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Extensionality in Bernays Set Theory

MITSURU YASUHARA

Gandy has shown, in [3], that the consistency of Bernays-Gödel set theory can be reduced to that of the theory without the axiom of extensionality. We verify a parallel result for the theory presented in [1], Appendix. Since one of the axioms (schemes) in [1] is the reflection principle, which is of rather different character than other axioms about set existence, some new considerations are required. Also we take a "top down" approach, which seems to work more quickly in this context than Gandy's "bottom up" one. At the end, we comment on other set theories with regard to extensionality and the "top down" approach.

The theory **B** is a single-sorted first-order theory with a binary predicate $\in$, a monadic function symbol σ, and a term forming operator $\{x/\ldots\}$. Using obvious abbreviations like $a \subseteq b$ and $a \cap b$, the nonlogical axioms of **B** can be stated as follows:

(Ex) [The axiom of extensionality] $a \simeq b \,\&\, a \in c \rightarrow b \in c$, *where* '$a \simeq b$' *is for* '$a \subseteq b \,\&\, b \subseteq a$'.
(CF) [The axioms of choice and Fundierung] $a \in c \rightarrow [\sigma(c) \in c \,\&\, a \notin \sigma(c)]$.
(Cp) [The axiom of impredicative comprehension] $c \in \{x/\phi(x)\} \leftrightarrow S(c) \,\&\, \phi(c)$, *where* '$S(c)$' *is for* '$\exists z c \in z$'.
(Rf) [The axiom of reflection] $\phi \rightarrow \exists y[ST(y) \,\&\, S(y) \,\&\, \phi^y]$.

Here, '$ST(y)$' is for '$\forall u, v(v \in y \,\&\, (u \subseteq v \vee u \in v) \rightarrow u \in y)$ [y is strongly transitive], and ϕ^y is the result of relativization of ϕ to y, i.e., any free variable a in ϕ is replaced by $a \cap y$ unless $S(a)$ is given, $\forall x \psi(x)$ is replaced by $\forall x(x \subseteq y \rightarrow \psi^y(x))$, $\exists x \psi(x)$ by $\exists x(x \subseteq y \,\&\, \psi^y(x))$, and $\{x/\psi(x)\}$ by $\{x/x \in y \,\&\, \psi^y(x)\}$.

(Eq) [*An axiom of equality*] $a \simeq b \rightarrow \sigma(a) \simeq \sigma(b)$.
(Em) $\neg\exists x\, x \in a \rightarrow \neg\exists x\, x \in \sigma(a)$.

(Actually, the last two axioms do not appear in [1]. Indeed, (Eq) is provable from (Ex), (Cp), and (Rf). The last determines the value of σ at $a = \varnothing$, which

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is not given by (CF). Both are natural requirements for σ to be a function. And so, we add them.)

By $\mathbf{B}^-$, we mean the theory that does not have (Ex).

We are going to define a transformation * *among formulas such that:*

(a) if $\mathbf{B} \vdash \phi$ then $\mathbf{B}^- \vdash \phi^*$

and

(b) if ϕ is a closed formula, then $(\phi \,\&\, \neg\phi)^*$ is $\phi^* \,\&\, \neg\phi^*$.

And thus, the relative consistency of $\mathbf{B}$ to $\mathbf{B}^-$ will be established.

Confusing syntax and semantics conveniently, we sometimes call a term $\{x/\phi(x)\}$ a class, and a set if $S(\{x/\phi(x)\})$ is provable. We will list below several formal definitions and theorems in $\mathbf{B}^-$. We omit quotation marks in definitions, and '$\mathbf{B}^- \vdash$' in theorems.

(1) V is for $\{x/x \simeq x\}$. $\varnothing$ is for $\{x/x \not\simeq x\}$.

(2) $S(\varnothing)$. $a \subseteq b \,\&\, S(b) \to S(a)$. These are proved in [1] without using (Ex).

(3) $TR(a)$ is for $\forall x\, x \in a \to x \subseteq a$. [$a$ is transitive.]

(4) $TR(\varnothing)$. Use (1) and (Cp).

(5) $tc(a)$ is for $\{x/\forall z[a \subseteq z \,\&\, TR(z) \to x \in z]\}$. [$tc(a)$ is the transitive closure of a.]

(6) $a \subseteq tc(a)$. $tc(\varnothing) \simeq \varnothing$. Both are obvious from relevant definitions.

(7) $a \in b \vee a \subseteq b \to tc(a) \subseteq tc(b)$. So, $a \simeq b \to tc(a) \simeq tc(b)$.

(8) $TR(tc(a))$.

Proof: We are to show, from $c \in b \in tc(a)$, that $c \in tc(a)$. So, assume $a \subseteq z \,\&\, TR(z)$. As $b \in tc(a)$, we have from (5) that $b \in z \,\&\, TR(z)$. So $b \subseteq z$ by (3). From $c \in b$, we have $c \in z$. Thus $c \in tc(a)$ from (5). So the proof is finished.

(9) $a \in tc(b) \leftrightarrow [a \in b \vee \exists y(y \in b \,\&\, a \in tc(y))]$.

Proof: The direction $\to$. Let ρ be $\{x/x \in b \vee \exists y(y \in b \,\&\, x \in tc(y))\}$. We aim at showing $b \subseteq \rho$ and $TR(\rho)$. For then, by $a \in tc(b)$ and (5), we have $a \in \rho$, hence the right hand side (RHS) holds by (Cp). That $b \subseteq \rho$ is obvious. In order to show $TR(\rho)$, we assume $d \in c$ and $c \in \rho$. The second assumption causes case distinctions.

Case $c \in b$. Then, as $c \subseteq tc(c)$ by (6), $d \in tc(c)$. Thus $c \in b \,\&\, d \in tc(c)$, whence $\exists y\, y \in b \,\&\, d \in tc(y)$. So $d \in \rho$.

Case $y \in b \,\&\, c \in tc(y)$ *for some* y. Then we have $d \in tc(y)$ from $d \in c$, $c \in tc(y)$, (8), and (3). So $\exists y\, y \in b \,\&\, d \in tc(y)$, from which $d \in \rho$ follows. Thus in either case $TR(\rho)$ is shown.

The direction $\leftarrow$. *Case* $a \in b$. By (6), $a \in tc(b)$.

Case for some y, $y \in b \,\&\, a \in tc(y)$. By (7), $tc(y) \subseteq tc(b)$. From $a \in tc(y)$, $a \in tc(b)$ results.

(10) $E(a)$ is for $\forall x,y(x \simeq y \,\&\, x \in a \to y \in a)$. [$a$ is extensional.] $E(\varnothing)$. $a \simeq b \,\&\, E(a) \to E(b)$.

(11) $M(a)$ is for $E(a) \,\&\, \forall x(x \in tc(a) \to E(x))$. [$a$ is in our inner model consisting of hereditarily extensional classes.]

(12) $M(\varnothing)$. Use (10), (11), and (6). So, $\exists x M(x)$. $a \simeq b \,\&\, M(a) \to M(b)$. Use (7), (10), and (11).

(13) $M(a) \leftrightarrow E(a) \,\&\, \forall x(x \in a \to M(x))$.

Proof: By (9) and (11), $M(a)$ is equivalent to $E(a) \,\&\, \forall x[x \in a \vee \exists y(y \in a \,\&\, x \in tc(y)) \to E(x)]$; which is equivalent, by logic, to $E(a) \,\&\, \forall x(x \in a \to E(x)) \,\&\, \forall x,y(y \in a \,\&\, x \in tc(y) \to E(x))$. The last conjunct is equivalent to $\forall x[x \in a \to \forall y(y \in tc(x) \to E(y))]$ after switching x and y. So the whole formula is equivalent to $E(a) \,\&\, \forall x\{x \in a \to [E(x) \,\&\, \forall y(y \in tc(x) \to E(y))]\}$, hence to $E(a) \,\&\, \forall x(x \in a \to M(x))$ by (11).

So far, we considered formal definitions and theorems in $\mathbf{B}^-$. Now we start considering meta-theorems and so forth as well.

(14) Meta-Definition For formulas and terms, we define their relativizations to $M(\ )$ inductively as follows: $\forall x\phi(x)$ is replaced by $\forall x M(x) \to \phi_M(x)$, $\exists x\phi(x)$ by $\exists x M(x) \,\&\, \phi_M(x)$, and $\{x/\phi(x)\}$ by $\{x/M(x) \,\&\, \phi_M(x)\}$. (Since two relativizations will be involved later, we put M as a subscript and y as a superscript as in (Rf).) When $a, \ldots, b$ is a list of free variables in ϕ, ϕ^* is, by definition, $M(a) \,\&\ldots\&\, M(b) \to \phi_M$. We usually denote this antecedent simply by H.

(15) Requirement (b), mentioned right after the list of axioms, is satisfied.

(16) $M(a) \,\&\, M(b) \to [a \simeq b \leftrightarrow (a \simeq b)_M]$. Assume H, that is $M(a) \,\&\, M(b)$. When $a \simeq b$, obviously $(a \simeq b)_M$, for this is $\forall x M(x) \to (x \in a \leftrightarrow x \in b)$. Note that $(a \simeq b)_M$ is also equivalent to $\forall x(M(x) \,\&\, x \in a \leftrightarrow M(x) \,\&\, x \in b)$. But under H, $M(x) \,\&\, x \in a \leftrightarrow x \in a$ by virtue of (13); similarly for b in place of a. So, $H \to [(a \simeq b)_M \to (a \simeq b)]$, also.

(17) $S_M(a) \leftrightarrow S(a) \,\&\, M(a)$.

Proof: Since $S_M(a)$ is $\exists z M(z) \,\&\, a \in z$, the direction $\to$ follows from (13). So, assume $M(a) \,\&\, S(a)$, and let α be $\{x/x \simeq a\}$. By (Cp), $b \in \alpha \leftrightarrow S(b) \,\&\, b \simeq a \ldots (^*)$. Thus $a \in \alpha$. We use (13) to show that $M(\alpha)$, or $E(\alpha) \,\&\, \forall x(x \in \alpha \to M(x))$. By $(^*)$, $M(a)$, and (12), the second conjunct is obvious. Assume that $b \simeq c \,\&\, b \in \alpha$. Then $S(c)$ by (2); and $c \simeq a$, as $\simeq$ is transitive. So $c \in \alpha$ by $(^*)$. Thus $E(\alpha)$ holds also.

(18) $M(a) \to M(\sigma(a))$. This is obvious from (CF) and (13), if $a \not\simeq \varnothing$; and from (Em) & (12), if $a \simeq \varnothing$.

(19) Meta-Theorem $\mathbf{B}^- \vdash (\mathrm{Ex})^* \,\&\, (\mathrm{CF})^* \,\&\, (\mathrm{Cp})^* \,\&\, (\mathrm{Eq})^* \,\&\, (\mathrm{Em})^*$.

Proof: By (16), we can use $a \simeq b$ and $(a \simeq b)_M$ interchangeably under H. In

(Ex)*, $M(c)$ is assumed, hence $E(c)$ by (11). And so, (10) finishes the task of showing (Ex)*. Since (CF)* is $M(a) \& M(c) \to$ (CF), this obviously follows from (CF) in $\mathbf{B}^-$. (Cp)* is $H \to [c \in \{x/\phi(x)\}_M \leftrightarrow S_M(c) \& \phi_M(c)]$. But $\{x/\phi(x)\}_M$ is, by definition, $\{x/M(x) \& \phi_M(x)\}$. So, $c \in \{x/\phi(x)\}_M$ is equivalent, by (Cp) of $\mathbf{B}^-$, to $S(c) \& M(c) \& \phi_M(c)$. So, use (17) to accomplish the task. (Eq)* follows from (Eq) and (16). Finally, (Em)* follows from (Em), because if $M(a)$ then $M(\sigma(a))$ by (18), whence $M(x) \& x \in a \leftrightarrow x \in a$ and $M(x) \& x \in \sigma(a) \leftrightarrow x \in \sigma(a)$ by (13).

(20) Let $\phi(a)$ be a formula and $\alpha(a)$ a term. Then the following are all provable in $\mathbf{B}^-$:

(I) $H \to M([\alpha(a)]_M)$
(II) $[a \simeq b \to \alpha(a) \simeq \alpha(b)]^*$
(III) $[a \simeq b \to (\phi(a) \leftrightarrow \phi(b))]^*$.

Proof by simultaneous induction on the formation of $\phi(a)$ and $\alpha(a)$: When $\alpha(a)$ is a, (I) and (II) are obvious. When it is $\sigma(a)$, (I) is given by (18), and (II) is (Eq)*. Assume $\alpha(a)$ is $\{x/\psi(x,a)\}$. Then, $[\alpha(a)]_M$ is $\{x/M(x) \& \psi_M(x,a)\}$. Thus $b \in [\alpha(a)]_M \to M(b)$ is obvious. That $E([\alpha(a)]_M)$ follows from the *IH* (= induction hypothesis) (III) on $\psi(x,a)$ and (Cp). So by (13), (I) is shown. (II) can be shown again by the *IH* on $\psi(x,a)$, (Cp), and (12). To show (III), it suffices to consider an atomic formula $\beta(a) \in \gamma(a)$, because connectives can be handled by a routine induction. By the *IH*, $H \& a \simeq b \to [\beta_M(a) \simeq \beta_M(b)] \& [\gamma_M(a) \simeq \gamma_M(b)]$ by (II); and $H \to M(\beta_M(a))$, etc. So, assume H, $a \simeq b$, and $\beta_M(a) \in \gamma_M(a)$. Then $\beta_M(a) \in \gamma_M(b)$. But $\gamma_M(b)$ is extensional, hence $\beta_M(b) \in \gamma_M(b)$. The converse implication can be shown similarly.

(21) Let ϕ be a formula. Then; $\mathbf{B}^- \vdash \phi^*$, if ϕ is a logical axiom, or the consequence of θ (and χ) such that $\mathbf{B}^- \vdash \theta^*$ (and $\mathbf{B}^- \vdash \chi^*$).

Proof: Case ϕ is tautologous. Then ϕ^* is also tautologous.

Case ϕ is $\forall x\psi(x) \to \psi(\alpha)$ where α is a term. Then ϕ^* is $H \to [\forall x(M(x) \to \psi_M(x)) \to \psi_M(\alpha_M)]$. Note that $\forall x(M(x) \to \psi_M(x)) \to (M(\alpha_M) \to \psi_M(\alpha_M))$ is a logical axiom. But $H \to M(\alpha_M)$ by (20), (I). Thus $\mathbf{B}^- \vdash \phi^*$.

Case ϕ is $\forall x\psi(x)$ and is obtained from $\psi(a)$ by the universalization. Let $b, \ldots, c$ be a list of free variables in ϕ (hence a does not appear in it), and let H be $M(b) \& \ldots \& M(c)$. Then $[\psi(a)]_M$ is $M(a) \& H \to \psi_M(a)$, and is provable in $\mathbf{B}^-$ by the *IH*. So, by the predicate calculus, we have $\mathbf{B}^- \vdash H \to \forall x(M(x) \to \psi_M(x))$, i.e., $\mathbf{B}^- \vdash \phi^*$.

Case ϕ is obtained from $\psi \to \phi$ and ψ by modus ponens. To simplify the notation, assume that b is the sole free variable that occurs in both ϕ and ψ, and that a and c occur only in ϕ and ψ, respectively. So, $(\psi \to \phi)^*$ is $M(a) \& M(b) \& M(c) \to (\psi_M \to \phi_M)$, and ψ^* is $M(b) \& M(c) \to \psi_M$. So by the predicate calculus, $M(a) \& M(b) \& M(c) \to \phi_M$. But c does not occur in ϕ_M. So $\mathbf{B}^- \vdash \exists x M(x) \& M(a) \& M(b) \to \phi_M$. But by (12), $\mathbf{B}^- \vdash \exists x M(x)$. Hence, $\mathbf{B}^- \vdash \phi^*$.

(22) We are still to show that $\mathbf{B}^- \vdash$ (Rf)*, that is, $\mathbf{B}^- \vdash H \to [\phi_M \to$

$\exists y M(y) \,\&\, ST_M(y) \,\&\, S_M(y) \,\&\, (\phi^y)_M]$. Assume $H \,\&\, \phi_M$. By (Rf) of $\mathbf{B}^-$, we have a y such that $ST(y) \,\&\, S(y) \,\&\, (\phi_M)^y$. Now let μ be $\{x/x \in y \,\&\, M(x)\}$.

(22.1) $M(\mu)$. To show this, we use (13). That $a \in \mu \to M(a)$ is clear. Assume $a \simeq b$ and $a \in \mu$. So by (Cp), $a \in y \,\&\, M(a)$. As $ST(y)$, and $a \simeq b$ hence $b \subseteq a$, we have $b \in y$ from $a \in y$. Also, by (12), $M(b)$. So $M(\mu)$.

(22.2) $TR(\mu)$. Assume $a \in b \in \mu$. Thus $b \in y \,\&\, M(b)$. As $ST(y)$ hence $TR(y)$, $a \in y$. Also $M(a)$ follows from $a \in b$ and $M(b)$ by (13). Thus $TR(\mu)$.

(22.3) $ST_M(\mu)$. Assume $M(a) \,\&\, M(b) \,\&\, b \in \mu \,\&\, (a \subseteq b \vee a \in b)$. If $a \in b$, then that $a \in \mu$ has already been shown. So assume $a \subseteq b$. But $b \in y$, and $ST(y)$ by assumptions. So $a \in y$. $M(a)$ is given in the assumption. Thus $a \in \mu$, hence $ST_M(\mu)$.

(22.4) $S(\mu)$. This follows from $S(y)$ by (2) and (Cp).

(22.5) $S_M(\mu)$: by (22.1), (22.4), and (17).

(22.6) $(\phi^\mu)_M$. The proof of this requires several steps and, indeed, follows from more general results. So we change the numbering system; (23) through (27) constitute the proof of facts from which (22.6) follows. But first note that, once this is shown, (22) is proved by (22.1), (22.3), (22.5), and (22.6).

(23) $TR(y) \,\&\, a \subseteq y \to \{[(a \subseteq b)^y \leftrightarrow (a \subseteq b)] \,\&\, [E(a) \leftrightarrow E^y(a)] \,\&\, [TR(a) \leftrightarrow TR^y(a)]\}$. This can be shown by routine checking.

(24) $TR(a) \,\&\, TR(b) \to TR(a \cap b)$.

(25) $TR(y) \,\&\, a \subseteq y \to [tc(a)]^y \simeq tc(a)$.

Proof: $b \in [tc(a)]^y$ is equivalent to $\forall z\, z \subseteq y \,\&\, (a \subseteq z)^y \,\&\, TR^y(z) \to b \in z$. By (23), $(a \subseteq z)^y \leftrightarrow a \subseteq z$, and $TR^y(z) \leftrightarrow TR(z)$ as $z \subseteq y$. So the whole formula is equivalent to $\forall z\; z \subseteq y \,\&\, a \subseteq z \,\&\, TR(z) \to b \in z$. Thus, in comparison with $b \in tc(a)$, this formula has an extra condition $z \subseteq y$. So, $tc(a) \subseteq [tc(a)]^y$ follows from $TR(y) \,\&\, a \subseteq y$. To show the converse inclusion, assume that $a \subseteq z \,\&\, TR(z)$. Since $a \subseteq y \,\&\, TR(y)$ is assumed, we have $a \subseteq y \cap z \,\&\, TR(y \cap z)$ by (24). So, $b \in [tc(a)]^y$ implies $b \in y \cap z$, hence $b \in z$. Thus $b \in tc(a)$.

(26) We use now y and μ of (22). So, $ST(y)$ and $TR(\mu)$. Under these assumptions,

$$a \subseteq \mu \,\&\, M(a) \leftrightarrow a \subseteq y \,\&\, M^y(a).$$

Proof: $M^y(a)$ is $E^y(a) \,\&\, \forall x[x \subseteq y \,\&\, (x \in tc(a))^y \to E^y(x)]$. So the RHS is equivalent to $a \subseteq y \,\&\, E(a) \,\&\, \forall x[x \subseteq y \,\&\, x \in tc(a) \to E(x)]$ by (23) and (25). But $a \subseteq y \,\&\, TR(y) \to tc(a) \subseteq y$. So $x \in tc(a) \to x \subseteq y$. Thus, RHS is equivalent to $a \subseteq y \,\&\, M(a)$. Assume $b \in a$. Then RHS implies $b \in y \,\&\, M(b)$ by (13). Thus $b \in \mu$ by the definition of μ. That is, $a \subseteq \mu$. This finishes the proof.

A proof of (22.6) is given by the following general theorem:

(27) Let y and μ be as in (22), and let α and ϕ be a term and a formula, respectively, in which y does not occur. Further, let H be $M(a) \,\&\, \ldots \,\&\, M(b)$

where $a, \ldots, b$ is a list of free variables in α or ϕ. (Hence $M(y)$ is *not* in H.) Then, $\mathbf{B}^- \vdash H \rightarrow$ [a] $M((\alpha^\mu)_M)$, [b] $(\alpha^\mu)_M \simeq (\alpha_M)^y$, and [c] $(\phi^\mu)_M \leftrightarrow (\phi_M)^y$.

Proof by induction: Note that [a] follows from (20), (I), even though y occurs in μ and $M(y)$ is not assumed. For, certainly $H \,\&\, M(m) \rightarrow M((\alpha^m)_M)$ is a case of (20), (I), where m is a new variable. Then, [a] is obtained by substituting μ for m and noting $M(\mu)$ by (22.1).

Case α is a. So, $(\alpha^\mu)_M$ is $(a \cap \mu)_M$ or $\{x/M(x) \,\&\, x \in a \,\&\, x \in \mu\}$ as $\mu_M \simeq \mu$; and $(\alpha_M)^y$ is $a \cap y$. So $b \in a \cap y$ implies $M(b)$ by H and (13). Also, $M(b) \,\&\, b \in y$ implies $b \in \mu$. Thus, [b] is shown by (Cp).

Case α is $\sigma(a)$. A similar proof shows [b] by virtue of (18).

Case α is $\{x/\psi(x)\}$. So, $(\alpha^\mu)_M$ is $\{x/x \in \mu_M \,\&\, M(x) \,\&\, (\psi^\mu)_M\}$. But $\mu_M \simeq \mu$. So by the *IH* on ψ, this class is $\simeq \{x/x \in \mu \,\&\, M(x) \,\&\, (\psi_M)^y\}$. Since $TR(y)$ and $TR(\mu)$, this class is $\simeq \{x/x \in y \,\&\, M^y(x) \,\&\, (\psi_M)^y\}$ by (26) and the definition of μ. This last is $(\alpha_M)^y$. Thus [b] holds.

To show [c], we check two cases when ϕ is atomic and is $\forall x\, \psi(x)$.

Case ϕ is $\beta \in \gamma$. A proof can be given that is similar to that of (20),(III) by using the IH on β and γ, and [a] and [b].

Case ϕ is $\forall x\, \psi(x)$. Then, $(\phi^\mu)_M$ is $\forall x\, \{(x \subseteq \mu)_M \,\&\, M(x) \rightarrow [(\psi(x))^\mu]_M\}$. But $M(x) \,\&\, (x \subseteq \mu)_M \leftrightarrow M(x) \,\&\, x \subseteq \mu$. So, this is equivalent to $\forall x\, \{M^y(x) \,\&\, x \subseteq y \rightarrow [(\psi(x))_M]^y\}$ by the *IH* and (26). This last is $(\phi_M)^y$.

In (22), $(\phi_M)^y$ was given. So by (27)[c], $(\phi^\mu)_M$ holds, that is, (22.6) is proved. So by (21), (19), and (22), we have: $\mathbf{B}^- \vdash \phi^*$. And so, our whole task is finished.

Comments

(1) Our "top down" approach does not work for Gödel-Bernays theory, because here the comprehension scheme is weaker, and the class $\{x/\phi(x)\}$ exists only if all quantifiers in $\phi(x)$ are bound to S. Thus the definition of the transitive closure $tc(a)$, (5), is not legitimate here. One can redefine $tc(a)$ as $\{x/\forall z\, S(z) \,\&\, a \subseteq z \,\&\, TR(z) \rightarrow x \in z\}$, for instance. But then, when a is a proper class, $tc(a)$ is V. Then the crucial formula (9) does not hold for $b = Od$. For, if $y \in Od$ then $tc(y) = y$. So the LHS is $a \in V$, while the RHS is $a \in Od$. Thus Gandy's "bottom up" approach is really necessary here.

(2) In Kelley-Morse theory, our approach works. The axioms consist of (Cp) as above and some axioms of set existence like replacement (cf. [2], for instance). (CP)* can be shown as above; and (replacement)* etc. can be shown as in [3], for what Gandy needed was formula (13) (which is based on (9)) and not the way $tc(a)$ was defined.

(3) What happens in Quine's theories **NF** and **ML**? (Cf., e.g., [5].) Indeed, this question was the motivation of the present work. If we succeed in "eliminating" the extensionality in **NF**, this theory is shown to be consistent in connection with [4]. Unfortunately, $TR(a)$ is not a stratified formula. Hence ϕ_M is not

stratified either, as M is ultimately based on TR. So, (Cp)* does not follow from (Cp) in **NF**. In **ML**, the "set-hood" axioms are the stumbling block for the same reason.

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66 Pine Street
Princeton, New Jersey 08542