we have $U \underset{d_1}{\perp} E$ for $E = C_0 \cup C_1 \cup B_1$. Hence, since $p \perp d_1$ and thus $\bar{p} \perp d_1$, it follows that

$$\bar{p} \vdash \bar{p}|E \ . \tag{2}$$

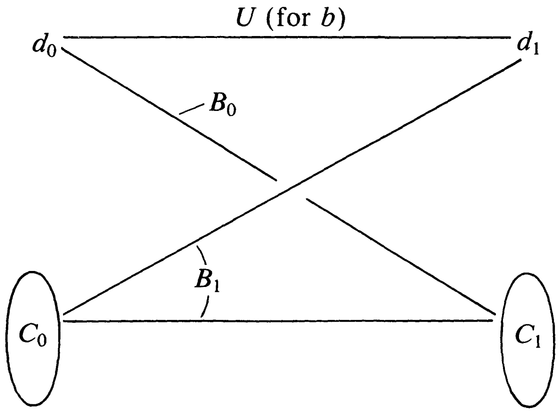


Figure 3.

By relative independence, we have $U \underset{d_0}{\perp} d_0C_1$, hence by $\bar{p} \perp d_0$ we have $\bar{p} \perp d_0C_1$. Again by relative independence, we have $UE \underset{d_0C_1}{\perp} B_0$. Since $\bar{p}|E$ (being parallel to $\bar{p}$) is $\perp$ to d_0C_1, it follows that

$$\bar{p}|E \vdash \bar{p}|(E \cup B_0) = \bar{p}|C \ .$$

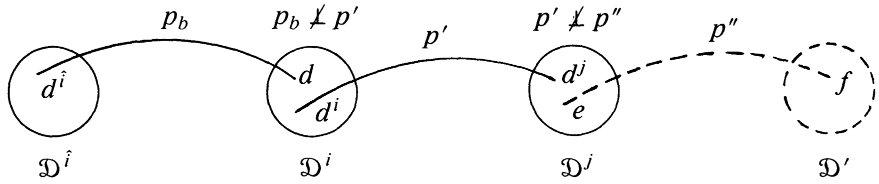
Together with (2), this shows what we wanted, that $\bar{p} \vdash \bar{p}|C$.

The existence of M_G with the required properties is now proved by using Claim 2, D.12″(iii), and D.12′ similarly to that of M_X in the proof of Lemma 2.2.

We now turn to the question of recovering a version of G from $(M, d^{\hat{\imath}})$ $(M = M_G)$.

We look at classes (0-classes and 1-classes) in M. The *distinguished* classes are, by definition, those of the form $\mathfrak{D}^i = [d^i]\,(i \in I)$, $\mathfrak{D}^j = [d^j]\,(j \in J)$. Since we have the element $d^{\hat{\imath}}$ distinguished, we can pick out the distinguished 1-classes by the following criterion: a class $\mathfrak{D}$ is a distinguished 1-class iff there is $d \in \mathfrak{D}$ and $b \in M$ such that $d^{\hat{\imath}}\, db \equiv d_0^0 d_1^0 b^0$ and p_b has dimension $<\lambda$ in M. The 'only if' part of this equivalence is clear. Assume that we have $d \in \mathfrak{D}$ and b as said and that $\mathfrak{D}$ is not a distinguished 1-class. By property (ii) of M, we have some $i \in I$ and $j \in J$ such that $p' =_{df} p_{b^{ij}} \not\perp p_b$. If $\mathfrak{D} \neq \mathfrak{D}^i$ (which is certainly the case when $t(d_0) \perp t(d_1)$) then $\{\mathfrak{D}^{\hat{\imath}}, \mathfrak{D}^i\} \cap \{\mathfrak{D}, \mathfrak{D}^j\} = \emptyset$, hence by Claim 1, $\mathfrak{D}^{\hat{\imath}} = \mathfrak{D}^i$, hence $i = \hat{\imath}$ and $d^i = d^{\hat{\imath}}$; by the triviality of $t(d_1^0)$, and by $\mathfrak{D} \neq \mathfrak{D}^j$, we have $d^j \underset{d^{\hat{\imath}}}{\perp} d$; since b^{ij} is dominated by $d^j/d^{\hat{\imath}}$, b by $d/d^{\hat{\imath}}$, we obtain $b^{ij} \underset{d^{\hat{\imath}}}{\perp} b$, hence $p' \perp p_b$ by C.8. It remains to consider the case $\mathfrak{D} = \mathfrak{D}^i$ (and $d_0^0 \equiv d_1^0$), but this leads immediately to a contradiction, as shown in Figure 4, which proves what we said about identifying the distinguished 1-classes. The distinguished 0-classes are then identified as those $\mathfrak{D}$ that are not distinguished 1-classes and for which there is a distinguished 1-class $\mathfrak{D}'$ such that some type over $(\mathfrak{D}, \mathfrak{D}')$ has dimension less than λ in M; this follows immediately from Claim 1.

For $i \in I$ and $J' \subset J$, we write $i\hat{G}J'$ iff $\langle i, j\rangle \in G$ for all but perhaps finitely many $j \in J'$. We also write $\mathfrak{D}_i\hat{G}\{\mathfrak{D}_j\colon j \in J'\}$ for $i\hat{G}J'$.



$\therefore p_b \not\perp p''$, contradiction.

Figure 4.

Claim 3 *For $\mathfrak{D}$ a distinguished 0-class and X a set of distinguished 1-classes, we have $\mathfrak{D}\hat{G}X$ iff the following holds: there is $d \in \mathfrak{D}$ and for all but finitely many $\mathfrak{D}' \in X$ there is $d' = d_{\mathfrak{D}'} \in \mathfrak{D}'$ such that some p over (d, d') has dimension $<\lambda$ in M.*

Proof of Claim 3: The 'only if' part of the claim is clear by construction. Suppose $d \in \mathfrak{D}$, $X' \subset X$, $X - X'$ is finite, and for each $\mathfrak{D}' \in X'$ we have $d' = d_{\mathfrak{D}'} \in \mathfrak{D}'$ and some $p' = p_{\mathfrak{D}'}$ over (d, d') with $dim(p, M) < \lambda$. By property (ii) of M, for each $\mathfrak{D}' \in X'$, there is $i \in I$, and there is $j = j_{\mathfrak{D}'} \in J$ such that $p_{\mathfrak{D}'} \not\perp p_{b^{ij}}$. By Claim 1, $\mathfrak{D}^i = \mathfrak{D}$, in particular, i is fixed. From now on, i will denote the unique index for which $\mathfrak{D}^i = \mathfrak{D}$. Let Y be the set of all $\mathfrak{D}^{j'}$ with $j' = j_{\mathfrak{D}'}$ for some $\mathfrak{D}' \in X'$. We have $\mathfrak{D}\hat{G}Y$. Now assume that it is not the case that $\mathfrak{D}\hat{G}X$. Then there is an infinite subset X'' of X' such that $X'' \cap Y = \varnothing$.

Let $K = \{j \in J\colon [d^j] \in Y\}$. We have that K is an infinite subset of J, for each $j \in K$, $p_j =_{df} p_{b^{ij}}$ is over (d^i, d^j) and for some $\hat{p}_j$ over $(d, \hat{d}^j)$, with $[\hat{d}^j] \in X'$,

$$p_j \not\perp \hat{p}_j \ . \tag{3}$$

It follows by the triviality of $t(d_1^0)$ that the system $\langle d^j \rangle_{j \in K} \ {}^\frown \ \langle \hat{d}^j \rangle_{j \in K}$ is independent.

Let $K_0 \subset K$ be finite such that for $D = \{d^j\colon j \in K\} \cup \{\hat{d}^j\colon j \in K\}$ and $D' = \{d^j\colon j \in K_0\} \cup \{\hat{d}^j\colon j \in K_0]$ we have

$$dd^i \underset{D'}{\perp} D \ .$$

Now, let $j \in K - K_0$. To emphasize the character of the elements involved, redenote: $d = \hat{d}_0$, $d^i = d_0$, $\hat{d}^j = \hat{d}_1$, $d^j = d_1$. By the last nonforking relation, together with $D' \perp d_1\hat{d}_1$ (by the independence of D), we conclude

$$d_0\hat{d}_0 \perp d_1\hat{d}_1 \ ;$$

note also that we have:

$$d_1 \perp \hat{d}_1 \ .$$

We immediately deduce the following three relations:

$$d_1 \underset{d_0\hat{d}_0}{\perp} \hat{d}_1 \ , \tag{4}$$

$$d_1 \underset{d_0}{\perp} \hat{d}_0 \ , \tag{5}$$

$$\hat{d}_1 \underset{\hat{d}_0}{\perp} d_0 \ . \tag{6}$$

$p = p_j = p_{b^{\nu}}$ is over (d_0, d_1), i.e., for $b = dom(p)$, $bd_0d_1 \equiv b^0d_0^0d_1^0$; $\hat{p} = \hat{p}_j$ is over $(\hat{d}_0, \hat{d}_1)$, i.e., for $\hat{b} = dom(\hat{p})$, $\hat{b}\hat{d}_0\hat{d}_1 \equiv b^0d_0^0d_1^0$. b is dominated by d_1/d_0, hence by $d_1/d_0\hat{d}_0$ too (by (5) and C.11(i)); similarly, $\hat{b}$ is dominated by $\hat{d}_1/d_0\hat{d}_0$. It follows by (4) that we have

$$b \underset{d_0\hat{d}_0}{\perp} \hat{b} \ . \tag{7}$$

Now, again by (5) and the fact that b is dominated by d_1/d_0, we infer $b \underset{d_0}{\perp} \hat{d}_0$, hence by $p \perp d_0$ and C.8, we get that $p \perp d_0\hat{d}_0$. By (7) and C.8, therefore, we have $p \perp \hat{p}$, a contradiction to (3).

Claim 3 clearly establishes that $(M_{G_1}, d^{\hat{\imath}}) \simeq (M_{G_2}, d^{\hat{\imath}})$ implies $G_1^* \simeq G_2^*$, as promised before. This completes the proof of Theorem 2.3.

3 Representations In this section we discuss a certain kind of analysis of models in terms of trees of 'small' models. The main result of the section is the quasi-uniqueness Theorem 3.4 asserting that any isomorphism of two models represented by certain trees induces a 'quasi-isomorphism' of those trees. In Section 5, this will enable us to give lower bounds on the number of models by considering models represented by trees that are pairwise nonquasi-isomorphic.

This section has no reference to the property *NDOP*. In the next section, *NDOP* will be used to show that every model has a representation.

In this section, the theory T is stable and countable. Although we talk about primary models and weight-1 types, we do not need existence statements concerning these items. The countability assumption is inessential.

An *ω-tree* is a partially ordered set order-isomorphic to a subposet of the poset of finite sequences of elements of a fixed set, ordered by 'initial segment of', and also, having a single root (minimal element). $\eta \lhd \nu$ denotes that ν is a successor of η (ν is farther from the root than η); $\langle\,\rangle$ denotes the root; $\nu^- = \eta$ if $\eta \lhd \nu$.

Definition 3.1

(i) An *ω-tree of sets* $\mathfrak{A} = \langle A_\eta \rangle_{\eta \in I}$ is given by an ω-tree I, and a set (subset of the T-universe) A_η such that $\eta < \nu$ implies $A_\eta \subset A_\nu$. $\mathfrak{A}$ is *independent* if it is independent with respect to the tree-ordering of I. Any model primary (a-primary) over $A_I = \bigcup_{\eta \in I} A_\eta$ is said to be *represented* (*a-represented*) by $\mathfrak{A}$.

(ii) A *normal tree of sets* is an ω-tree $\langle A_\eta \rangle_{\eta \in I}$ of sets such that: (a) for any $\eta \in I$, the system $\langle A_\nu \rangle_{\nu \rhd \eta}$ is independent over A_η, (b) $\eta \lhd \nu \lhd \tau$ implies $A_\nu \underset{A_\eta}{<} A_\tau$ (see C.14).

(iii) A *representation* is a normal tree $\mathfrak{N} = \langle N_\eta \rangle_{\eta \in I}$ of models such that $N_{\langle\,\rangle}$ is prime over $\varnothing$, and for $\eta \lhd \nu$, N_ν is of the form $N_\eta(a_\nu)$ (prime over $N_\eta \cup \{a_\nu\}$) for some a_ν *SR* over N_η. $\mathfrak{N}$ is a *representation of* any model primary over the set $N_I =_{df} \bigcup_{\eta \in I} N_\eta$.

(iv) An *a-representation* is a normal tree $\mathfrak{N} = \langle N_\eta \rangle_{\eta \in I}$ of a-models such that $N_{\langle\,\rangle}$ is a-prime over $\varnothing$, and for $\eta \lhd \nu$, N_ν is of the form $N_\eta[a_\nu]$ (a-prime over $N_\eta \cup \{a_\nu\}$) for some a_ν regular over N_η. $\mathfrak{N}$ is an *a-representation of M* if M is a-primary (a-prime) over N_I.

Remark: Notice that in a representation as in (iii), if $\eta \lhd \nu$, then N_ν is of weight 1 over N_η, and of course every N_ν is countable.

Let $\mathfrak{A}$ be a normal tree, with the notation of the definition. $S, S', \ldots$ will denote an arbitrary convex (downward closed) subset of I, $\langle\,\rangle \in S$; $S^+ = \{\nu \in I\colon \nu^- \in S\}$, $S^* = S^+ - S$. For any subset J of I, $A_J =_{df} \bigcup \{A_j\colon j \in J\}$. Sometimes we even write simply J for A_J; e.g. we may write

$$S \underset{\eta}{\perp} \nu \text{ to mean } A_S \underset{A_\eta}{\perp} A_\nu \ .$$

Theorem 3.2 *For S_0, S_1, S_2, S, S' convex subsets of I, we have*

(i) $A_{S^+ \cap S'} \underset{A_S}{\leq} A_{S'}$ *provided* $S \subset S'$;

(ii) $\mathfrak{A}$ *is an independent tree, hence* $A_{S_1} \underset{A_{S_0}}{\perp} A_{S_2}$ *provided* $S_0 \supset S_1 \cap S_2$.

Proof:

Step 1. Let $\rho \in I$. A *ρ-set* is a subset $R \subset I$ such that $\rho \in R$, for all $\nu \in R$ we have $\rho \leq \nu$, and if $\nu \in R$ and $\rho \leq \mu \leq \nu$, then $\mu \in R$.

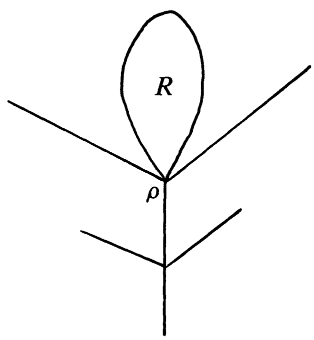


Figure 5.

R has *height* $\leq n$ if for all $\nu \in R$, distance $(\rho, \nu) = card\{\mu\colon \rho < \mu \leq \nu\}$ is $\leq n$. By induction on $n < \omega$, we (simultaneously) prove the (conjunction of the) following two statements:

$(1)_n$ $\eta \lhd \rho$, R a ρ-set of height $\leq n \Rightarrow \rho \underset{\eta}{\leq} R$ (the last is, of course, an abbreviation for $A_\rho \underset{A_\eta}{\leq} A_R$).

$(2)_n$ Whenever $\eta \in I$, and R_ρ is a ρ-set of height $\leq n$ for all $\rho \rhd \eta$, the system $\langle R_\rho \rangle_{\rho \rhd \eta}$ is independent over η.

For $n = 0$, there is nothing to prove.

Let $n \geq 1$; assume $(1)_{n-1}$ and $(2)_{n-1}$.

We prove $(1)_n$.

Denote, for any $\nu \rhd \rho$, $\nu \in R$, the set $\{\mu \in R: \mu \geq \nu\}$ by R_ν; R_ν is a ν-set of height $\leq n - 1$. Apply $(2)_{n-1}$ to conclude that $\langle R_\nu \rangle_\nu$ is an independent system over ρ, and $(1)_{n-1}$ to conclude that

$$\nu \underset{\rho}{<} R_\nu \ .$$

We have $\rho \underset{\eta}{<} \nu$.

Hence by C.14 (i), $\rho \underset{\eta}{<} R_\nu$; i.e., $R_\nu/\rho \perp \eta$ (i.e., $t(A_{R_\nu}/A_\rho)$ is $\perp$ to every type over A_η).

By the independence of the R_ν over η, and C.14(ii), we conclude

$$\underbrace{\bigcup \{R_\nu \colon \nu \in R \ \& \ \nu \rhd \rho\}}_{=R}/\rho \perp \eta$$

i.e., $\rho \underset{\eta}{<} R$, as required for $(1)_n$.

We now prove $(2)_n$.

For any finite set X of successors of η, we prove that $\langle R_\rho \rangle_{\rho \in X}$ is independent over η; we do this by induction on $|X|$. For $|X| = 0$, 1, there is nothing to prove; assume $|X| \geq 2$. Pick $\rho_0 \in X$; by induction hypothesis, $\langle R_\rho \rangle_{\rho \in \underbrace{X-\{\rho_0\}}_{X'}}$ is independent over η.

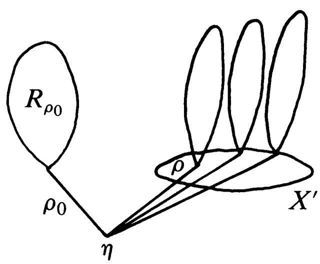


Figure 6.

It remains to show that

$$R_{\rho_0} \underset{\eta}{\perp} \bigcup_{\rho \in X'} R_\rho \tag{1}$$

holds. By $(1)_n$ (already proved) we have

$$R_{\rho_0}/\rho_0 \perp \text{all types over } \eta \ .$$

Since $X' \underset{\eta}{\perp} \rho_0$ by 3.1 (ii)(a), we have

$$R_{\rho_0}/\rho_0 \perp X'/\rho_0 \ .$$

It follows that we have

$$R_{\rho_0} \underset{\rho_0}{\perp} X' \ ,$$

i.e.,

$$R_{\rho_0} \underset{\rho_0}{\perp} X \ . \tag{2}$$

Now, by $(1)_n$ again, $R_\rho/\rho \perp$ all types over η, hence

$$R_\rho/\rho \perp \rho_0/\eta \quad (\rho \in X') \ . \tag{3}$$

The $R_\rho (\rho \in X')$ are independent over X' since they are independent over η; also $R_\rho \underset{\rho}{\perp} X'$ (similarly to (2)); therefore from (3) and $X' \underset{\eta}{\perp} \rho_0$, we get (see C.5(i)):

$$\bigcup_{\rho \in X'} R_\rho/X' \perp \rho_0/X' \ ;$$

hence,

$$\left.\begin{array}{l} \bigcup_{\rho \in X'} R_\rho \underset{X'}{\perp} \rho_0 \ ; \\ \\ \bigcup_{\rho \in X'} R_\rho \underset{X'}{\perp} X \ . \end{array}\right\} \quad \text{i.e.,} \tag{4}$$

By $X' \underset{\eta}{\perp} \rho_0$, $\rho_0 \underset{\eta}{<} R_{\rho_0}$ ($(1)_n$), C.8, and (2), we have that R_{ρ_0}/X is orthogonal to all types over X', in particular to $\bigcup_{\rho \in X'} R_\rho/X'$. Using (4), we get from this that

$$R_{\rho_0}/X \perp \bigcup_{\rho \in X'} R_\rho/X, \quad \text{hence} \quad R_{\rho_0} \underset{X}{\perp} \bigcup_{\rho \in X'} R_\rho \ .$$

The last relationship combined with (2) gives

$$R_{\rho_0} \underset{\rho_0}{\perp} \bigcup_{\rho \in X'} R_\rho \ .$$

$X' \underset{\eta}{\perp} \rho_0$ and (4) gives

$$\rho_0 \underset{\eta}{\perp} \bigcup_{\rho \in X'} R_\rho \ ,$$

and the last two facts give (by 'transitivity' again), (1) as required.

This completes the proof of $(1)_n$ and $(2)_n$.

In summary, in Step 1 we proved:

$(1)_\infty$ $\eta \lhd \rho \Rightarrow A_\rho \underset{A_\eta}{<} A_{\geq \rho}$ (with $\geq \rho = \{\nu: \nu \geq \rho\}$)

$(2)_\infty$ $\langle A_{\geq \rho} \rangle_{\rho \rhd \eta}$ is an independent system over A_η.

Step 2. Now, we prove

$$A_\rho \underset{A_{\rho^-}}{\perp} A_{\ngeq \rho} \quad (\text{or } \rho \underset{\rho^-}{\perp}' \ngeq \rho; \text{ here } \ngeq \rho = \{\nu: \nu \ngeq \rho\})$$

as follows; see also Figure 7.

Let $\rho_0 = \rho \rhd \rho_1 \rhd \rho_2 \rhd \ldots \rhd \rho_{n-1} \rhd \rho_n = \langle\,\rangle$ be all the predecessors of ρ; put $R_0 = \{\nu: \nu \geq \rho\}$, and $R_i = \{\nu: \nu \geq \rho_i, \nu \ngeq \rho_{i-1}\}$. By $(2)_\infty$, we have

$$\underbrace{R_0 \cup \ldots \cup R_i}_{\{\nu: \nu \geq \rho_i\}} \underset{\rho_{i+1}}{\perp} R_{i+1} \ ;$$

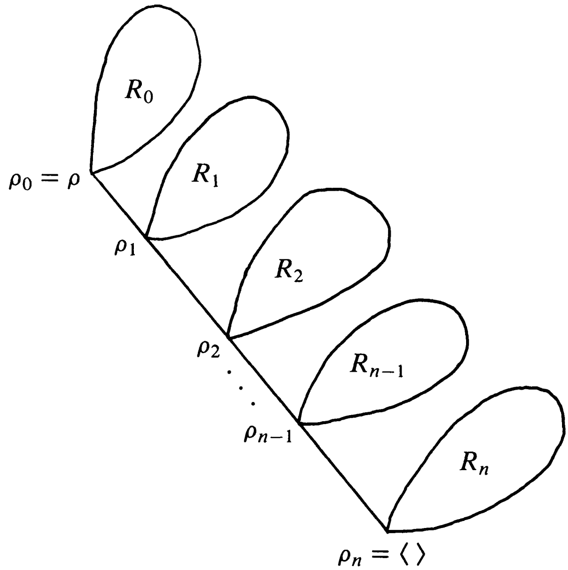


Figure 7.

hence,

$$R_0 \underset{R_1\cup\ldots\cup R_i}{\perp} R_1 \cup \ldots \cup R_{i+1} \; (0 \leq i < n) \ .$$

By (repeated) transitivity, the last gives

$$R_0 \underset{\rho_1}{\perp} R_1 \cup \ldots \cup R_n$$

which is identical to our claim.

By A.13, assertion (ii) of the theorem follows.

Step 3. Now, we prove the first assertion of the theorem, more precisely the special case

$$A_{S^+} \underset{A_S}{<} A_{S^{++}} \ . \tag{5}$$

Let $\sigma \in S^{+*}$; by Step 2, we know

$$\sigma^- \underset{\sigma^{--}}{\perp} S \tag{6}$$

and

$$\sigma \underset{\sigma^-}{\perp} S^+ \ . \tag{7}$$

Since (by 3.1 (ii)(b)), we have $\sigma/\sigma^- \perp \sigma^{--}$, by C.8 and (6), we conclude

$$\sigma/\sigma^- \perp S \ .$$

Suppose $p \in S(A_S)$; the last fact plus (7) yields that

$$\sigma/S^+ \perp p \ .$$

This is true for all $\sigma \in S^{+*}$; by Step 2, $\langle A_\sigma : \sigma \in S^{+*} \rangle$ is independent over A_{S^+}, hence

$$S^{+*}/S^+ \perp p$$

or equivalently,

$$S^{++}/S^+ \perp p \ ;$$

since $p \in S(A_S)$ was arbitrary, (5) follows.

Now, (5) applied to S^+, $S^{++}, \ldots$ in place of S gives

$$S^{++} \underset{S^+}{<} S^{+++},\ S^{+++} \underset{S^{++}}{<} S^{++++}, \ldots$$

and a fortiori,

$$S^{++} \underset{S}{<} S^{+++},\ S^{+++} \underset{S}{<} S^{++++}, \ldots \ .$$

By transitivity C.14(i), we conclude that

$$S^+ \underset{S}{<} I \ .$$

Applying the last to the subset S' of I instead of I (take the 'induced' subtree of sets), we obtain the assertion (i) of the Theorem.

Let $\mathfrak{A} = \langle A_\eta : \eta \in I \rangle$ be a normal tree again; let us write $p_\nu = t(A_\nu / A_{\nu^-})$.

Corollary 3.3 *If $\nu^- \neq \mu^-$ and p_ν, p_μ are stationary, then $p_\nu \perp p_\mu$.*

Proof: Since $p_\nu \perp$ all types over A_μ, for $\mu \leq \nu^{--}$, the conclusion is clear if $\mu < \nu$ or $\nu < \mu$. Thus we may assume that $\mu \not< \nu$, $\nu \not< \mu$. Suppose $\nu^- \neq \mu^-$; e.g., $\mu^- \not\leq \nu^-$; in particular, $\mu^- \neq \langle\,\rangle$. Apply 3.2(i) as follows: Let

$$\begin{aligned} S' &= (\leq \nu^-) \cup (\leq \mu)[(\leq \mu) = \{\rho \in I : \rho \leq \mu\}] \ , \\ S &= (\leq \nu^-) \cup (< \mu^-) \ ; \end{aligned}$$

then

$$S^+ \cap S' = (\leq \nu^-) \cup (\leq \mu^-)$$

(μ is not in S^+, since $\mu^- \not\leq \nu^-$).

Therefore,

$$A_\mu / A_{(\leq \nu^-) \cup (\leq \mu^-)} \perp A_\nu / A_{(\leq \nu^-) \cup (< \mu^-)} \ .$$

The type on the left-hand side is an *nf* extension of p_μ, the other is an *nf* extension of p_ν, both by 3.2(ii), since $\mu \not\leq \nu^-$ and $\nu \not\leq \mu^-$. Since p_μ, p_ν are stationary, $p_\mu \perp p_\nu$.

With the notation above, a *class* of $\mathfrak{A}$ is any set of the form

$$J_{\nu_0} = \{\nu : \nu^- = \nu_0^- \text{ and } p_\nu \not\perp p_{\nu_0}\} \ (\nu_0 \neq \langle\,\rangle) \ .$$

If the types p_ν are of weight 1 (as in case of representations), relation $\not\perp$ is an equivalence relation among the p_ν ($\nu \in I$). By 3.3 the classes are exactly the equivalence classes of this equivalence.

Theorem 3.4 (Quasi-uniqueness of representations) *Let M^ℓ be represented by the normal tree $\mathfrak{N}^\ell = \langle N_\eta^\ell \rangle_{\eta \in I^\ell}$ of countable models such that N_ν^ℓ is of weight 1 over N_η^ℓ (for all $\eta \lhd \nu$ in I^ℓ; $\ell = 0, 1$). Let*

$$f\colon M^0 \rightrightarrows M^1$$

be an isomorphism. Then there is a partial $1-1$ correspondence

$$h \subset I^0 \times I^1$$
$$(\langle \nu, \hat{\nu} \rangle, \langle \nu, \hat{\nu} \rangle \in h \Rightarrow \hat{\nu} = \hat{\nu}'; \langle \nu, \hat{\nu} \rangle, \langle \nu', \hat{\nu} \rangle \in h \Rightarrow \nu = \nu')$$

between elements of I^0 and I^1 such that (writing "$\nu \mapsto \hat{\nu}$" for "$\langle \nu, \hat{\nu} \rangle \in h$"), we have

(i) *when $\nu \mapsto \hat{\nu}$, $f(p_\nu) \not\perp p_{\hat{\nu}}$; here $p_\nu = t(N_\nu^0/N_{\nu^-}^0)$, $p_{\hat{\nu}} = t(N_{\hat{\nu}}^1/N_{\hat{\nu}^-}^1)$*
(ii) *if $\nu \mapsto \hat{\nu}$, $\tau \mapsto \hat{\tau}$, then $\nu < \tau$ iff $\hat{\nu} < \hat{\tau}$*
(iii) *for any class J of $\mathfrak{N}^0$, we have that there are only countably many $\nu \in J$ for which $\nu \mapsto \hat{\nu}$ is undefined; similarly for classes of $\mathfrak{N}^1$.*

Proof:

Step 1. For the purposes of Lemma 3.5 below we use the data of the theorem but assume only that $f\colon M^0 \to M^1$ is an elementary embedding. The lemma contains the main step of the construction of the assignment $\nu \mapsto \hat{\nu}$. We leave out the upper index 1 sometimes; $M = M^1$, $N_I = N_I^1$, etc.

Given a countable subset B of M^i ($i = 0, 1$) there is a countable model $\tilde{B}$, $B \subset \tilde{B} \subset M^i$, such that M^i is atomic over $N_{I^i}^i \cup \tilde{B}$ (see B.6). We fix such an assignment $B \mapsto \tilde{B}$.

Let us use the following notation and terminology. Let $B_\nu = f(N_\nu^0)$. Let now $\eta \lhd \nu$ in I^0 be fixed, and write $B = B_\eta$, $A = B_\nu$. Let $S \subset I(= I^1)$ be countable and convex. We say that ν and S are *properly related* if the following two conditions hold:

$$\tilde{B} \underset{N_S}{\perp} N_I \ , \tag{8}$$

$$A \underset{B}{\perp} \tilde{B} \cup N_S \ , \tag{9}$$

and, in particular,

$$A \underset{\tilde{B}}{\perp} N_S \ . \tag{9'}$$

Note that, given η, we can choose S satisfying (8); and then clearly (e.g., by D.2(i)), all but countably many $\nu \rhd \eta$ will satisfy ν and S will be properly related.

A *0-class* (*1-class*) is a class of the normal tree $\mathfrak{N}^0$ ($\mathfrak{N}^1$) (see above). A node η of I^0 (I^1) is called *proper* if there is at least one uncountable class above η (whose members are $> \eta$).

Lemma 3.5 *Suppose that ν and S are properly related; use the notation above.*

(i) *There is some $\hat{\nu}_0 \in S^*$ such that for the 1-class $\hat{J} = J_{\hat{\nu}_0}$ containing $\hat{\nu}_0$, we have*

$$A \underset{\tilde{B} \cup N_S}{\not\perp} N_{S^* \cap \hat{J}}$$

and $f(p_\nu) \not\perp p_{\hat{\nu}_0}$.
Clearly, $\hat{J}$ *is uniquely determined by* ν *alone.*
(ii) *There is* $\hat{\nu} \in S^* \cap \hat{J}$ *such that*

$$A \underset{\tilde{B} \cup N_S}{\perp} N_{S^* - \{\hat{\nu}\}} .$$

Any $\hat{\nu} \in S^*$ *with this property is said to be S-related to* ν.
(iii) *Suppose, in addition, that* $S \subset S'$, $\nu < \tau$, *and* τ *and* S' *are properly related. Then, if* $\hat{\nu}$ *is S-related to* ν, $\hat{\tau}$ *is S'-related to* τ, *we have* $\hat{\nu} < \hat{\tau}$, *and also*

$$A \underset{\tilde{B} \cup N_S}{\not\perp} N_{\hat{\nu}} .$$

(iv) *Let* ν *be a proper node in* I^0. *Suppose that* ν *and* S_i *are properly related* $(i = 1, 2)$ *and* $\hat{\nu}_i$ *is* S_i*-related to* ν $(i = 1, 2)$. *Then* $\hat{\nu}_1 = \hat{\nu}_2$.

Proof of Lemma 3.5: We first claim

$$A \underset{\tilde{B} \cup N_S}{\not\perp} N_{S^*} . \tag{10}$$

Suppose

$$A \underset{\tilde{B} \cup N_S}{\perp} N_{S^*} . \tag{11}$$

Recall (3.2(i)) the relation

$$N_{S^+} \underset{N_S}{<} N_I ;$$

i.e., $t(N_I/N_{S^+}) \perp N_S$.

Using (8), we infer $t(N_I/N_{S^+}) \perp \tilde{B} \cup N_S$ (by C.8). $t(A/\tilde{B} \cup N_{S^+})$ *dnf* over $\tilde{B} \cup N_S$ by (11), $t(N_I/\tilde{B} \cup N_{S^+})$ *dnf* over N_{S^+} by (8); therefore, the last orthogonality relation implies

$$A \underset{\tilde{B} \cup N_{S^+}}{\perp} N_I .$$

By (11) again,

$$A \underset{\tilde{B} \cup N_S}{\perp} N_I ,$$

and then, by (9′),

$$A \underset{\tilde{B}}{\perp} N_I .$$

Since $\tilde{B}$ is a model, and M is atomic over $\tilde{B} \cup N_I$, it follows (see C.12), that

$$A \underset{\tilde{B}}{\perp} M ;$$

hence by (9),

$$A \underset{B}{\perp} M ,$$

contradicting $A \subset M$. This proves (10).

Let J' be the set of $\hat{\nu} \in S^*$ such that $q =_{df} t(A/B) \perp p_{\hat{\nu}}$. If $\hat{\nu} \in S^*$, by 3.2 we have $N_{\hat{\nu}} \underset{N_{\hat{\nu}^-}}{\perp} N_S$; since by (8), $N_{\hat{\nu}} \underset{N_S}{\perp} \tilde{B} \cup N_S$, we have $N_{\hat{\nu}} \underset{N_{\hat{\nu}^-}}{\perp} \tilde{B} \cup N_S$: hence $q \perp t(N_{\hat{\nu}}/\tilde{B} \cup N_S)$ for $\hat{\nu} \in J'$. It follows that

$$q \perp t(N_{J'}/\tilde{B} \cup N_S) \tag{12}$$

since $\langle N_{\hat{\nu}} \rangle_{\hat{\nu} \in J'}$ is independent over $\tilde{B} \cup N_S$ (by 3.2(ii) and (8)). If we had $J' = S^*$, (12) (and (9)) would contradict (10). Let $\hat{\nu}_0 \in S^* - J'$; by definition, $f(p_\nu) \not\perp p_{\hat{\nu}_0}$; i.e., we have the second assertion of (i). Since $f(p_\nu)$ is of weight 1, for any $\hat{\nu} \in S^* - J'$ we must have $p_{\hat{\nu}} \not\perp p_{\hat{\nu}_0}$; hence $S^* - J' \subset \hat{J} =_{df}$ the 1-class containing $\hat{\nu}_0$.

Assume $A \underset{\tilde{B} \cup N_S}{\perp} N_{S^* \cap \hat{J}}$; we'll deduce a contradiction. We have $N_{J'} \underset{\tilde{B} \cup N_S}{\perp} N_{S^* \cap \hat{J}}$; this follows from 3.2(ii) and (8). Hence by the assumption and (12) we conclude $A \underset{\tilde{B} \cup N_{S \cup (S^* \cap \hat{J})}}{\perp} N_{J'}$; i.e., $A \underset{\tilde{B} \cup N_{S \cup (S^* \cap \hat{J})}}{\perp} N_{S^*}$ (since $S^* - J' \subset \hat{J}$); by the assumption again, this would give $A \underset{\tilde{B} \cup N_S}{\perp} N_{S^*}$, contradicting (10).

We have verified part (i) of the lemma.

The system $\langle N_{\hat{\nu}} \rangle_{\hat{\nu} \in S^*}$ is independent over $\tilde{B} \cup N_S$; hence by D.3′, there is $\hat{\nu} \in S^*$ satisfying the assertion of (ii). By (i), we must have $\hat{\nu} \in \hat{J}$, as said in (ii).

Now, with $\hat{\nu}$ S-related to ν, let $\tau > \nu$ and $\hat{\tau} \in I$, such that $\hat{\tau} \notin S$. Let $\hat{\nu}'$ be the unique index $\leq \hat{\tau}$ such that $\hat{\nu}' \in S^*$. Assume that

$$A \underset{\tilde{B} \cup N_S}{\perp} N_{\hat{\nu}'} \ .$$

Claim *Under the notation and conditions of the last paragraph, we have*

$$f(p_\tau) \perp p_{\hat{\tau}} \ .$$

First assume $\hat{\tau}^- \geq \hat{\nu}'$ and consider the following diagram of sets, Figure 8, with the arrows containments.

$$\begin{tikzcd}
& f(N^0_\tau) \cup \tilde{B} \cup N_S & & N_{\hat{\tau}} \cup \tilde{B} \cup N_S & \\
& f(N^0_{\tau^-}) \cup \tilde{B} \cup N_S \arrow[u] & & N_{\hat{\tau}^-} \cup \tilde{B} \cup N_S \arrow[u] & \\
\text{may be equal} & A \cup \tilde{B} \cup N_S \arrow[u] & & N_{\hat{\nu}'} \cup \tilde{B} \cup N_S \arrow[u] & \text{may be equal} \\
& & \tilde{B} \cup N_S \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Figure 8.

This is a normal tree of sets: the only independence relation (at the branching) involved in this is our assumption; the orthogonality relations follow from those in $\mathfrak{N}^0$ and $\mathfrak{N}^1$ and (9) as well as $N_{\nu'} \underset{N_{\hat{\nu}'^-}}{\perp} \tilde{B} \cup N_S$, which is a consequence of (8). By 3.3, the Claim follows. The remaining case $\hat{\tau} = \hat{\nu}'$ is similar.

Using the Claim, we see (iii) as follows. Assume the assumptions of (iii). Then, since $\hat{\tau} \in (S')^*$, we have $\hat{\tau} \notin S$. Now, if $\hat{\nu} \neq \hat{\nu}'$ ($\hat{\nu}'$ as in the Claim) then by (ii) ($\hat{\nu}$ being S-related to ν), we find that the assumption of the Claim holds, hence $f(p_\tau) \perp p_{\hat{\tau}}$, contradicting the fact that $\hat{\tau}$ is S'-related to τ (see (i)); hence $\hat{\nu}' = \hat{\nu}$; i.e., $\hat{\nu} < \hat{\tau}$ we claimed in (iii) ($\hat{\nu} = \hat{\tau}$ is impossible since $f(p_\nu) \perp f(p_\tau)$). Now we know $\hat{\nu}' = \hat{\nu}$; if we had $A \underset{B \cup N_S}{\perp} N_\nu$, then by the Claim again, we would get the contradiction $f(p_\tau) \perp p_{\hat{\tau}}$; this proves the second assertion in (iii). This completes the proof of part (iii).

Assume the hypotheses of part (iv). Since ν is proper, we can find $\nu' \geq \nu$ with uncountably many successors. Find a countable convex S' such that $S_1 \cup S_2 \subset S' \subset I$, and

$$B_{\nu'} \underset{N_{S'}}{\perp} N_I \ .$$

Since ν' has uncountably many successors, there is $\tau \rhd \nu'$ such that τ and S' are properly related. Let $\hat{\tau}$ be S'-related to τ. Applying (iii) once to ν, τ, $S_1 \subset S'$ and $\hat{\nu}_1$, once to ν, τ, $S_2 \subset S'$ and $\hat{\nu}_2$, we obtain that $\hat{\nu}_1 < \hat{\tau}$ and $\hat{\nu}_2 < \hat{\tau}$. We also know that $\hat{\nu}_1$, $\hat{\nu}_2$ belong to the same 1-class (by (i)); in particular, $\hat{\nu}_1^- = \hat{\nu}_2^-$. It follows that $\hat{\nu}_1 = \hat{\nu}_2$.

We can make the following 'global' use of Lemma 3.5. It is easy to see that we can assign a countable convex set $S_\eta \subset I$ to every $\eta \in I^0$ such that $\nu < \eta$ implies $S_\eta \subset S_\nu$ and such that

$$\tilde{B}_\eta \underset{N_{S_\eta}}{\perp} N_I$$

for all $\eta \in I^0$; let us call such a system $\langle S_\eta \rangle_{\eta \in I^0}$ a *spine* for f, and let us fix a spine. Define a partial map $\nu \mapsto \hat{\nu}$ from I^0 to I^1 as follows. For $\nu = \langle\,\rangle$, $\hat{\nu}$ is not defined. For other ν, $\hat{\nu}$ is defined if and only if ν is properly related to $S = S_{\nu^-}$, and $\hat{\nu}$ is then chosen to be an index in I such that $\hat{\nu}$ is S-related to ν; if ν is proper, $\hat{\nu}$ is uniquely determined by this property (see (iv)). Note that, given any $\eta \in I^0$, for all but countably many $\nu \rhd \eta$, $\nu \mapsto \hat{\nu}$ will be defined; moreover, the map is order preserving: $\nu \mapsto \hat{\nu}$, $\tau \mapsto \hat{\tau}$, $\nu < \tau \Rightarrow \hat{\nu} < \hat{\tau}$ (see (iii)). The map $\nu \mapsto \hat{\nu}$ induces a map from uncountable 0-classes to 1-classes: to any uncountable 0-class J, we have a unique 1-class $\hat{J}$ such that $\nu \in J$, $\hat{\nu} \in \hat{J}$ imply $f(p_\nu) \not\perp p_{\hat{\nu}}$ and such that: if $\nu \mapsto \hat{\nu}$ and $\nu \in J$, then $\hat{\nu} \in \hat{J}$. Moreover, if $J \mapsto \hat{J}$ under this induced map, and J is uncountable, $\hat{J}$ has to be uncountable too: supposing $\hat{J}$ countable, letting η be the root of J, and using the attendant notation introduced prior to Lemma 3.5, we could find a countable $J' \subset J$ such that (with $S = S_\eta$)

$$N_{\hat{J}} \cup \tilde{B} \cup N_S \underset{B_{J'}}{\perp} B_J \ ,$$

hence for $\nu \in J - J'$, by

$$B_{J'} \underset{B}{\perp} B_\nu (= A)$$

we have

$$N_{\hat{J}} \cup \tilde{B} \cup N_S \underset{B}{\perp} B_\nu \ ,$$

hence (9) as well as $A \underset{\tilde{B} \cup N_S}{\perp} N_{\hat{J}}$, contradicting (i). By the construction, if $J \mapsto \hat{J}$ as above, η is the root of J, $\hat{\eta}$ is the root of $\hat{J}$, then $\hat{\eta} \in S_\eta$.

Step 2. Now, assume $f\colon M^0 \xrightarrow{\sim} M^1$ is an isomorphism; we are going to use the above for both f and f^{-1}. Fix spines $\langle S_\eta\colon \eta \in I^0\rangle$ for f, and $\langle S_{\hat{\eta}}\colon \hat{\eta} \in I^1\rangle$ for f^{-1}; by the above we have partial maps

$$\begin{aligned} &\nu \overset{h_0}{\mapsto} \hat{\nu} \quad (I_0 \to I_1) \\ &\hat{\nu} \overset{h_1}{\mapsto} \dot{\nu} \quad (I_1 \to I_0) \end{aligned}$$

defined from these spines. Our goal is to show that, possibly after disregarding some further indices, these two correspondences become inverses of each other.

Recall that h_0 and h_1 induce a map from uncountable 0-classes to uncountable 1-classes and one from uncountable 1-classes to uncountable 0-classes and that, actually, these can be defined without reference to the maps h_0 and h_1. Since the relations

$$\begin{aligned} f(p_\nu) &\not\perp p_{\hat{\nu}} \\ f^{-1}(p_{\hat{\nu}}) &\not\perp p_\nu \end{aligned}$$

are equivalent, it follows that the two maps of classes are inverses of each other.

We now fix our attention on a pair of uncountable classes J and $\hat{J}$ corresponding to each other via $h_0(h_1)$, with respective roots η and $\hat{\eta}$; J is a 0-class, $\hat{J}$ is a 1-class.

As before, we write $B_\nu = f(N_\nu^0)$, $B = B_\eta$; we omit the upper index 1; we also write $N = N_{\hat{\eta}}$. The set $C = f^{-1}(N_{\hat{\eta}})$ ("symmetric" to B) is included in the countable model $\tilde{C} \subset M^0$ (such that M^0 is primary over $N_{I^0}^0\tilde{C}$); we let $\bar{N} = f(\tilde{C})$ (since we want to map everything over to the "1-side"). Let us put $S_0^0 = S_{\hat{\eta}}$, $S_0 = S_\eta$ (given by the spines for f and f^{-1}), and let us define, by induction on $n < \omega$, the countable convex subsets $S_0^0 \subset \ldots \subset S_n^0 \subset S_{n+1}^0 \subset \ldots$ of I^0, and similarly $S_n \subset I$, such that

$$\tilde{B}\bar{N}N_{S_n} \underset{B_{S_{n+1}^0}}{\perp} B_{I^0}$$

and

$$\tilde{B}\bar{N}B_{S_{n+1}^0} \underset{N_{S_{n+1}}}{\perp} N_I \ .$$

If we put $S^0 = \bigcup_{n<\omega} S_n^0$, $S = \bigcup_{n<\omega} S_n$ and $D = \tilde{B}\bar{N}B_{S^0}N_S$, then we have

$$D \underset{B_{S^0}}{\perp} B_{I^0} \tag{13}$$

$$D \underset{N_S}{\perp} N_I \ . \tag{14}$$

Put $J' = J - S^0 = (S^0)^* \cap J$, $\hat{J}' = \hat{J} - S = S^* \cap \hat{J}$. By $B_{J'} \underset{B}{\perp} B_{S^0}$ (a consequence of the independence of the tree $\langle B_\nu\rangle_{\nu \in I^0}$) and (13), we get

$$B_{J'} \underset{B}{\perp} D \ . \tag{15}$$

In particular, $B_\nu \underset{B}{\perp} \tilde{B}N_S$ for every $\nu \in J'$; i.e., ν and S are properly related.

Similarly,

$$N_{\hat{J}'} \underset{N}{\perp} D \ . \tag{16}$$

Let $\nu \in J'$. By 3.5(i), we have $B_\nu \underset{\tilde{B}N_S}{\not\perp} N_{\hat{J}'}$. This implies

$$B_\nu \underset{D}{\not\perp} N_{\hat{J}'} \quad (\nu \in J') \ ; \tag{17}$$

indeed, we have the implications

$$\left.\begin{array}{r} B_\nu \underset{D}{\perp} N_{\hat{J}'} \\ (14) \Rightarrow D \underset{\tilde{B}N_S}{\perp} N_{\hat{J}'} \end{array}\right\} \Rightarrow B_\nu \underset{\tilde{B}N_S}{\perp} N_{\hat{J}'} \ .$$

Similarly, we obtain

$$N_{\hat{\nu}} \underset{D}{\not\perp} B_{J'} \quad (\hat{\nu} \in \hat{J}') \ . \tag{18}$$

Now, let ν be a proper node in J'. Since $B_\nu \underset{B}{\perp} \tilde{B}N_S$ implies $B_\nu \underset{B}{\perp} \tilde{B}N_{S_\eta}$, ν and S_η are properly related; i.e., $\nu \in dom(h_0)$. By 3.5(ii), there is $\hat{\nu} \in \hat{J}'$, $\hat{\nu}$ S-related to ν; by 3.5(iv), $\hat{\nu} = h_0(\nu)$, and by 3.5(iii), we have $B_\nu \underset{\tilde{B}N_S}{\not\perp} N_{\hat{\nu}}$. Just as we inferred (17), we now conclude

$$B_\nu \underset{D}{\not\perp} N_{\hat{\nu}} \quad (\nu \in J' \text{ proper}, \ \nu \overset{h_0}{\mapsto} \hat{\nu}) \ , \tag{19}$$

and symmetrically,

$$N_{\hat{\nu}} \underset{D}{\perp} B_{\dot{\nu}} \quad (\hat{\nu} \in \hat{J}' \text{ proper}, \ \hat{\nu} \overset{h_1}{\mapsto} \dot{\nu}) \ . \tag{20}$$

Define J_1 ($\hat{J}_1$) to be the set of proper nodes in J' (in $\hat{J}'$). We *claim* that h_0 maps J_1 into $\hat{J}_1$, h_1 maps $\hat{J}_1$ into J_1, and in fact, on these sets h_0 and h_1 are inverses of each other. If $\nu \in J_1$, there is some uncountable 0-class above it; the 1-class corresponding to this (under h_0) is uncountable too; since h_0 is order-preserving, it follows that $\hat{\nu} = h_0(\nu)$ is proper too; in other words, h_0 maps J_1 into $\hat{J}_1$. Similarly, h_1 maps $\hat{J}_1$ into J_1. Now, let $\nu \in J_1$ and let us consider $\hat{\nu}$ and $\dot{\nu}$ such that

$$\nu \overset{h_0}{\mapsto} \hat{\nu}$$
$$\hat{\nu} \overset{h_1}{\mapsto} \dot{\nu} \ .$$

Now, both (19) and (20) apply. Since $w(N_{\hat{\nu}}/D) = w(N_{\hat{\nu}}/N) = 1$ (see (16)), it follows that $B_\nu \underset{D}{\not\perp} B_{\dot{\nu}}$. Since $\langle B_\nu \rangle_{\nu \in J'}$ is independent over B, and we have (15), we conclude that $\nu = \dot{\nu}$ must be the case. This shows that $h_1 \circ h_0$ is the identity on J_1. Also, by symmetry $h_0 \circ h_1$ is the identity on $\hat{J}_1$. This proves the *claim*.

We will show that $|J' - J_1| = |\hat{J}' - \hat{J}_1|$. This will follow (see D.5(ii)) from the fact that both systems $\langle B_\nu \rangle_{\nu \in J' - J_1}$ and $\langle N_{\hat{\nu}} \rangle_{\hat{\nu} \in \hat{J}' - \hat{J}_1}$ are *maximal* independent over $DN_{\hat{J}_1}$ in their union, and each member of these systems has weight 1 over $DN_{\hat{J}_1}$. In fact, using the bijection $\nu \mapsto \hat{\nu}$ of J_1 and $\hat{J}_1$ such that $B_\nu \underset{D}{\not\perp} N_{\hat{\nu}}$, we find by D.5′(iv) and C.11(iv) that $\langle B_\nu \rangle_{\nu \in J' - J_1} \frown \langle N_{\hat{\nu}} \rangle_{\hat{\nu} \in \hat{J}_1}$

is independent over D, hence $\langle B_\nu\rangle_{\nu\in J'-J_1}$ is independent over $DN_{\hat{J}_1}$. Clearly, $\langle N_{\hat{\nu}}\rangle_{\hat{\nu}\in\hat{J}'-\hat{J}_1}$ is independent over $DN_{\hat{J}_1}$. If, contrary to the first maximality assertion, we had $\tilde{\nu}\in\hat{J}'-\hat{J}_1$ such that $\langle B_\nu\rangle_{\nu\in J'-J_1} \frown \langle N_{\tilde{\nu}}\rangle$ is independent over $DN_{\hat{J}_1}$, then we would have $B_{J'-J_1} \underset{DN_{\hat{J}_1}}{\perp} N_{\tilde{\nu}}$, hence by $N_{\hat{J}_1} \underset{D}{\perp} N_{\tilde{\nu}}$, also

$$B_{J'-J_1}N_{\hat{J}_1} \underset{D}{\perp} N_{\tilde{\nu}} \ . \tag{21}$$

By D.5′(iv) and C.11(iv), replacing each B_ν by $N_{\hat{\nu}}$ ($\nu\mapsto\hat{\nu}$, $\nu\in J_1$) in $\langle B_\nu\rangle_{\nu\in J'}$, we get that $\vec{A} =_{df} \langle B_\nu\rangle_{\nu\in J'-J_1} \frown \langle N_{\hat{\nu}}\rangle_{\hat{\nu}\in\hat{J}_1}$ is independent over D. Together with (21), this means that $\vec{A} \frown \langle N_{\tilde{\nu}}\rangle$ is independent over D. But then, replacing each $N_{\hat{\nu}}$ by B_ν ($\hat{\nu}\mapsto\nu$, $\hat{\nu}\in\hat{J}_1$) in $\vec{A}$, we get that $\langle B_\nu\rangle_{\nu\in J'} \frown \langle N_{\tilde{\nu}}\rangle$ is independent over D, in contradiction to (18). Another application of D.5′(iv) and C.11(iv) shows that $B_\nu \underset{D}{\perp} N_{\hat{J}_1}$ for $\nu\in J'-J_1$; this proves that $w(B_\nu/DN_{\hat{J}_1}) = w(B_\nu/D) = 1$; also, it can be used, together with (17), to show the second maximality assertion.

In summary, we also have $|J'-J_1| = |\hat{J}'-\hat{J}_1|$.

To finish the proof of Theorem 3.4, we only have to make a few remarks. We define the 1-1 correspondence of the theorem as follows. Elements in I^0 (in I^1) whose class is countable will not take part in the correspondence. Otherwise, if J is an uncountable 0-class, $\hat{J}$ is the corresponding 1-class, we let J' and $\hat{J}'$ be the subsets as above (the difference $J-J'$, $\hat{J}-\hat{J}'$ are countable); the correspondence will map J' bijectively onto $\hat{J}'$. In particular, on the set $J_1\subset J'$ of proper nodes, the correspondence is just h_0, and it is onto $\hat{J}_1$. On the remaining set $J'-J_1$, the correspondence is any bijection onto $\hat{J}'-\hat{J}_1$. Since h_0 is order-preserving, and since all elements ν of $J'-J_1$ ($\hat{J}'-\hat{J}_1$) are improper, and thus the correspondence for nodes above ν is undefined, and since we have disturbed h_0 on $J'-J_1$ only by going to another element with the same predecessor, the correspondence is order-preserving.

The reader will notice that the whole proof can be repeated for a-representations of a-models practically without change. In fact, the only thing to watch is the matter of cardinalities. For simplicity, we still assume T is countable. Let $\lambda_0=\aleph_0$ if T is t.t., $\lambda_0=2^{\aleph_0}$ otherwise (see the beginning of Section 2). Then we have

Theorem 3.4′ *Let $\mathfrak{N}^\ell = \langle N^\ell_\eta\rangle_{\eta\in I^\ell}$ be an a-representation (see* 3.1*(iv)) of the a-model M^ℓ ($\ell=0,1$), and let $f\colon M^0 \cong M^1$ be an isomorphism. Then there is a partial* 1-1 *correspondence $\nu\mapsto\hat{\nu}$ between I^0 and I^1 such that*

(i) *when $\nu\mapsto\hat{\nu}$, we have $f(p_\nu)\perp p_{\hat{\nu}}$ (p_ν, $p_{\hat{\nu}}$ are as before)*
(ii) *if $\nu\mapsto\hat{\nu}$, $\tau\mapsto\hat{\tau}$, then $\nu<\tau$ iff $\hat{\nu}<\hat{\tau}$*
(iii) *for any class J of $\mathfrak{N}^0$, we have that there are only countably many $\nu\in J$ for which $\nu\mapsto\hat{\nu}$ is undefined; similarly for classes of $\mathfrak{N}^1$.*

The proof of the theorem for the case $\lambda_0=\aleph_0$ (T is t.t.), or with 'countably many' replaced by 'λ_0 many' in part (iii), is literally the same as that of 3.4. The arguments necessary for the remaining case will not be given here; c.f. the end of [6].

4 Consequences of NDOP Throughout this section, T is superstable and satisfies *NDOP*.

Proposition 4.1 *Let M be a-prime over N_I for an independent ω-tree of a-models $\langle N_\eta \rangle_{\eta \in I}$. Then $p \not\perp M$ implies $p \not\perp N_\nu$ for some $\nu \in I$.*

Proof: First assume that I is finite. If I is linearly ordered, the assertion is trivial. Otherwise, we use induction on $|I|$. We put $I = S_1 \cup S_2$ such that S_1, S_2 are convex, $S_1 \cap S_2 = S_0$ is linearly ordered, and $S_1 \neq I$, $S_2 \neq I$. N_{S_0} is the same as N_τ for τ the maximal element of S_0; let us denote N_{S_0} by M_0. By independence,

$$N_{S_1} \underset{M_0}{\perp} N_{S_2} \ ,$$

hence (see C.12(ii)) if M_i is a-prime over N_{S_i} $(i = 1, 2)$, we have

$$M_1 \underset{M_0}{\perp} M_2 \ .$$

By the arguments in the proof of 1.4, we can find M_i a-prime over N_{S_i} $(i = 1, 2)$ such that the given M is a-prime over $M_1 \cup M_2$. By *NDOP*, e.g., $p \not\perp M_1$. By induction hypothesis $(|S_1| < |I|)$, there is $\eta \in S_1$ such that $p \not\perp N_\eta$, concluding the case when I is finite.

In the general case, let $q \in S(M)$ such that $p \not\perp q$; since T is superstable, there is finite $B \subset M$ such that q *dnf* over B; hence $p \not\perp q \upharpoonright B$ and thus $p \not\perp B$. There is a finite convex $I' \subset I$ such that B is a-atomic over $N_{I'}$, hence $B \subset M'$ for some a-prime M' over $N_{I'}$. Since $p \not\perp M'$, the first case applies and proves what we want.

Proposition 4.1′ *Let T be t.t. Let M be prime over N_I for an independent ω-tree $\langle N_\eta \rangle_{\eta \in I}$ of models. Then $p \not\perp M$ implies $p \not\perp N_\nu$ for some $\nu \in I$.*

Proof: The same as that of 4.1, using the equivalent formulation 1.3 of *NDOP*.

Proposition 4.2 *Suppose M is a-prime over N_I for an independent ω-tree $\langle N_\eta \rangle_{\eta \in I}$ of a-models. Then M is a-minimal over N_I; i.e., there is no a-model M' such that $N_I \subset M' \subsetneqq M$.*

Proof: Suppose such M' existed. Then, of course, M' is a-prime over N_I as well. Let $p \in S(M')$ be a nonalgebraic type realized in M. By 4.1, $p \not\perp N_\eta$ for some $\eta \in I$. By D.11(v), there is regular $q \in S(N_\eta)$ such that $p \not\perp q$. By D.5′(i), $q|M'$ is realized in M as well, say by $a \in M - M'$. We have $a \underset{N_\eta}{\perp} N_I$, hence by C.12(ii), $a \underset{N_\eta}{\perp} M$ since M is a-prime over N_I; contradiction to $a \in M - N_\eta$.

Proposition 4.2′ *Let T be t.t. Suppose M is a-prime over N_I for an independent ω-tree $\langle N_\eta \rangle_{\eta \in I}$ of models. Then M is minimal over N_I.*

Proof: The same as that of 4.2, cf. D.17.

Theorem 4.3 (Existence of a-representation) *Every a-model has an a-representation.*

Proof: Let M be an a-model. We construct an ω-tree I and a tree $\langle N_\eta \rangle_{\eta \in I}$ of a-models, by proceeding by an induction on the level of η. We define $N_{\langle\,\rangle}$ to be any a-prime model $\subset M$. Supposing that we have defined the indices η as well as the indexed models N_η up to and including level n, let η be an index of level n, and consider $N_\eta \subset M$. Let $\langle a_\nu \rangle_{\nu \in I_0}$ be a system of elements $a_\nu \in M$ such that a_ν/N_η is regular, $a_\nu/N_\eta \perp N_{\eta^-}$ if η^- is defined, $\langle a_\nu \rangle_{\nu \in I_0}$ is independent over N_η, and maximal among all such systems. We define the set of successors of η to be I_0 (or some set in 1-1 correspondence with I_0); we put $N_\nu = N_\eta[a_\nu]$, some model contained in M a-prime over $N_\eta \cup \{a_\nu\}$. This completes the construction of $\langle N_\eta \rangle_{\eta \in I}$; it is clearly an a-representation. Suppose now that M is not a-prime over N_I, in particular, M is not a-minimal over N_I: there is an a-model M' such that $N_I \subset M' \subsetneqq M$; of course, M' can be chosen to be a-prime over N_I. Let p be a type $\in S(M')$ realized in $M - M'$. By 4.1, there is η such that $p \not\perp N_\eta$. Choose η to be of minimal level in addition. There is regular $q \in S(N_\eta)$ such that $p \not\perp q$; $q|M'$ is realized in $M - M'$, say by a. For the case $\eta \neq \langle\,\rangle$, we have $p \perp N_{\eta^-}$, hence $q \perp N_{\eta^-}$. Notice that $a \underset{N_\eta}{\perp} M'$. The system $\langle a_\nu \rangle_{\nu \rhd \eta} {}^\frown \langle a \rangle$ is independent over N_η, and it consists of elements of M regular over N_η, with types over N_η that are $\perp$ to N_{η^-} if $\eta \neq \langle\,\rangle$; this is a contradiction to the maximality of $\langle a_\nu \rangle_{\nu \rhd \eta}$.

Theorem 4.3′ (T t.t.) (Existence of representations) *Every model has a representation.*

Proof: The same as that of 4.3.

5 Numerical computations In this section we derive estimates for $I(T, \aleph_\alpha)$, the number of isomorphism types of models of T in power $\aleph_\alpha$, for T countable and t.t., and for $I_a(T, \aleph_\alpha)$, the number of isomorphism types of a-models of T in power $\aleph_\alpha$, for T countable and superstable. The countability assumption on T is only for convenience. The results for theories satisfying *NDOP* are obtained in an identical way for the two cases, and they are deduced in an elementary way from 3.4, the quasi-uniqueness result (to obtain the lower bounds), and 4.3′, the existence of representations (to obtain the upper bounds). Of course, theories with *DOP* are dealt with in 2.3.

Until further notice we assume now that T is t.t. and countable.

A *concrete n-chain* is a sequence $N_0 \subset N_1 \subset \ldots \subset N_n$ of models such that N_0 is prime over $\varnothing$, and $N_{k+1} = N_k(a_k)$ for some a_k *SR* over N_k such that $a_k/N_k \perp N_{k-1}$ if $k > 0$. Two concrete n-chains $\langle N_k \rangle_{k \le n}$, $\langle N'_k \rangle_{k \le n}$ are *isomorphic* if there is an isomorphism of N_n onto N'_n that maps each N_k onto $N'_k (k \le n)$; an (*abstract*) *n-chain* (*represented by* $\langle N_k \rangle_{k \le n}$) is an (the) isomorphism type $[\gamma]$ of some n-chain γ (of $\langle N_k \rangle_{k \le n}$).

The *basic tree* of T (denoted $\mathcal{C}$ or $\mathcal{C}(T)$) is the poset whose underlying set is the set of all chains (n-chains for all $n < \omega$), with the obvious ordering: $[\langle N_k \rangle_{k \le n}] \le [\langle N'_k \rangle_{k \le n'}]$ iff $n \le n'$ and $\langle N_k \rangle_{k \le n} \simeq \langle N'_k \rangle_{k \le n}$. $\mathcal{C}$ is an ω-tree.

The *depth* of T, $d(T)$ is the usual well-foundation rank of the basic tree $\mathcal{C}$; $rank_{\mathcal{C}}(c) = sup\{rank_{\mathcal{C}}(c') + 1 \colon c'$ a successor of $c\}$ for c in $\mathcal{C}$, and $d(T) = rank_{\mathcal{C}}(root_{\mathcal{C}})$ (*sup* of a set of ordinals is the smallest ordinal greater or equal to all ordinals in the set). $d(T) = \infty$ if $\mathcal{C}$ is not well-founded (in which case T is called *deep*); $d(T)$ is an ordinal if $\mathcal{C}$ is well-founded (in which case T is called *shallow*).

Theorem 5.1 *Suppose T is t.t. and countable and suppose T has NDOP. We have the estimates*

$$I(T, \aleph_\alpha) \leq \begin{cases} \min(2^{\aleph_\alpha}, \beth_{d(T)-1}(|\omega+\alpha|^{\aleph_0})) & \text{if } 1 \leq d(T) < \omega \\ \min(2^{\aleph_\alpha}, \beth_{d(T)+1}(|\omega+\alpha|)) & \text{if } d(T) \geq \omega \end{cases}$$

$$I(T, \aleph_\alpha) \geq \begin{cases} \min(2^{\aleph_\alpha}, \beth_{d(T)-1}(|\alpha|)) & \text{if } 1 \leq d(T) < \omega \text{ and } \alpha \geq \omega \\ \min(2^{\aleph_\alpha}, \beth_{d(T)+1}(|\alpha|)) & \text{if } d(T) \geq \omega \text{ and } \alpha \geq \omega\ . \end{cases}$$

In particular, if $d(T) \geq \omega$ and $\alpha \geq \omega$, we have the equality

$$I(T, \aleph_\alpha) = \min(2^{\aleph_\alpha}, \beth_{d(T)+1}(|\alpha|))\ .$$

Proof: First we deal with the upper bounds. A ($\mathcal{C}$-) *labeled tree I* is an ω-tree together with a function f: $I \to \mathcal{C}$ such that f is order- and level-preserving; in particular, if $\eta \lhd \nu$ in I, $f(\eta) \lhd f(\nu)$ in $\mathcal{C}$. In other words, to every node of I on level n, say, we attach an n-chain, so that the chain attached to η is the 'restriction' of the chain attached to ν, if $\eta < \nu$. Two labeled trees are *isomorphic* if there is a tree-isomorphism between them that also preserves labels.

It is practically obvious that labeled trees capture exactly the notion of 'isomorphism type of representations'. More particularly, let $\mathfrak{N}^\ell = \langle N^\ell_\eta \rangle_{\eta \in I^\ell}$ ($\ell = 0, 1$) be two normal trees of models. An *isomorphism* h: $\mathfrak{N}^0 \xrightarrow{\sim} \mathfrak{N}^1$ is, by definition, an isomorphism $\bar{h}$: $I^0 \xrightarrow{\sim} I^1$ of trees (posets) together with a family $\langle h_\eta \rangle_{\eta \in I^0}$ of isomorphisms h_η: $N^0_\eta \xrightarrow{\sim} N^1_{\bar{h}(\eta)}$ such that if $\eta < \nu$, then h_ν extends h_η. Given a representation $\mathfrak{N} = \langle N_\eta \rangle_{\eta \in I}$, its *type* is the labeled tree $I^{\mathfrak{N}}$ with underlying tree I and with label on η, the isomorphism type of the concrete chain $N\langle\,\rangle \subset N_{\eta_0} \subset N_{\eta_1} \subset \ldots \subset N_\eta$, where $\langle\,\rangle < \eta_0 < \eta_1 < \ldots < \eta$ are all the $\eta' \in I$ with $\eta' \leq \eta$. It is clear that two representations are isomorphic iff their types are isomorphic labeled trees. It is also clear that any $\mathcal{C}$-labeled tree is the type of some representation.

It is an important but easy observation that if M^ℓ is represented by $\mathfrak{N}^\ell$ ($\ell = 0, 1$) and $\mathfrak{N}^0 \simeq \mathfrak{N}^1$ (in the sense defined above), then $M^0 \simeq M^1$. To see this, first note that $\mathfrak{N}^0 \simeq \mathfrak{N}^1$ implies $N^0_{I^0} \simeq N^1_{I^1}$; in fact if $h = \langle \bar{h}, \langle h_\eta \rangle_{\eta \in I^0} \rangle$: $\mathfrak{N}^0 \xrightarrow{\sim} \mathfrak{N}^1$, then $\bigcup_{\eta \in I^0} h_\eta$: $N^0_{I^0} \xrightarrow{\sim} N^1_{I^1}$. This follows from the independence property (3.2(ii)) of normal trees, together with the following obvious general fact: if $\langle A^\ell_i \rangle_{i \in I}$ is independent over A^ℓ ($\ell = 0, 1$), h: $A^0 \xrightarrow{\sim} A^1$ (an elementary isomorphism), h_i: $A^0_i \xrightarrow{\sim} A^1_i$, $h_i \supset h$ ($i \in I$), $t(A^\ell_i/A)$ is stationary, then

$$\bigcup_{i \in I} h_i\colon \bigcup_{i \in I} A^0_i \xrightarrow{\sim} \bigcup_{i \in I} A^1_i\ .$$

The observation now follows from the uniqueness of prime models (B.7). From the above discussion we now see that the existence theorem (4.3′) implies that

$$I(T, \aleph_\alpha) \leq I(\mathcal{C}, \aleph_\alpha)$$

where the last quantity is the number of isomorphism types of $\mathcal{C}$-labeled trees.

Hence, to complete the proof of the first part of the theorem, it suffices to estimate $I(\mathcal{C}, \aleph_\alpha)$ from above suitably.

Let $\mathcal{C}$ temporarily be any countable tree of ("well-foundation") rank δ; let $I(\mathcal{C}, \leq \aleph_\alpha)$ denote the number of nonisomorphic $\mathcal{C}$-labeled trees of power $\leq \aleph_\alpha$. We prove by induction on δ that

$$I(\mathcal{C}, \leq \aleph_\alpha) \leq \left\{ \begin{array}{ll} 1 & \text{if } \delta = 0 \\ \beth_{\delta-1}(|\omega + \alpha|^{\aleph_0}) & \text{if } 1 \leq \delta < \omega \\ \beth_{\delta+1}(|\omega + \alpha|) & \text{if } \delta \geq \omega \end{array} \right\} \tag{1}$$

For $\delta = 0$, the assertion is obvious. Let $\delta > 0$.

Let $\mathcal{C}[1]$ be the first level of $\mathcal{C}$ (successors of the root). If I is a $\mathcal{C}$-labeled tree, $\eta \in I[1]$ (the first level of I), then η determines a labeled tree I_η with root η, labeled by $\mathcal{C}_{f(\eta)}$ (the subtree of $\mathcal{C}$ with root $f(\eta)$, $f(\eta)$ being the label on η, $f(\eta) \in \mathcal{C}[1]$). It is clear that the isomorphism type of I is determined by how many times each isomorphism type of a $\mathcal{C}_c$-labeled tree ($c \in \mathcal{C}[1]$) is represented as $I_\eta (\eta \in I[1])$. More precisely, let X be the set of isomorphism types of $\mathcal{C}_c$-labeled trees of cardinality $\leq \aleph_\alpha$, collectively for all $c \in \mathcal{C}[1]$; for each $x \in X$, let $[x]$ be the cardinality of the set $\{\eta \in I[1]: I_\eta \text{ is in the class } x\}$; if we do this for two $\mathcal{C}$-labeled trees I^0 and I^1, then

$$I^0 \simeq I^1 \Leftrightarrow [x]_{(I^0)} = [x]_{(I^1)} \text{ for all } x \in X \ ;$$

this is clear. Since $[x]$ may range over all cardinals (including finite ones) $\leq \aleph_\alpha$, $[x]$ ranges over a set of power $|\omega + \alpha|$. It follows that

$$I(\mathcal{C}, \leq \aleph_\alpha) \leq |\omega + \alpha|^{|X|} \ . \tag{2}$$

Let us denote the expression on the right-hand side of (1) by $g(\delta)$.

By induction hypothesis, for $X_c =$ the set of isomorphism types of $\mathcal{C}_c$-labeled trees ($c \in \mathcal{C}[1]$), we have (since $\operatorname{rank}(\mathcal{C}_c) < \delta$)

$$|X_c| \leq \sup_{\gamma<\delta} g(\gamma) \ ;$$

hence, since $X = \bigcup_{c \in \mathcal{C}[1]} X_c$ and $\mathcal{C}[1]$ is countable,

$$|X| \leq \aleph_0 \cdot \sup_{\gamma<\delta} g(\gamma) \ . \tag{3}$$

Since, as it is easily seen, the function g satisfies $g(\delta) = |\omega + \alpha|^{\aleph_0 \cdot \sup_{\gamma<\delta} g(\gamma)}$, (2) and (3) imply (1). As we noted above, (1) implies the upper bounds in the theorem.

The opposite inequality is slightly trickier to deduce; the method works only for $\alpha \geq \omega$. Of course, we now use the "quasi-uniqueness" theorem, Theorem 3.4.

The labels do not help us any more; hence we talk about (bare) trees I instead of labeled trees.

In 3.4, the following *quasi-isomorphism* of tree is implicit: $I^0 \underset{q}{\simeq} I^1$ iff there is a partial 1-1 correspondence h between I^0 and I^1 that preserves and reflects order (but not necessarily level) and such that for every $\eta \in I^0$ (every $\eta \in I^1$), for all but countably many successors ν of η, $h(\nu)$ ($h^{-1}(\nu)$) is defined; we also write $h: I^0 \underset{q}{\rightrightarrows} I^1$.

We want to find a class of trees in which quasi-isomorphisms preserve levels.

Let us call a tree I *ample* if for all $\eta \in I$, if an isomorphism type of trees occurs among the subtrees with roots the successors of η, then it occurs uncountably often; in symbols, for all $\nu_0 \rhd \eta$,

$$|\{\nu \rhd \eta : I_\nu \simeq I_{\nu_0}\}| \geq \aleph_1 \ .$$

For ample I^0, I^1, if $h: I^0 \underset{q}{\rightrightarrows} I^1$, and $\hat{\eta} = h(\eta)$, then $\mathrm{rank}_{I^0}(\eta) = \mathrm{rank}_{I^1}(\hat{\eta})$; this is easy to check by induction.

Here is a trick (verbal communication by Shelah), to ensure that a quasi-isomorphism preserves levels. Suppose $\alpha \geq \omega$ and let

$$\bar{X} = \langle X_n : n < \omega \rangle$$

be a family of sets of uncountable cardinalities $\leq \aleph_\alpha$ such that the X_n are pairwise disjoint and each is of cardinality $|\alpha|$. A tree I is *constrained by* $\bar{X}$ if for each $n < \omega$, if $\eta \in I[n]$ (level n in I) and $\mathrm{rank}_I(\eta) = 1$, then the number of successors of η is a cardinal in X_n. We claim that if I^0, I^1 are ample trees both constrained by $\bar{X}$, and $h: I^0 \underset{q}{\rightrightarrows} I^1$, then h preserves levels, at least for elements of rank > 0: if $\hat{\eta} = h(\eta)$, $\mathrm{rank}_{I^0}(\eta) > 0$, and $\eta \in I^0[n]$, then $\hat{\eta} \in I^1[n]$. The proof is by induction on $\mathrm{rank}_{I^0}(\eta)$. If this equals 1, then so does $\mathrm{rank}_{I^1}(\hat{\eta})$, and the assertion is a direct consequence of the trees being constrained by $\bar{X}$. Let $\mathrm{rank}_{I^0}(\eta) > 1$. Then η has an immediate successor ν such that $\mathrm{rank}_{I^0}(\nu) \geq 1$, and by ampleness, there is such ν such that $\hat{\nu} = h(\nu)$ is defined; by induction hypothesis, $\mathrm{level}_{I^1}(\hat{\nu}) = \mathrm{level}_{I^0}(\nu)$; since $\hat{\eta} < \hat{\nu}$, it follows that $\mathrm{level}_{I^1}(\hat{\eta}) \leq \mathrm{level}_{I^0}(\eta)$; by symmetry, $\mathrm{level}_{I^0}(\eta) \leq \mathrm{level}_{I^1}(\hat{\eta})$, hence the two levels are equal.

Now, we estimate the number $q(\alpha, \delta)$ of quasi-isomorphism types of $\bar{X}$-constrained ample trees of power $\aleph_\alpha$ and of rank $\leq \delta$, for $\alpha \geq \omega$ and $\bar{X}$ as above. We claim that

$$q(\alpha, \delta) \geq \begin{cases} \mathit{min}(2^{\aleph_\alpha}, \beth_{\delta-1}(|\alpha|)) & \text{if } 1 \leq \delta < \omega \\ \mathit{min}(2^{\aleph_\alpha}, \beth_{\delta+1}(|\alpha|)) & \text{if } \omega \leq \delta \ . \end{cases} \tag{4}$$

Let $g(\gamma) = \beth_{\gamma-1}(|\alpha|)$ for $1 \leq \gamma < \omega$, $g(\gamma) = \beth_{\gamma+1}(|\alpha|)$ for $\gamma \geq \omega$; let $g^*(\delta) = \sup_{\gamma<\delta} g(\gamma)$. For $\delta = 1$, there is nothing to prove; assume $\delta > 1$; suppose the assertion is true for $\gamma < \delta$. Hence, there are at least $\mathit{min}(2^{\aleph_\alpha}, g^*(\delta))$ pairwise non-quasi-isomorphic ample trees of cardinality $\aleph_\alpha$ and of rank $< \delta$ constrained by $\bar{X}' = \langle X_{n+1} : n < \omega \rangle$ (with $\bar{X} = \langle X_n : n < \omega \rangle$ the given constraint). Let Y be a set of pairwise non-quasi-isomorphic trees with all these properties such that $|Y| = \mathit{min}(\aleph_\alpha, g^*(\delta))$. Now, consider an arbitrary assignment f with $\mathit{dom}(f) = Y$ such that for each $J \in Y$, $f(J)$ is an uncountable cardinality $\leq \aleph_\alpha$; the number of such assignments is $|\alpha|^{|Y|} = \mathit{min}(2^{\aleph_\alpha}, |\alpha|^{g^*(\delta)}) = \mathit{min}(2^{\aleph_\alpha}, g(\delta))$. Given such an f, one can construct a tree $I = I^f$ such that the subtrees I_η of I determined by elements η in $I[1]$ are, up to q-isomorphism, all from Y, and in fact, each tree J from Y occurs exactly $f(J)$ many times, up to q-isomorphism, as some I_η ($\eta \in I[1]$). Clearly, I^f is constrained by $\bar{X}$, it is ample, its rank is $\leq \delta$, and its power is exactly $\aleph_\alpha$.

We claim that $I^f \underset{q}{\rightrightarrows} I^{f'}$ implies that $f = f'$. In fact, if h is the quasi-isomorphism, it maps the first level (except countably many elements) of I^J onto the first level (except countably many elements) of $I^{f'}$, and if $h(\eta) = \hat{\eta}$ ($\eta \in I^f[1]$), then it induces a quasi-isomorphisms $I^f_\eta \underset{q}{\rightrightarrows} I^{f'}_{\hat{\eta}}$; hence, h maps the set $\{\eta: I^f_\eta \underset{q}{\rightrightarrows} J\}$ onto $\{\hat{\eta}: I^{f'}_{\hat{\eta}} \underset{q}{\rightrightarrows} J\}$ modulo countably many exceptions, for each $J \in Y$; i.e., $f(J) = f'(J)$ for all $J \in Y$ as claimed.

This proves (4).

The estimate (4) now implies the corresponding estimate in the theorem. Let $\delta = d(T)$; let I be any tree of rank $\leq \delta$. Make I into a $\mathfrak{C}$-labeled tree by choosing an arbitrary labeling f: $I \to \mathfrak{C}$ (this is possible since rank$(I) \leq$ rank$(\mathfrak{C})$; if rank$(\mathfrak{C}) = \infty$, this is clear; if rank $(\mathfrak{C}) < \infty$, induct on rank$(\mathfrak{C})$). Consider any representation $\mathfrak{N} = \langle N_\eta \rangle_{\eta \in I}$ whose type is I; let M_I be the model represented by $\mathfrak{N}$. If $M_I \simeq M_{I'}$, then by 3.4 the (bare) trees I and I' are quasi-isomorphic. Hence, we get for $I(T, \aleph_\alpha)$ the same lower estimates as (4) for $q(\alpha, \delta)$.

For a deep theory T, when $d(T) = \infty$, the theorem says that $I(T, \aleph_\alpha)$ is the maximal number, $2^{\aleph_\alpha}$, whenever $\alpha \geq \omega$. One can improve this result by removing the restriction $\alpha \geq \omega$.

Theorem 5.2 *Suppose T is t.t. and countable. Suppose T is deep; i.e., $d(T) = \infty$. Then $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ for $\alpha > 0$.*

Proof: The assumption $d(T) = \infty$ implies that there is an infinite sequence $N_0 \subset N_1 \subset \ldots \subset N_n \subset \ldots$ such that $\langle N_k \rangle_{k \leq n}$ is a concrete n-chain for all $n < \omega$. Let $N_1 = N_0(a_1)$, and let B be a finite subset of N_0 such that $t(a_1/N_0)$ is based on B (i.e., $a_1 \underset{B}{\perp} N_0$ and $t(a_1/B)$ is stationary). Let us fix an arbitrary uncountable cardinal $\kappa = \aleph_\alpha$. We consider models that are represented by some $\mathfrak{N} = \langle N_\eta \rangle_{\eta \in I}$ with I an ample (see the proof of Theorem 5.1) ω-tree of cardinality κ and such that for $\langle\, \rangle = \eta_0 \lhd \eta_1 \lhd \ldots \lhd \eta_n$ in I, $\langle N_{\eta_k} \rangle_{k \leq n} \simeq \langle N_k \rangle_{k \leq n}$; i.e., such that the type of $\mathfrak{N}$ is a labeled tree in which we have used only the labels $c_n = [\langle N_k \rangle_{k \leq n}]$ $(n < \omega)$; *wlog*, we also assume that $N_{\langle\, \rangle}$ is literally the same as N_0. Let M^0, M^1 be two such models, represented by appropriate $\mathfrak{N}^0$, $\mathfrak{N}^1$, and assume that f is an isomorphism $M^0 \rightrightarrows M^1$ which is the identity on B. Given any ν in I^0 on level 1, and $\hat{\nu}$ in I^1 on level 1, we have that $p_\nu = t(N_\nu/N_{\langle\, \rangle}) \not\perp t(a_1/B)$, and $p_{\hat{\nu}} \not\perp t(a_1/B)$; hence, since f is the identity on B, $f(p_\nu) \not\perp p_{\hat{\nu}}$. Consider now the "quasi-isomorphism" h derived from f according to 3.4. We must have that any $\nu \in I^0$ on level 1 (if it is in the domain of h) corresponds by h to some $\hat{\nu}$ in I' on the first level; this is because the condition $f(p_\nu) \not\perp p_{\hat{\nu}}$ determines the class of $\hat{\nu}$ in I', and we have seen that indices $\hat{\nu}$ on level 1 (that obviously form a single class, by the way) do satisfy this condition. Next, remember that if $\nu \mapsto \hat{\nu}$ under h, then $\text{rank}_{I^0}(\nu) = \text{rank}_{I^1}(\hat{\nu})$ (by the ampleness, see the proof of Theorem 5.1 above).

By these remarks, it is now easy to construct 2^κ nonisomorphic models of power κ. With any large enough set $X \subset \kappa$, associate an ample tree $I = I_X$ of size κ such that the set of ranks of elements on level 1 in I is precisely X. Let M_X be the model whose type is I_X with the labeling with the c_n as above. Our remarks above show that if M_X and $M_{X'}$ are B-isomorphic, then $X = X'$. Since B is finite, we obtain the conclusion of the theorem.

For T t.t. and countable, $\mathcal{C}(T)$ is clearly a countable tree. Therefore, obviously, if $d(T) < \infty$, then $d(T)$ is a countable ordinal. Using this remark, we now have a corollary about countable t.t. theories not mentioning the property *NDOP*.

Corollary 5.3 *Suppose T is countable and t.t. Then $\alpha > 0$ and $I(T, \aleph_\alpha) < 2^{\aleph_\alpha}$ imply $I(T, \aleph_\beta) < \beth_{\omega_1}(|\omega + \beta|)$, for any ordinal β.*

Proof: Let $\alpha > 0$. $I(T, \aleph_\alpha) < 2^{\aleph_\alpha}$ implies, on the one hand, that T has *NDOP* (by 2.3; now $\lambda_0 = \aleph_0$), and on the other hand, that $d(T) < \infty$ (by 5.2). Thus, the conclusion follows from $d(T) < \omega_1$ and 5.1.

Assume now that T is, more generally, s.s. In the context of a-models, we have a notion of depth analogous to the one introduced above. In fact, we will show in the next section that in case T is t.t. the two notions of depth coincide; anticipating that result, we do not make a notational distinction between the two. In some detail: we have the notion of 'concrete n-chain' as before, with N_0 a-prime (over $\emptyset$), and $N_{k+1} = N_k[a_k]$ (replacing $N_{k+1} = N_k(a_k)$); using the resulting modified notion of 'n-chain', we arrive at the (modified) 'basic tree' $\mathcal{C}$, and we put $d(T) =$ rank of $\mathcal{C}$.

Theorem 5.4 *Suppose T is s.s. and countable, and suppose that T has NDOP. Then we have*

$$I_a(T, \aleph_\alpha) \le \begin{cases} \min(2^{\aleph_\alpha}, \beth_{d(T)-1}(|\omega + \alpha|^{\aleph_0})) & \text{if } 1 \le d(T) < \omega \\ \min(2^{\aleph_\alpha}, \beth_{d(T)+1}(|\omega + \alpha|)) & \text{if } d(T) \ge \omega \end{cases}$$

$$I_a(T, \aleph_\alpha) \ge \begin{cases} \min(2^{\aleph_\alpha}, \beth_{d(T)-1}(|\alpha|)) & \text{if } 1 \le d(T) < \omega,\ \alpha \ge \omega, \text{ and } \aleph_\alpha \ge \lambda_0 \\ \min(2^{\aleph_\alpha}, \beth_{d(T)+1}(|\alpha|)) & \text{if } d(T) \ge \omega,\ \alpha \ge \omega, \text{ and } \aleph_\alpha \ge \lambda_0\ . \end{cases}$$

In particular,

$$I_a(T, \aleph_\alpha) = \min(2^{\aleph_\alpha}, \beth_{d(T)+1}(|\alpha|)) \text{ if } d(T) \ge \omega,\ \alpha \ge \omega, \text{ and } \aleph_\alpha \ge \lambda_0\ .$$

Also, $I_a(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ whenever $d(T) = \infty$, $\alpha > 0$, and $\aleph_\alpha \ge \lambda_0$. As a corollary, we have that $\alpha > 0$ and $I_a(T, \aleph_\alpha) < 2^{\aleph_\alpha}$ imply $I_a(T, \aleph_\beta) < \beth_{d(T)+1}(|\omega + \beta|)$ whenever T is countable superstable.

The proofs of the various parts of Theorem 5.4 are identical to those of the corresponding results given above, using the corresponding results for a-models in previous sections.

6 A discussion of depth In this section T is assumed to be superstable. We start by restating the definition of the depth of T in a manner slightly different from the one in the last section. For brevity, let $\mathcal{P}$ denote the class of pairs $\langle M, M' \rangle$ of a-models M, M', such that $M \subsetneqq M'$, and M' is finitely a-generated over M ($M' = M[a]$ for some a). We define the function $d(-)$ on $\mathcal{P}$ such that

$d(M'/M) = d(\langle M, M' \rangle)$ is an ordinal or ∞ as follows. $d(-)$ is uniquely determined by the following condition:

$d(M'/M) \geq \alpha + 1 \Leftrightarrow$ for some M'' such that
$\langle M', M'' \rangle \in \mathcal{P}$, we have $t(M''/M') \perp M$ and $d(M''/M') \geq \alpha$.

(Of course, we have the conventions: $\alpha < \infty$ for all $\alpha \in Ord$, and also $\infty + 1 = \infty$.)

For a finite tuple a, and a-model M, $d(a/M) = d(M[a]/M)$. Note that we clearly have that $a/M \precsim a'/M$ implies $d(a/M) = d(a'/M)$ (M an a-model).

Proposition 6.1 (T *is s.s.*) *Suppose $M \subset N$ are a-models, and $a \underset{M}{\perp} N$. Then*

(i) *$d(a/M) \leq d(a/N)$; and*
(ii) *Supposing that T has NDOP, we have $d(a/M) = d(a/N)$.*

Proof: In this proof, M and N denote a-models, $M \subset N$.

ad(*i*). This is a straightforward computation. We prove by induction on α that $a \underset{M}{\perp} N$ and $d(a/M) \geq \alpha$ imply $d(a/N) \geq \alpha$. It suffices to consider the case $\alpha = \beta + 1$. Assume $a \underset{M}{\perp} N$ and $d(a/M) \geq \beta + 1$; i.e., for some b, $b/M[a] \perp M$ and $d(b/M[a]) \geq \beta$. Without loss of generality, $b \underset{M[a]}{\perp} N$. It follows that $b \underset{M[a]}{\perp} N[a]$; hence by induction hypothesis, $d(b/N[a]) \geq \beta$. Since we have $M[a] \underset{M}{\perp} N$, $b/M[a] \perp M$ implies $b/M[a] \perp N$, hence $b/N[a] \perp N$. This shows that b is a witness for $d(a/N) \geq \beta + 1$.

ad(*ii*). The proof is a similar induction. Suppose $a \underset{M}{\perp} N$ and $d(a/N) \geq \beta + 1$; let b be such that $b/N[a] \perp N$ and $d(b/N[a]) \geq \beta$. Since the a-prime model over $M[a] \cup N$ is a-prime over $N \cup \{a\}$ (as is easily seen by $a \underset{M}{\perp} N$), we may assume that $N[a]$ is a-prime over $M[a] \cup N$. We claim that there is a type $q' \in S(M[a])$ such that $q' \precsim q = t(b/N[a])$. Indeed, let $q \precsim \bigotimes_{i<n} r_i$ ($r_i \in S(N[a])$) be a regular decomposition of q (see D.10). Let $i < n$ be any index. $q \perp N$ implies $r_i \perp N$. By *NDOP* (note that $M[a] \underset{M}{\perp} N$) we have $r_i \not\perp M[a]$; i.e., $r_i \precsim r_i'$ for some regular $r_i' \in S(M[a])$ (see D.11(v), D.5'). Since this was true for all $i < n$, $q \precsim q' =_{df} \bigotimes_{i<n} r_i'$, as claimed. If b' realizes $q'|N[a]$, then, by $q \precsim q'|N[a]$, we clearly have that $d(b'/N[a]) = d(b/N[a]) \geq \beta$. By the induction hypothesis, $d(b'/M[a]) \geq \beta$. $b'/M[a] \perp M$, since $b'/M[a]$ is parallel to $b'/N[a]$, and the latter type is orthogonal to the even larger set N. These facts tell us that b' witnesses $d(a/M[a]) \geq \beta + 1$, as desired.

Proposition 6.1 is an encouragement to define the depth of any stationary type, at least for the case when T is s.s. and has *NDOP*, as we will assume from now on. For stationary q, $d(q)$ is $d(a/M)$ for an a-model M such that $dom(q) \subset M$ and $a \models q|M$. Proposition 6.1 tells us that $d(a/M)$ does not depend on the choice of M (within the conditions stated), or in other words, that $d(-)$ is a parallelism invariant.

Proposition 6.2 *Suppose T is s.s. and has NDOP.*

(i) *$p \lhd q$ implies $d(p) \leq d(q)$.*
(ii) *For any stationary p, $d(p) = max\{d(r) : r \lhd p, r$ is regular$\}$.*

Proof: ad(i). Exercise.

ad(ii). Notice that up to $\precapprox$, there are only finitely many regular r such that $r \lhd p$ (see D.10, D.11), hence there are only finitely many values in the set after the '*max*' operation. Therefore, it suffices to prove that $d(p) \geq \beta + 1$ implies that there is r regular such that $r \lhd p$ and $d(r) \geq \beta + 1$, under the assumption that the same statement is true with $\beta + 1$ replaced by β (induction hypothesis). So, assume M is an a-model, $q \in S(M[a])$, $d(q) \geq \beta$ and $q \perp M$ (here p is parallel to a/M). By the induction hypothesis, there is $s \in S(M[a])$ such that s is regular and $d(s) \geq \beta$. By the decomposition theorem, $M[a]$ is a-prime over $\bigcup_{i<n} M[a_i]$ for an M-independent system $\langle a_i \rangle_{i<n}$ of elements realizing regular types over M. By *NDOP*, $s \not\perp M[a_i]$ for some $i < n$, hence for some $s' \in S(M[a_i])$, $s' \precapprox s$; by 6.1, $d(s') = d(s) \geq \beta$. Let $r = t(a_i/M)$; then s' witnesses $d(r) \geq \beta + 1$, as desired.

Note that if we put $\mathcal{P}_1 = \{\langle M, M'\rangle \in \mathcal{P} : M' = M[a]$ for some a with a/M regular$\}$, and we define $d_1(-)$ on $\mathcal{P}_1$ just as d was defined on $\mathcal{P}$ (in particular, with "$\langle M', M''\rangle \in \mathcal{P}_1$" replacing "$\langle M', M''\rangle \in \mathcal{P}$"), then 6.2 tells us that $d(M'/M) = d_1(M'/M)$ for $\langle M, M'\rangle \in \mathcal{P}_1$.

Let us write $d(M) = sup\{d(p) + 1 : p \in S(M)\}$, with M any a-model. We have $d(M) = sup\{d_1(p) + 1 : p \in S(M),\ p$ regular$\}$. It is now obvious that $d(T)$ as defined in the last section, in the second version with "a-models", is the same as $d(M_0)$ for M_0 the a-prime model (over $\emptyset$).

It is not hard to see that, in fact, $d(M) = d(M_0) = d(T)$ for any a-model M: on the one hand, with $M_0 \subset M$, it is clear that $d(M_0) \leq d(M)$; on the other hand, using an a-representation of M, we can easily see that for every regular $p \in S(M)$, there is a regular $p' \in S(M_0)$ such that $d_1(p) \leq d_1(p')$, proving $d(M) \leq d(M_0)$ (for a similar argument, see the proof of Proposition 6.4 below).

Let us now turn to the case when T is t.t. and has *NDOP*. We could define $d_0(a/M)$ for arbitrary tuples a and arbitrary models M analogously to $d(a/M)$; but we do not seem to be able to prove the analog of 6.1(ii) for the resulting notion. Instead, we restrict our attention to strongly regular a/M, and introduce the analog of d_1 (see above). Accordingly, let $\mathcal{P}_0$ be the class of all pairs $\langle M, M'\rangle$ such that $M' = M(a)$ for some a with a/M strongly regular. Note (see D.22) that $\langle M, M'\rangle \in \mathcal{P}_0$ iff M' is 'minimal' over M in the weak sense that $M \subsetneqq M'$ and for every N with $M \subsetneqq N \subset M'$ there is an M-isomorphism of N onto M'. Note also that if a/M, a'/M are both *SR* and $a/M \not\perp a'/M$, then $a/M \underset{RK}{\sim} a'/M$, and in fact, $M(a)$ is M-isomorphic to $M(a')$.

Define $d_0(-)$ to be the unique function from $\mathcal{P}_0$ to $Ord \cup \{\infty\}$ such that for any $\alpha \in Ord$,

$$d_0(M'/M) \geq \alpha + 1 \Leftrightarrow \text{there is } \langle M', M''\rangle \in \mathcal{P}_0 \text{ such that } t(M''/M') \perp M \text{ and } d_0(M''/M') \geq \alpha.$$

As before, we put $d_0(a/M) = d_0(M(a)/M)$ for a/M *SR*. In this context, note that if $M \subset M'$, then $t(b/M') \perp M$ is equivalent to $t(M'(b)/M') \perp M$ (see C.4(v) and C.12(i)).

Proposition 6.3 *Suppose T is t.t., a/M is SR and* $a \underset{M}{\perp} N$. *Then*

(i) $d_0(a/M) \le d_0(a/N)$, *and*
(ii) *if T has NDOP, then* $d_0(a/N) = d_0(a/M)$.

Proof: The proof is quite similar to that of Proposition 6.1. We omit the proof of (i).

ad(ii). Suppose $d(a/N) \ge \beta + 1$, and the appropriate induction hypothesis. Let b be such that $b/N(a)$ is *SR*, $b/N(a) \perp N$, and $d_0(b/N(a)) \ge \beta$. We may assume (by $a \underset{M}{\perp} N$) that $N(a)$ is prime over $M(a) \cup N$. By *NDOP* (see 1.3), $M(a) \underset{M}{\perp} N$ and $b/N(a) \perp N$, we have $b/N(a) \not\perp M(a)$. By D.17, there is $q \in S(M(a))$ such that q is *SR* and $q \not\perp b/N(a)$. Let b' realize $q|N(a)$. As we noted above, we now have that $(N(a))(b)$ and $(N(a))(b')$ are $N(a)$-isomorphic, so since $d_0((N(a))(b)/N(a)) \ge \beta$, so is $d_0((N(a))(b')/N(a)) \ge \beta$. By the induction hypothesis, and since $b' \underset{M(a)}{\perp} N(a)$, we conclude $d_0(b'/M(a)) \ge \beta$. Since also $b'/M(a) \perp M$, clearly, we obtain $d_0(a/M) \ge \beta + 1$ as desired.

Proposition 6.4 *Suppose T is t.t. and has NDOP. Let a/M be SR. Then*

$$d_0(a/M) = d_1(a/M) = d(a/M) \ .$$

Proof: Note that $d(a/M)$ makes sense, for an arbitrary model M, via the extension made after 6.1 of the function $d(-)$ to any stationary type. Similarly for $d_1(a/M)$, since a/M is regular.

Let us prove $d_0(a/M) \le d(a/M)$. Assume $d_0(a/M) \ge \beta + 1$, and the appropriate induction hypothesis. Let N be an a-model extending M such that $a \underset{M}{\perp} N$; we want to show that $d(a/N) \ge \beta + 1$. By 6.3, $d_0(a/N) \ge d_0(a/M) \ge \beta + 1$, hence there is a nonalgebraic type $q \in S(N(a))$ such that $q \perp N$ and $d_0(q) \ge \beta$. Let $q' = q|N[a]$ (where $N[a] \supset N(a)$). We still have $q' \perp N$, and by the induction hypothesis, $d(q) = d(q') \ge d_0(q) \ge \beta$; hence, $d(a/N) \ge \beta + 1$, as required.

Next, we prove $d_1(a/M) \le d_0(a/M)$. Assume $d_1(a/M) \ge \beta + 1$, and the appropriate induction hypothesis. For an a-model N extending M with $a \underset{M}{\perp} N$, we have some b such that $b/N[a]$ is regular, $b/N[a] \perp N$, and $d_1(b/N[a]) \ge \beta$. In $(N[a])[b] - N[a]$, there is some b' such that $b'/N[a]$ is *SR* (see D.16). Since $b'/N[a]$ is dominated by $b/N[a]$, we have that $b'/N[a] \precsim b/N[a]$, hence $(N[a])[b']$ is $N[a]$-isomorphic to $(N[a])[b]$, and $b'/N[a] \perp N$. It follows that $d_1(b'/N[a]) = d_1(b/N[a]) \ge \beta$. By the induction hypothesis, and $b'/N[a]$ being *SR*, $d_0(b'/N[a]) \ge \beta$. Next, apply 4.3′ with a slight twist: let $\mathfrak{N} = \langle N_\eta \rangle_{\eta \in I}$ be a representation of $N[a]$ "over N": for $\eta \lhd \nu$, $\langle N_\eta, N_\nu \rangle \in \mathcal{P}_0$, M is prime over N_I and $N_{\langle\,\rangle} = N$ (instead of $N_{\langle\,\rangle}$ being a prime model over $\varnothing$); and, in addition, for some $\hat{\nu} > \langle\,\rangle$, $N_{\hat{\nu}} = N(a)$. It is clear that the proof of 4.3′ will give us such a representation of $N[a]$ "over N" as well. Since for every $b \in N[a] - N$, we have $a \underset{N}{\not\perp} b$ (see B.9), it follows that the only successor of $\langle\,\rangle$ is $\hat{\nu}$ for which $N_{\hat{\nu}} = N(a)$. Now, by 4.1′, there is some $\nu \in I$ such that $b'/N[a] \not\perp N_\nu$. Let ν be minimal (closest to $\langle\,\rangle$) with the given property; necessarily, $\nu \ge \hat{\nu}$. We have some $q \in S(N_\nu)$ *SR* and nonorthogonal to $b'/N[a]$ (see D.17); hence by the remarks made before the definition of d_0, $d_0(q|N[a]) = d_0(b'/N[a]) \ge$

β and by 6.3(ii), $d_0(q) \geq \beta$. The 'concrete chain' $N_{\langle\rangle} \subset N_{\eta_1} \subset \ldots \subset N_{\eta_{n-1}} \subset N_\nu \subset N_\nu(b')$ with $\langle\rangle$, $\eta_1, \ldots, \eta_{n-1}$ being all the elements of I preceding ν ($n \geq 1$, $\eta_1 = \hat{\nu}$) clearly shows that $d_0(N_{\eta_1}/N_{\langle\rangle}) = d_0(N(a)/N)$ is at least $d_0(b'/N_\nu) + n$ (note that $d_0(N_{\eta_i}/N_{\eta_{i-1}}) \geq d_0(N_{\eta_{i+1}}/N_{\eta_i}) + 1$, with $\eta_0 = \langle\rangle$, $\eta_n = \nu$, for all i with $0 < i < n$, because of the definition of a representation, and also, $d(N_\nu/N_{\eta_{n-1}}) \geq d(b'/N_\nu) = 1$ since $b'/N_\nu \perp N_{\eta_{n-1}}$). We have proved that $d_0(a/N) \geq d_0(b'/N_\nu) + n = d_0(q) + n \geq d_0(q) + 1 \geq \beta + 1$, as desired.

Let us define $d_0(M)$, for T t.t. and having *NDOP*, as $sup\{d(M'/M) + 1 : (M, M') \in \mathcal{P}_0\}$. Now, it is clear that $d(T)$ as defined in Section 5 is the same as $d_0(M_0)$, with M_0 the prime model (over $\varnothing$). Also, $d_0(M_0) = d(M_0^*)$, with M_0^* the a-prime model (over $\varnothing$). In fact, $d_0(M_0) \leq d_0(M_0^*)$, and $d_0(M_0^*) = d(M_0^*)$ follow directly from 6.3 (note that $d_0(r|M_0^*) = d_0(r)$ for all *SR* $r \in S\ (M_0)$), and 6.4, respectively. On the other hand, an argument similar to the one employed in the proof of 6.4 shows that $d_0(M_0^*) \leq d_0(M_0)$ (by considering a representation of M_0^* "over M_0"). We have established that the two possible definitions of $d(T)$, for T t.t. and having *NDOP*, introduced in Section 5 give actually the same value to $d(T)$.

We note that the definition of depth in [6] is slightly different. Shelah's $d(M'/M)$ (for $(M, M') \in \mathcal{P}_1$) is our $d(M'/M)$ if the latter is $< \omega$, and our $d(M'/M) + 1$ if our $d(M'/M) \geq \omega$ (compare 4.1 Definition in [6]; instead of $d(M'/M)$, Shelah writes $dp(M, M', a)$ for a/M regular, $M' = M[a]$). Similarly, Shelah's $Dp(T) =$ our $d(T)$ if the latter is $< \omega$, our $d(T) + 1$ otherwise.

Finally, let us point out that the so-called nonmultidimensional theories represent those on the lowest level of the depth-hierarchy. T is called *nonmultidimensional* (n-md) if every nonalgebraic type is nonorthogonal to the empty set, $p \not\perp \varnothing$. This is equivalent to saying that there is a cardinal λ ($< card(\mathfrak{C})$) such that there is no family of pairwise orthogonal nonalgebraic types of cardinality $> \lambda$. Indeed, for any type q, there can be only $w(q)$ many pairwise orthogonal types which are all $\not\perp$ to q (see D.2(viii)); applying this to types $q \in S(\varnothing)$, we immediately obtain that T being n-md implies that λ as stated exists. Conversely, if there is a nonalgebraic type $p \perp \varnothing$, then C.6(i) allows us to construct arbitrarily large families of pairwise $\perp$ types (in fact, consisting of isomorphic copies of p itself).

For T s.s., T being n-md is the same as to say that the number of equivalence classes of the equivalence relation $\precapprox$ is a set-cardinal (rather than a proper class-cardinal); and in fact, it is the same as to say that this number is $\leq 2^{|T|}$; this is easily seen.

Let us now assume that T is s.s. Notice that $d(T) \geq 1$, simply because T has infinite models. Also notice that $d(T) = 1$ implies that T has *NDOP*: if we had an instance of *DOP* as in Definition 1.1, then with some finite $A \subset M$, p would be based on $M_0 \cup A$; hence we would have for $M_0' = M_0[A]$ that $p|M_0' \perp M_0$, contradicting $d(M_0) = 1$.

Finally, note that for T s.s., $d(T) = 1$ iff T is n-md. It is clear that T n-md implies $d(T) = 1$. On the other hand, if $d(T) = 1$ and M_0 is the a-prime model, then every nonalgebraic type is $\not\perp M_0$: this is clear since up to parallelism, every stationary type is one over some M_0' with $(M_0, M_0') \in \mathcal{P}$. It follows that "$\lambda$ exists" as in the alternative definition of n-md, hence T is n-md.

NOTE

1. See AMS Abstracts 82T-03-324 (June 1982), and 83T-03-110, 83T-03-111 (February 1983) as well as papers in the preprint collection, *A Fall 82 Collection of Old and New Preprints,* by S. Shelah.

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