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Generalized Quantifiers and the Square of Opposition

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1 Introduction Work by Rescher and Gallagher [11], and more recently by Geach [5], Peterson [9], Thompson [12], and Peterson and Carnes [10], provides strong evidence that syllogistic logic and the method of Venn diagrams can be extended to accommodate sentences of such forms as

(1.1) Almost all S are P
Most S are P
Many S are P
Few S are P.

This suggests that we should be able to modify the first-order predicate calculus to provide for renderings of such sentences, and of arguments involving such sentences. Recently Barwise and Cooper [2] have given the formal syntax and semantics for a family of such modifications. In this family of languages, as in the other works cited above, 'almost all', 'most', 'many', and 'few' are treated as quantifiers analogous in certain important respects to 'all' and 'some'. But Barwise and Cooper, unlike the other authors cited, do not treat these as mere ad hoc additions to our stock of quantifiers. Instead they treat them as merely a few from among an indefinitely large class of quantifiers, including 'the' (in its use in definite descriptions), 'both', 'at least seven', 'infinitely many', 'all but three', 'with at most three exceptions', and a host of others.

The generality of the treatment given by Barwise and Cooper suggests that if we are to continue to explore the logical properties of generalized quantifiers as viewed from a perspective like that of traditional logic then we should no longer be content to do so piecemeal. There are, in fact, strong reasons for studying generalized quantifiers from a traditional perspective, for (as we shall

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see) the syntax and semantics that generalized quantifiers seem to demand resist introduction of the kind of instantiation and generalization rules that we may take as characteristic of (natural deduction treatments of) the standard modern predicate calculus, but do lend themselves to rules of immediate inference and of syllogistic inference like those of traditional logic. This is not to say that generalized quantifiers should be studied exclusively from the traditional perspective, of course, but only that such study may well yield some insights, particularly about the relation of logic to ordinary language, that we would not readily discover by other means. In this paper I initiate a general study of quantifiers from a traditional perspective by studying rules of immediate inference, and associated squares of opposition, appropriate to various classes of generalized quantifiers.

2 Syntax Consider the following sentence:

(2.1) All men are greedy.

For purposes of rendering this sentence in the notation of standard logic, we are accustomed to paraphrase 2.1 as:

(2.2) Take any object x: if x is a man then x is greedy.

We then render this as:

(2.3) $(\forall x)[Mx \supset Gx]$.

Now let us try to parallel this analysis with an analysis of:

(2.4) Most men are greedy.

To effect a similar analysis, we would have to find some way in which to complete

(2.5) Take most objects x: . . .

so as to provide a reasonable paraphrase of 2.4; or, at the very least, we would have to find or devise some connective to put in place of '?' in

(2.6) $(\mu x)[Mx\ ?\ Gx]$.

But there seems to be no satisfactory way of completing 2.5, and none of the sixteen possible truth-functional connectives gives satisfactory results when put in place of '?'.

One way of describing the difficulty is to say that, in claiming that most men are greedy, we are not talking about most objects, but only about most men. This calls to mind the fact that we do have an alternative way of rendering 2.1, using relativized or restricted quantification:

(2.7) $(\forall x{:}Mx)[Gx]$.

If we adopt this, rather than 2.3, as our model, we can render 2.4 as

(2.8) $(\mu x{:}Mx)[Gx]$

and sidestep the need to find some connective to replace the '?' in 2.6. We know, of course, that in the case of the universal and existential quantifiers the use of restricted quantification is eliminable in favor of unrestricted quantification, and it is usual to think of unrestricted quantification as more fundamental. However, we can also eliminate unrestricted universal and existential quantification in favor of their restricted versions, and thus it is open to us to think of relativized quantification as more fundamental. It has been argued (for example by McCawley [7], pp. 118–123) that in the syntax of natural language the restricted form is fundamental. If that is so, then perhaps the availability of an equivalent unrestricted version may be a misleadingly special feature of 'all' and 'some'. Perhaps for most quantifiers, 'most' included, only a restricted form is available. It is fundamental both to the syntax and to the semantics of the approach taken by Barwise and Cooper to assume that all quantification is fundamentally restricted quantification.

It will be useful to introduce some terminology for discussing the syntax of such expressions as 2.8. There are several component expressions for which we may wish to have designations:

(2.9a) μ
(2.9b) μx
(2.9c) $(\mu x{:}Mx)$
(2.9d) Mx
(2.9e) Gx.

Of these, each of 2.9a–c might lay claim to the title 'quantifier', but it would be needlessly confusing to award the title to all three. Each choice of one of the three as rightful owner of the title has disadvantages, but on balance it seems to me best to reserve the term 'quantifier' for 2.9a. (In this judgment I differ from Barwise and Cooper.) I shall use the following terminology:

(2.10a) quantifier
(2.10b) prefix (of the quantifier phrase)
(2.10c) quantifier phrase
(2.10d) sortal phrase
(2.10e) matrix (of the formula).

This terminology has the advantage of preserving the pattern established by existing use of the terms 'noun', 'noun phrase', 'preposition', 'prepositional phrase', and the like. In somewhat less formal usage, I shall sometimes refer to Mx as the *subject*, and to Gx as the *predicate* of 2.8, and refer to the subject and predicate as the *terms* of a formula.

As the formal language to be studied, I take a language *GQ* for generalized quantification that contains denumerably many quantifier constants, denumerably many one-place predicate constants, denumerably many n-place relation constants (for each n greater than one), denumerably many individual constants, denumerably many individual variables, the usual sentential connectives, three operator constants (suffixed superscript prime and asterisk, and prefixed superscript minus sign), and five punctuation marks (left and right parentheses, left and right brackets, and the colon). We give a recursive definition of quantifiers:

(2.11) If Q is any quantifier constant, then Q is a quantifier;
if Q is any quantifier, then (Q'), (Q^*), and $({}^{-}Q)$ are quantifiers.

Normally the parentheses can be omitted from the designation of complex quantifiers, as we shall see. In the recursive definition of formulas, only the last clause is novel, compared to standard logic:

(2.12) If P is any one-place predicate constant, and t is any individual constant or individual variable, then Pt is a formula;
if R is any n-place relation constant and $t_1, \ldots, t_n$ are any n individual constants or variables (not necessarily distinct), then $Rt_1 \ldots t_n$ is a formula;
if A and B are any formulas, then $\sim A$, $(A \& B)$, $(A \vee B)$, $(A \supset B)$, and $(A \equiv B)$ are formulas;
if Q is any quantifier, x is any individual variable, and A and B are any formulas, then $(Qx{:}A)[B]$ is a formula.

I will wait until after the presentation of the semantics for *GQ* to give rules for the system.

3 Semantics If we wish to elaborate on a sentence such as

(3.1) Most senators are males

one way in which to do so is this:

(3.2) The senators that are males constitute most of the senators.

This kind of paraphrase is the key to the formal semantics I shall offer for quantifiers.[1] The idea it suggests, recast in a way that foreshadows the set-theoretic language we are accustomed to use for truth-conditions, is this: there is no one senator, or even any one group of senators, that is meant when we speak of most senators (we don't mean the alphabetically first fifty-one of them, for example); rather, there are various collections of senators any one of which would count as constituting most senators, and it happens that the group of male senators is one of them. In our models for the formal system *GQ*, we must assign values not only to the individual, predicate, and relational constants of the language, but also to the quantifier constants. The value assigned to a quantifier whose intended interpretation is *most*, for example, should do the job of telling us which groups of senators do, and which do not, count as most of them. Similarly, it must tell us which subsets of the set of prime numbers do, and which do not, count as containing most prime numbers. In general, for each set that might serve as the extension of a predicate the value assigned to the quantifier should enable us to tell which of that set's subsets would and which would not count as containing most of the elements of that set. The sort of thing that can do this job is a function which, given any subset of the domain as its argument, returns as its value an appropriate collection of the subsets of that argument. I will call such a function a *quantificational function*, and we can define quantificational functions formally as follows:

(3.3) F is a quantificational function for a set D iff $F: \mathcal{P}D \to \mathcal{P}\mathcal{P}D$ and for each $E \subseteq D$, $F(E) \subseteq \mathcal{P}E$.

Given any quantificational function F on D, we can define three other related quantificational functions F', F^*, and ^-F on D, as follows:

(3.4) for each $E \subseteq D$

$$F'(E) = \{E_0 \subseteq E: E - E_0 \in F(E)\}$$
$$F^*(E) = \{E_0 \subseteq E: E - E_0 \notin F(E)\}$$
$$^-F(E) = \{E_0 \subseteq E: E_0 \notin F(E)\}.$$

We can now define a model for GQ to be an ordered pair $\langle D, V\rangle$ consisting of a nonempty domain D and a valuation V satisfying the following conditions:

(3.5a) if t is any individual constant or individual variable, $V(t) \in D$
(3.5b) if P is any one-place predicate constant, $V(P) \subseteq D$
(3.5c) if R is any n-place relation constant, $V(R) \subseteq D^n$
(3.5d) if Q is any quantifier constant, $V(Q)$ is a quantificational function on D
(3.5e) if Q is any quantifier, with $F = V(Q)$, then $V(Q') = F'$, $V(Q^*) = F^*$, and $V(^-Q) = {}^-F$.

We require one auxiliary notion before we can state the truth-conditions for formulas in GQ.

(3.6) If $M = \langle D, V\rangle$ is any model, with $e \in D$ and x an individual variable, then by $M[e/x]$ we mean the model $\langle D, V[e/x]\rangle$ whose domain is the same and whose valuation agrees in value with V for every argument except (possibly) x, for which $V[e/x](x) = e$.

If A is any formula of GQ and M is any model for GQ, we abbreviate the claim that A is true in the model M by writing that $M \models A$, and define this by the following truth-conditions:

(3.7a) if A is Pt, where P is any one-place predicate constant and t is any individual constant or individual variable, $M \models A$ iff $V(t) \in V(P)$
(3.7b) if A is $Rt_1 \ldots t_n$, where R is any n-place relation constant and $t_1, \ldots, t_n$ are any n individual constants or individual variables, $M \models A$ iff $\langle V(t_1), \ldots, V(t_n)\rangle \in V(R)$
(3.7c) if A is $\sim B$ for some formula B, $M \models A$ iff it is false that $M \models B$
(3.7d) if A is $(B \& C)$ for some formulas B and C, $M \models A$ iff $M \models B$ and $M \models C$
(3.7e) if A is $(B \vee C)$ for some formulas B and C, $M \models A$ iff $M \models B$ or $M \models C$
(3.7f) if A is $(B \supset C)$ for some formulas B and C, $M \models A$ iff either it is false that $M \models B$ or it is true that $M \models C$
(3.7g) if A is $(B \equiv C)$ for some formulas B and C, $M \models A$ iff $M \models (B \supset C)$ and $M \models (C \supset B)$
(3.7h) if A is $(Qx{:}B)[C]$ for some quantifier Q, some variable x, and some formulas B and C, $M \models A$ iff $D(B) \cap D(C) \in F(D(B))$, where $D(B)$ is $\{e \in D: M[e/x] \models B\}$ and $D(C)$ is $\{e \in D: M[e/x] \models C\}$ and F is $V(Q)$.

Perhaps it would be helpful to look at an example or two, in order to see how the semantics works. Suppose S and P are one-place predicate constants and $M = \langle D, V\rangle$ is a model in which V assigns the set of all senators in D as the value of S, and the set of all procrastinators in D as the value of P. Then Sx will be true in M iff the value assigned to x is a senator. $D(Sx)$ will be just the set of all elements of D such that if they had been the value assigned to x, Sx would have been true in M. In short, $D(Sx)$ will be just $V(S)$, i.e., the set of senators. Similarly $D(Px)$ will be the set of values of x for which Px would be true in M, i.e., the procrastinators, i.e., $V(P)$. $D(Sx) \cap D(Px)$ is the set of senators who are procrastinators. Suppose Q is a quantifier, with $V(Q) = F$, and suppose F is such that, for each $E \subseteq D$, $F(E) = \{E\}$. Then $F(D(Sx))$, i.e., $F(V(S))$, will be just $\{V(S)\}$. So $(Qx{:}Sx)[Px]$ will be true in M iff the set of senators who are procrastinators is the set of all senators, namely $V(S)$, i.e., iff all senators are procrastinators. $F'(V(S))$ will turn out to be just $\{\varnothing\}$, so $(Q'x{:}Sx)[Px]$ will be true in M iff no senators are procrastinators. $F^*(V(S))$ is $\wp V(S) - \{\varnothing\}$, so $(Q^*x{:}Sx)[Px]$ will be true in M iff some senators are procrastinators. ${}^-F(V(S))$ is $\wp V(S) - \{V(S)\}$, and $({}^-Qx{:}Sx)[Px]$ will be true in M iff some senators are not procrastinators.

The quantificational function F just considered corresponds to the universal quantifier. If a quantificational function F_1 is such that, for each $E \subseteq D$, $F_1(E) = \{E_0 \subseteq E{:}|E_0| > |E - E_0|\}$, and if $V(Q_1) = F_1$, then $(Q_1x{:}Sx)[Px]$ will be true in M iff the number of senators who are procrastinators exceeds the number who are not, i.e., iff most senators are procrastinators. Suppose F_2 is such that, for each $E \subseteq D$, $F_2(E) = \{E\}$ if $|E| = 1$, and $F_2(E) = \varnothing$ otherwise; then if $V(Q_2) = F_2$ we will find that $(Q_2x{:}Sx)[Px]$ is true in M iff there is exactly one senator and he or she is a procrastinator, i.e., iff *the* senator is a procrastinator.

By the definitions of F', F^*, and ${}^-F$ (for given quantificational function F) that were given in 3.4, we find that in general $(F')' = (F^*)^* = {}^-({}^-F) = F$. Moreover: $(F')^* = (F^*)' = {}^-F$; ${}^-(F') = ({}^-F)' = F^*$; and ${}^-(F^*) = ({}^-F)^* = F'$. Corresponding results hold for quantifiers, by 3.5e. For example, since $(F')^* = {}^-F$, we find that for every M, S, and P, $M \vDash (((Q')^*)x{:}S)[P]$ iff $M \vDash (({}^-Q)x{:}S)[P]$. As a consequence we can dispense with the parentheses previously required by 2.11. Corresponding syntactic results can be shown to follow from the rules presented in the next section.

In view of the close semantic and syntactic connections between a given quantifier Q and the associated quantifiers Q', Q^*, and ${}^-Q$, I call them *cognate* quantifiers, and describe each as a cognate of (itself and) the others. I refer to Q' as the *obverse* of Q, to Q^* as the *dual* of Q, and to ${}^-Q$ as the *contradictory* of Q.

4 General quantifier rules One of the interesting problems in the theory of generalized quantifiers is that of formulating general rules of inference applicable to all quantifiers. At first glance it might appear that there could be none. How could any rule, even a wildly gerrymandered one, be equally applicable to 'all', 'some', 'most', and 'the'? But there are a few, impressions to the contrary notwithstanding. One such rule (or pair of rules, depending on how you choose to count) arises from the fact that in clause 3.7h of the truth-conditions the truth value of the quantified formula is made to depend only on the extensions of the terms of the formula. As a consequence, for any quantifier Q, any variable x, and any formulas A, A_1, A_2, B, B_1, and B_2 we have the following rule(s):

(4.1a) $\dfrac{\vdash A_1 \equiv A_2}{\therefore (Qx{:}A_1)[B] \equiv (Qx{:}A_2)[B]}$ Rule E (extensionality)

(4.1b) $\dfrac{\vdash B_1 \equiv B_2}{\therefore (Qx{:}A)[B_1] \equiv (Qx{:}A)[B_2].}$

That is, provably equivalent sortal phrases may be substituted for one another, and so may provably equivalent matrices.

Another rule arises from the fact that the extension of the matrix affects the truth of the quantified formula only to the extent that it intersects the extension of the sortal phrase. Thus the set-theoretic equality $D(B) \cap D(C) = D(B) \cap (D(B) \cap D(C))$ gives rise to the following rule, for any quantifier Q, any variable x, and any formulas B and C:

(4.2) $(Qx{:}B)[C] :: (Qx{:}B)[(B \,\&\, C)].$ Rule A (absorption)

Here the quadruple dot (according to a notation first introduced by Kahane [6]) indicates that the expressions on either side may be substituted for one another *salve veritate* in any (extensional) context. I call this the rule of absorption by analogy with the propositional calculus rule so designated in Copi [4]. This rule corresponds to the fact that we can paraphrase the claim that most men are greedy as the claim that most men are greedy men, and similarly with 'all', 'few', 'the', etc.

A third rule arises from the fact that, barring collision of bound variables, one bound variable may be substituted uniformly for another in any context. Thus, for any quantifier Q, any variables x and y, and any formulas Ax and Bx, if Ax and Bx contain no occurrences of y, and if Ay and By are the results of substituting occurrences of y for all free occurrences of x in Ax and Bx respectively, then:

(4.3) $(Qx{:}Ax)[Bx] :: (Qy{:}Ay)[By].$ Rule B (bound variables)

Three more rules, which could have been recast as definitions, correspond to the three provisions of clause 3.5e in the definition of a model. For any quantifier Q, any variable x, and any formulas A and B:

(4.4a) $(Q'x{:}A)[B] :: (Qx{:}A)[\sim B]$ Rule O (obversion)
(4.4b) $(Q'x{:}A)[\sim B] :: (Qx{:}A)[B]$
(4.4c) $({}^{-}Qx{:}A)[B] :: (Q^*x{:}A)[\sim B]$
(4.4d) $({}^{-}Qx{:}A)[\sim B] :: (Q^*x{:}A)[B]$
(4.5a) $(Q^*x{:}A)[B] :: \sim(Qx{:}A)[\sim B]$ Rule D (duality)
(4.5b) $(Q^*x{:}A)[\sim B] :: \sim(Qx{:}A)[B]$
(4.5c) $({}^{-}Qx{:}A)[B] :: \sim(Q'x{:}A)[\sim B]$
(4.5d) $({}^{-}Qx{:}A)[\sim B] :: \sim(Q'x{:}A)[B]$
(4.6a) $({}^{-}Qx{:}A)[B] :: \sim(Qx{:}A)[B]$ Rule C (contradiction)
(4.6b) $(Q^*x{:}A)[B] :: \sim(Q'x{:}A)[B].$

It would of course suffice to take one version of each of these rules; the remaining versions are merely useful equivalents.

Other rules can be derived from the rules given so far. For example, using Rules E and A, we can easily show that:

(4.7) $(Qx{:}A)[B] :: (Qx{:}A)[A \equiv B]$ since $\vdash (A \;\&\; B) \equiv (A \;\&\; (A \equiv B))$

(4.8) $(Qx{:}{\sim}A)[B] :: (Qx{:}{\sim}A)[{\sim}(A \equiv B)]$ since $\vdash ({\sim}A \equiv B) \equiv {\sim}(A \equiv B)$

(4.9) $(Qx{:}A)[B] :: (Qx{:}A)[A \supset B]$ since $\vdash (A \;\&\; B) \equiv (A \;\&\; (A \supset B))$

(4.10) $(Qx{:}A)[A] :: (Qx{:}A)[A \vee B]$ since $\vdash A \equiv (A \;\&\; (A \vee B))$

(4.11) $(Qx{:}A)[A] :: (Qx{:}A)[A \vee {\sim}A]$ (corollary to 4.10)

(4.12) $(Qx{:}A)[A] :: (Qx{:}A)[B \vee {\sim}B]$ (corollary to 4.11)

(4.13) $(Qx{:}A \;\&\; {\sim}A)[B] :: (Qx{:}A \;\&\; {\sim}A)[C]$ since $\vdash ((A \;\&\; {\sim}A) \;\&\; B) \equiv ((A \;\&\; {\sim}A) \;\&\; C)$

(4.14)
$$\frac{\begin{array}{l}\vdash (A \supset B)\\ (Qx{:}A)[B]\end{array}}{\therefore (Qx{:}A)[A]}$$
since if $\vdash (A \supset B)$, then $\vdash (A \;\&\; B) \equiv A$

It is an open question[2] whether there are any rules independent of 4.1–4.6 and sound under the semantics for *GQ*. The semantics for *GQ* could be augmented by special restrictions on the quantificational functions to be associated with particular quantifier constants. If that were done some of the quantifier constants could be treated as logical constants, and would have special rules associated with them. This would be a reasonable treatment for 'all', 'the', 'both', 'the *n*', 'at least *n*', 'at most *n*', 'exactly *n*', and even 'most', to name only a few. But for many quantifier expressions in natural language, such as 'many', 'several', and 'almost all', this would be an unreasonable treatment. We should no more expect to be able to give precise semantics for 'several' than we expect to give semantics for 'red' that will distinguish its logical role from that of all other color words. In *GQ* all quantifier constants are treated as nonlogical constants, and none have special semantic restrictions or special rules of inference associated with them. This enables us to use *GQ* to study the general properties of quantifiers.

5 ***Relations among categorical schemata*** Given any quantifier Q, we may distinguish thirty-two cognate categorical schemata whose logical relationships to one another are of interest, especially in connection with the study of squares of opposition. These are exhibited in Table 1. For the sake of brevity, the variable of quantification, and the colon, are omitted from the specification of the schemata. The variable of quantification can be assumed to be the same throughout.

Table 1: Cognate

$(QX)[Y]$	$(QX)[{\sim}Y]$	$(Q{\sim}X)[Y]$	$(Q{\sim}X)[{\sim}Y]$
$(Q'X)[Y]$	$(Q'X)[{\sim}Y]$	$(Q'{\sim}X)[Y]$	$(Q'{\sim}X)[{\sim}Y]$
$(Q^*X)[Y]$	$(Q^*X)[{\sim}Y]$	$(Q^*{\sim}X)[Y]$	$(Q^*{\sim}X)[{\sim}Y]$
$({}^-QX)[Y]$	$({}^-QX)[{\sim}Y]$	$({}^-Q{\sim}X)[Y]$	$({}^-Q{\sim}X)[{\sim}Y]$

Each of the forms in the second row is equivalent to one of the forms in the first row by obversion. For example, $(Q'X)[Y]$ is equivalent to $(QX)[\sim Y]$. Each of the forms in the third row is equivalent to the negation of the form immediately above it in the second row, and therefore to the negation of one of the forms in the first row. Each of the forms in the bottom row is equivalent to the negation of the form directly above it in the top row. Thus all further logical relations among the thirty-two cognate categorical schemata may be reduced to relations among the eight schemata in the top row and, indeed, to relations between $(QX)[Y]$ and its cognate schemata (including itself) in the top row.

If A and B are any two of the cognate categorical schemata from Table 1 (not necessarily distinct, and not necessarily from the top row) then it is of interest to examine, for any given quantifier Q, which of fifteen possible relations may obtain between them. The fifteen relations of interest are given in Table 2.

These relations among schemata are not independent of one another, as is indicated in Figure 1 in which arrows between relations indicate that having the relation at the base of the arrow entails having the relation at its tip.

Some of the nominally possible relations between cognate categorical schemata A and B are in fact not possible for certain choices of A and B. For example, by Rule C it is impossible to have $(QX)[Y] \equiv (^{-}QX)[Y]$, or to have $(Q'X)[Y] \equiv (Q^*X)[Y]$. But in addition to cases of this sort, we find it is impossible to have $(QX)[Y] \not\mapsto (QX)[\sim Y]$. For suppose otherwise: then by appropriate instantiation in the definition we can get both

$$(QS)[P] \;\&\; \sim(QS)[\sim P]$$

and also

$$(QS)[\sim P] \;\&\; \sim(QS)[\sim\sim P].$$

But this is a contradiction. By similar reasoning we find that for no two schemata A and B chosen from the same row in Table 1 do we ever have either $A \not\mapsto B$ or $A \not\mapsfrom B$.

Moreover, some of the nominally distinct relations between cognate schemata turn out to be equivalent for certain choices of schemata A and B. Thus, for example:

$$\begin{aligned} & \forall XY[(QX)[Y] \supset (QX)[\sim Y]] \\ \text{iff}\quad & \forall XY[(QX)[\sim Y] \supset (QX)[\sim\sim Y]] \\ \text{iff}\quad & \forall XY[(QX)[\sim Y] \supset (QX)[Y]] \qquad \text{(by Rule E)} \\ \text{iff}\quad & \forall XY[(QX)[Y] \equiv (QX)[\sim Y]]. \end{aligned}$$

Categorical Schemata

$(QY)[X]$	$(QY)[\sim X]$	$(Q\sim Y)[X]$	$(Q\sim Y)[\sim X]$
$(Q'Y)[X]$	$(Q'Y)[\sim X]$	$(Q'\sim Y)[X]$	$(Q'\sim Y)[\sim X]$
$(Q^*Y)[X]$	$(Q^*Y)[\sim X]$	$(Q^*\sim Y)[X]$	$(Q^*\sim Y)[\sim X]$
$(^{-}QY)[X]$	$(^{-}QY)[\sim X]$	$(^{-}Q\sim Y)[X]$	$(^{-}Q\sim Y)[\sim X]$

Table 2. Possible Logical Relations Between Cognate Categorical Schemata

Symbol	Name	Definition	Possible Combinations A: t B: t	A: t B: f	A: f B: t	A: f B: f
$A\,\mathbf{L}\,B$	loose relationship	$\forall XY[A \vee {\sim}A]$	x	x	x	x
$A\,\mathbf{S}\,B$	subcontrariety	$\forall XY[A \vee B]$	x	x	x	
$A \leftarrow B$	superalternation	$\forall XY[B \supset A]$	x	x		x
$A\,\mathbf{T}_1\,B$	pre-truth	$\forall XY[A]$	x	x		
$A \rightarrow B$	subalternation	$\forall XY[A \supset B]$	x		x	x
$A\,\mathbf{T}_2\,B$	post-truth	$\forall XY[B]$	x		x	
$A \leftrightarrow B$	equivalence	$\forall XY[A \equiv B]$	x			x
$A\,\mathbf{T}\,B$	co-truth	$\forall XY[A \,\&\, B]$	x			
$A\,\mathbf{C}\,B$	contrariety	$\forall XY[{\sim}A \vee {\sim}B]$		x	x	x
$A - B$	contradictoriness	$\forall XY[A \equiv {\sim}B]$		x	x	
$A\,\mathbf{F}_2\,B$	post-falsity	$\forall XY[{\sim}B]$		x		x
$A \nrightarrow B$	anti-subalternation	$\forall XY[A \,\&\, {\sim}B]$		x		
$A\,\mathbf{F}_1\,B$	pre-falsity	$\forall XY[{\sim}A]$			x	x
$A \nleftarrow B$	anti-superalternation	$\forall XY[{\sim}A \,\&\, B]$			x	
$A\,\mathbf{F}\,B$	co-falsity	$\forall XY[{\sim}A \,\&\, {\sim}B]$				x

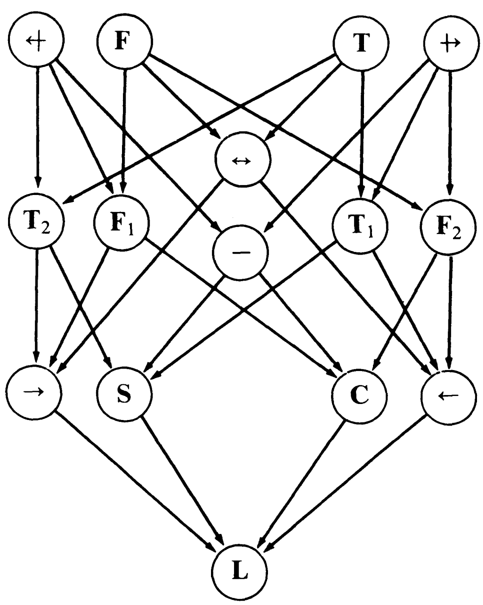

Figure 1. Connections among relations.

It follows that

$$\begin{array}{ll} & (QX)[Y] \rightarrow (QX)[\sim Y] \\ \text{iff} & (QX)[Y] \leftarrow (QX)[\sim Y] \\ \text{iff} & (QX)[Y] \leftrightarrow (QX)[\sim Y]. \end{array}$$

By similar argument it is easily shown that, for any schemata A and B from the same row of Table 1, $A \rightarrow B$ iff $A \leftarrow B$ iff $A \leftrightarrow B$; and $A\,\mathbf{T}_1 B$ iff $A\,\mathbf{T}_2 B$ iff $A\,\mathbf{T}\,B$; and $A\,\mathbf{F}_1 B$ iff $A\,\mathbf{F}_2 B$ iff $A\,\mathbf{F}\,B$.

Table 3 shows which relations are possible between cognate schemata with the same quantifier, i.e., between schemata from the same row of Table 1. For this purpose it suffices to show which relations are possible between the schema $(QX)[Y]$ and its cognates from the top row of Table 1. An entry in the table other than '0' or '$\Uparrow$' designates a particular quantifier (the semantics for which is given following the table) such that if Q is interpreted as that quantifier the pair of schemata at the head of the entry's column will bear the relation corresponding to the entry's row in Table 3, and no stronger relation. The relations are listed in monotonically decreasing order of strength, as shown in Figure 1. A '0' indicates that the pair of schemata in question cannot bear the given relation to one another, and '$\Uparrow$' indicates that the pair of schemata in question can only bear the given relation when they bear some stronger relation given above. Rows (for $\nrightarrow$ and $\nleftarrow$) which would only contain '0's, and rows (for $\rightarrow$, $\leftarrow$, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{F}_1$, and $\mathbf{F}_2$) which would only contain '$\Uparrow$'s have been omitted. For the sake of brevity, punctuation has been omitted from the representations of the schemata at the head of columns. (The '0's in columns 2 and 6–8, and only those, depend on the use of Rule A.)

The various quantifiers designated in Table 3 as illustrations of the possibility of the relations exhibited can be defined semantically by the conditions that, in any model $M = \langle D, V\rangle$:

$$V(Q_F) = \{\langle E, \varnothing\rangle : E \subseteq D\}$$
$$V(Q_T) = \{\langle E, \mathcal{P}E\rangle : E \subseteq D\}$$

Table 3. Distinct Possible Logical Relations Between Cognate Categorical Schemata with the Same Quantifier

	1	2	3	4	5	6	7	8
A	QXY	QXY	QXY	QXY	QXY	QXY	QXY	QXY
B	QXY	$QX\sim Y$	$Q\sim XY$	$Q\sim X\sim Y$	QYX	$QY\sim X$	$Q\sim YX$	$Q\sim Y\sim X$
$A\,\mathbf{F}\,B$	Q_F	Q_F	Q_F	Q_F	Q_F	Q_F	Q_F	Q_F
$A\,\mathbf{T}\,B$	Q_T	Q_T	Q_T	Q_T	Q_T	Q_T	Q_T	Q_T
$A \leftrightarrow B$	$Q_\forall$	Q_α	Q_β	Q_β	$Q_\exists$	Q_γ	Q_γ	$Q_\forall$
$A - B$	0	0	Q_a	Q_a	0	0	0	0
$A\,\mathbf{S}\,B$	$\|$	Q_ϵ^*	Q_α	Q_α	Q_δ	Q_δ	Q_δ	Q_δ'
$A\,\mathbf{C}\,B$	$\|$	Q_ϵ	Q_α^*	Q_α^*	${}^{-}Q_\delta$	Q_δ^*	Q_δ^*	Q_δ^*
$A\,\mathbf{L}\,B$	$\|$	$Q_\forall$	$Q_\forall$	$Q_\forall$	$Q_\forall$	$Q_\forall$	$Q_\forall$	$Q_\exists$

$$
\begin{aligned}
V(Q_\forall) &= \{\langle E,\{E\}\rangle: E \subseteq D\}\\
V(Q_\exists) &= \{\langle E,\wp E - \{\varnothing\}\rangle: E \subseteq D\}\\
V(Q_\alpha) &= \{\langle E,\wp E\rangle: E \subset D\} \cup \{\langle D,\varnothing\rangle\}\\
V(Q_\beta) &= \{\langle E,\varnothing\rangle: \varnothing \subset E \subset D\} \cup \{\langle \varnothing,\wp\varnothing\rangle, \langle D,\wp D\rangle\}\\
V(Q_\gamma) &= \{\langle E,\wp E\rangle: E \subseteq D \,\&\, |D| > 2\} \cup \{\langle E,\varnothing\rangle: E \subseteq D \,\&\, |D| \leq 2\}\\
V(Q_\delta) &= \{\langle E,\wp E\rangle: E \subset D\} \cup \{\langle D,\{D\}\rangle\}\\
V(Q_\epsilon) &= \{\langle E,\{E\}\rangle: E \subseteq D \,\&\, |E| = 1\} \cup \{\langle E,\varnothing\rangle: E \subseteq D \,\&\, |E| \neq 1\}\\
V(Q_a) &= \{\langle E,\wp E\rangle: V(a) \in E \subseteq D\} \cup \{\langle E,\varnothing\rangle: V(a) \notin E \subseteq D\},
\end{aligned}
$$

where a is some individual constant of the language.

It is easily verified that each of these is, as defined, a quantifier. In particular, $Q_\forall$ and $Q_\exists$ are the ordinary universal and existential quantifiers and Q_ϵ is the quantifier corresponding to Russellian definite descriptions. Q_F and Q_T are the degenerate quantifiers that produce only false and true formulas, respectively. The remaining quantifiers are more exotic and bear only uninterestingly complex relations to expressions common in ordinary discourse. It is also simple to verify that each of these quantifiers induces relations among schemata as indicated in the table, and induces no stronger relation between the same schemata. For example, as is indicated in column 5, row 5, $(Q_\delta X)[Y]$ and $(Q_\delta Y)[X]$ are subcontraries, because: it is possible for $(Q_\delta X)[Y]$ to be false; but that can occur only when the extension of X is the whole of the domain D but the extension of Y is not, and in such circumstances $(Q_\delta Y)[X]$ is true; moreover it is possible for both to be true, e.g., when neither X nor Y has the whole domain D as its extension. By Rule C, $(Q_\delta X)[Y] \equiv {\sim}({}^{-}Q_\delta X)[Y]$ and $(Q_\delta Y)[X] \equiv {\sim}({}^{-}Q_\delta Y)[X]$, so from the fact that $(Q_\delta X)[Y]$ and $(Q_\delta Y)[X]$ are subcontraries we can immediately infer that $({}^{-}Q_\delta X)[Y]$ and $({}^{-}Q_\delta Y)[X]$ are contraries. The proofs for other quantifiers are similar.

To show the impossibility of those relations indicated as such by a '0' in Table 3, it is convenient to let F (as distinguished from **F**, which designates a relation between categorical schemata) be any logically false formula, say $(Px \,\&\, {\sim}Px)$, and then let T be ${\sim}F$. To show that for no quantifier Q can $(QX)[Y]$ and $(QX)[{\sim}Y]$ be contradictories, assume otherwise. Then for some Q we must have $\forall XY[(QX)[Y] \equiv {\sim}(QX)[{\sim}Y]]$. In particular, then, $(QF)[T] \equiv {\sim}(QF)[{\sim}T]$. But by Rule A $(QF)[T] \equiv (QF)[F \,\&\, T]$, and by Rule E $(QF)[F \,\&\, T] \equiv (QF)[F]$. On the other hand ${\sim}(QF)[{\sim}T] \equiv {\sim}(QF)[F]$. So $(QF)[F] \equiv {\sim}(QF)[F]$, which is a contradiction. The other impossibility proofs are similar.

It should be noted that Q_T and Q_F are not merely *examples* of quantifiers that induce the relations of co-truth and co-falsity, respectively. They are essentially the *only* such quantifiers. For if Q is such that $(QX)[Y]$ is co-true with some other schema, then it is such that $(QX)[Y]$ is true for all X and Y, and is therefore equivalent to $(Q_T X)[Y]$; similarly for Q_F.

6 Relations between cognate quantifiers If **R** is any of the fifteen relations defined in Table 2, and if Q_1 and Q_2 are any cognate quantifiers, then let $Q_1 \mathbf{R} Q_2$ abbreviate the claim that $(Q_1X)[Y] \mathbf{R} (Q_2X)[Y]$. We can then think of this either as expressing a relation between certain cognate schemata or as expressing a relation directly between the two cognate quantifiers. The six possible relations between Q and Q' are, in the light of Rule O, given in column 2 of Table 3. Using Rules O, D, and C we can verify that the relation between Q and Q', or rather the strongest relation that holds between them, determines the strongest relations that hold between Q and Q^*, between Q' and ^-Q, between Q^* and ^-Q, between Q and ^-Q, and between Q' and Q^*, as recorded in the following results:

(6.1) $Q\mathbf{F}Q'$ iff $Q \nleftarrow Q^*$ iff $Q' \nleftarrow {}^-Q$ iff $Q^*\mathbf{T}\,{}^-Q$ iff $Q \nleftarrow {}^-Q$ iff $Q' \nleftarrow Q^*$
(6.2) $Q\mathbf{T}Q'$ iff $Q \nrightarrow Q^*$ iff $Q' \nrightarrow {}^-Q$ iff $Q^*\mathbf{F}\,{}^-Q$ iff $Q \nrightarrow {}^-Q$ iff $Q' \nrightarrow Q^*$
(6.3) $Q \leftrightarrow Q'$ iff $Q - Q^*$ iff $Q' - {}^-Q$ iff $Q^* \leftrightarrow {}^-Q$
(6.4) $Q\mathbf{S}Q'$ iff $Q \leftarrow Q^*$ iff $Q' \leftarrow {}^-Q$ iff $Q^*\mathbf{C}\,{}^-Q$
(6.5) $Q\mathbf{C}Q'$ iff $Q \rightarrow Q^*$ iff $Q' \rightarrow {}^-Q$ iff $Q^*\mathbf{S}\,{}^-Q$
(6.6) $Q\mathbf{L}Q'$ iff $Q\mathbf{L}Q^*$ iff $Q'\mathbf{L}\,{}^-Q$ iff $Q^*\mathbf{L}\,{}^-Q$
(6.7) if $Q\mathbf{L}Q'$, $Q\mathbf{C}Q'$, $Q\mathbf{S}Q'$, or $Q \leftrightarrow Q'$ then $Q - {}^-Q$ and $Q' - Q^*$.

*7 **Squares of opposition*** By a square of opposition I mean a display of the strongest logical relations among cognate quantifiers Q, Q', Q^*, and ^-Q in which the quantifiers are placed at the vertices of a square and their relationships are shown along the edges and diagonals, and in which quantifiers that are obverses of one another appear in the same row and quantifiers that are duals of one another are placed in the same column. I count two squares of opposition as being of the same type if they have the same logical relations along corresponding edges and diagonals, or if one can be rotated 180° about a vertical or horizontal axis, or about an axis orthogonal to the page, in a way that brings this about. Thus, for example, the squares of opposition in Figure 2 are all of the same type.

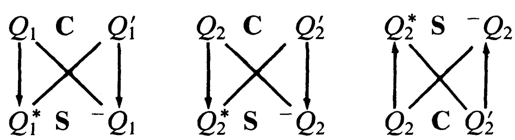

Figure 2. Some squares of opposition of the same type.

Using these definitions, and the results given in 6.1–6.7, it follows that there are exactly four types of square of opposition, as shown in Figure 3. I follow tradition by omitting explicit representation of the trivial relation **L**.

The standard universal quantifier fits the Boolean square of opposition. If we hold to a strict terminology, according to which no modifications of the definitions of the logical relations are to be entertained, then we will not be able to say (as is often, and correctly, said in a less strict terminology) that, in the presence of the assumption that the terms involved are nonempty, the universal quantifier and its cognates form a classical square of opposition. To say that involves amending the definition of contrariety, for example, so that the schemata $(QX)[Y]$ and $(Q'X)[Y]$ are contraries iff for all *nonempty* X and Y, $\sim(QX)[Y] \vee \sim(Q'X)[Y]$. If we disallow such redefinitions, it may at first seem intuitively unlikely that there will be *any* quantifiers that fit a classical square. However, as we can determine from Table 3, there are: the quantifier

Q_ϵ, corresponding to Russellian definite descriptions, i.e., the quantifier 'the', is one! The quantifier 'most', construed in accordance with the semantics offered for it at the end of Section 3 above, is another. From Table 3 we can also see that Q_α fits a semi-degenerate square of opposition. In addition, the quantifier 'some, but not all' fits such a square. Finally the quantifier Q_F and its dual Q_T fit a degenerate square of opposition, and are the only quantifiers that do.

Type I: Boolean

Type II: Classical

Type III: Semi-degenerate

Type IV: Degenerate

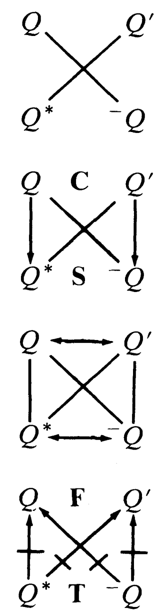

Figure 3. The four types of square of opposition.

8 Properties of quantifiers We noted in Section 6 that relations between cognate categorical schemata could be reconstrued as relations between cognate quantifiers. But since cognate quantifiers are interdefinable, it follows that these relations can also be reconstrued as properties of quantifiers. In Table 4 the entries (other than '0' and '$\Uparrow$', which retain the sense given time in Table 3) are designations for the properties Q has by virtue of the fact that the schemata cited above the entry have the relation cited to its left.

The properties **E2**–**E8** can be given more or less familiar names, following the terminology in Bird ([3], p. 55). **E2** is the property of obvertibility, **E3** partial invertibility, **E4** invertibility, **E5** convertibility, **E6** obverted convertibility, **E7** partial contraposability, and **E8** contraposability. The properties **A3** and **A4** are not normally discussed, let alone named, but might be called partial anti-invertibility and anti-invertibility, respectively, to maintain parallelism with established vocabulary. Similarly, the properties **S2**–**S8** arre not commonly discussed, but might be given parallel names by calling **S2** obverse subcontrariety, **S3** partial inverse subcontrariety, **S4** inverse subcontrariety, **S5** converse subcontrariety, **S6** obverted converse subcontrariety, **S7** partial contrapositive subcon-

Table 4. Logical Properties of Quantifiers

	1	2	3	4	5	6	7	8
A	QXY	QXY	QXY	QXY	QXY	QXY	QXY	QXY
B	QXY	$QX{\sim}Y$	$Q{\sim}XY$	$Q{\sim}X{\sim}Y$	QYX	$QY{\sim}X$	$Q{\sim}YX$	$Q{\sim}Y{\sim}X$
A**F**B	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
A**T**B	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
$A \leftrightarrow B$	**E1**	**E2**	**E3**	**E4**	**E5**	**E6**	**E7**	**E8**
$A - B$	0	0	**A3**	**A4**	0	0	0	0
A**S**B	$\Uparrow$	**S2**	**S3**	**S4**	**S5**	**S6**	**S7**	**S8**
A**C**B	$\Uparrow$	**C2**	**C3**	**C4**	**C5**	**C6**	**C7**	**C8**
A**L**B	$\Uparrow$	**L2**	**L3**	**L4**	**L5**	**L6**	**L7**	**L8**

trariety, and **S8** contrapositive subcontrariety. By replacing 'subcontrariety' by 'contrariety', we have names for the properties **C2–C8**. I will not attempt to provide appropriate names for the vacuous properties **L2–L8**.

For each of these properties, Table 3 provides a quantifier which exemplifies that property and exemplifies no stronger property from the same column. The type of square of opposition that a quantifier fits is determined solely by its properties from column 2, but other properties, such as contraposability, are normally taken to be significant in connection with the square of opposition, and yet are independent of the properties in column 2. For example, $Q_\exists$ and Q_δ both have the property **L2**, and neither has any stronger property from column 2. It follows that both generate squares of opposition of Type I. But Q_δ also has the property **S5** (converse subcontrariety) while $Q_\exists$ has no stronger property from column 5 than **L5**. Thus we may think of these two quantifiers as generating squares of opposition of different subtypes within the same type. It therefore becomes of interest to determine into what subtypes squares of opposition may fall, based on the other properties quantifiers may possess.

The question what subtypes there are can be answered by answering two other questions: What combinations of properties may a quantifier have? and How are the properties of a given quantifier related to those of its cognates? This latter question arises because the collection of properties that Q has may determine the collection Q' has, and if in some cases these are different collections, then since Q and Q' fall within the same square of opposition we will want to say that the two collections of properties determine the same subtype. Three observations simplify the process of answering this second question. First, every quantifier has the property **E1** trivially. The only question is whether it has one of the two stronger properties **T** or **F**. So we may neglect **E1**.

Second, we can show directly from the definitions of the properties involved that Q has a given property **P6** from the sixth column iff it has the corresponding property **P7** from the seventh (i.e., Q has **E6** iff it has **E7**, has **S6** iff it has **S7**, etc.). Hence one of these two columns may be considered redundant, and may be ignored. I choose to ignore column 7. Third, we can show that Q has a given property **P3** from the third column iff it has the corresponding property **P4** from the fourth. This is so because, by 4.7 and 4.8, all and only those combinations of truth values which can be given to $(QS)[P]$ and $(Q \sim S)[P]$ can be given to $(QA)[B]$ and $(Q \sim A)[\sim B]$, where A is S and B is $S \equiv P$. Hence we may ignore one of these columns, and I choose to ignore column 3. Elementary use of Rules E, O, D, and C now suffices to give the results in Table 5.

Table 5. Connections Among Properties of Cognate Quantifiers

Q	Q'	Q^*	^{-}Q	Q	Q'	Q^*	^{-}Q	Q	Q'	Q^*	^{-}Q	Q	Q'	Q^*	^{-}Q
F	**F**	**T**	**T**	**E4**	**E4**	**E4**	**E4**	**S5**	**S8**	**C8**	**C5**	**L6**	**L6**	**L6**	**L6**
T	**T**	**F**	**F**	**A4**	**A4**	**A4**	**A4**	**C5**	**C8**	**S8**	**S5**	**E8**	**E5**	**E5**	**E8**
E2	**E2**	**E2**	**E2**	**S4**	**S4**	**C4**	**C4**	**L5**	**L8**	**L8**	**L5**	**S8**	**S5**	**C5**	**C8**
S2	**S2**	**C2**	**C2**	**C4**	**C4**	**S4**	**S4**	**E6**	**E6**	**E6**	**E6**	**C8**	**C5**	**S5**	**S8**
L2	**L2**	**L2**	**L2**	**E5**	**E8**	**E8**	**E5**	**C6**	**C6**	**S6**	**S6**				

Thus, for example, if **S5**(Q), i.e., if Q has the property **S5**, then **S8**(Q'), **C8**(Q^*), and **C5**(^{-}Q). Note too that if (as is the case with the existential quantifier) Q is convertible, then ^{-}Q will be too, while Q' and Q^* will be contraposable. That is, it would be impossible to produce a quantifier which was convertible but whose dual was not contraposable.

The process of sorting out the various possible subtypes of squares of opposition is simplified by establishing some elementary results concerning the consequences of having one or more of the properties **E2**–**E8**, **A4**. There are several such results which can be proven using only our six rules. In reporting these results below I use (for example) '**E5**' to abbreviate '**E5**(Q)', so that in each of these results I am reporting connections among the properties of a single quantifier. I abbreviate further by letting, for example, '**P2** iff **P4**' abbreviate '**L2** iff **L4**, and **S2** iff **S4**, and **C2** iff **C4**, and **E2** iff **E4**, and **A2** iff **A4**'.

(8.1) If **E2**: **P5** iff **P6** iff **P8**.
(8.2) If **E4**: **E2**; and
A4 and **C5** are impossible.
(8.3) If **E5**: **P2** iff **P4** iff **P6** iff **P8**;
and **A4** is impossible.
(8.4) If **E6**: **E2**, **E4**, **E5**, and **E8**.
(8.5) If **E8**: **P2** iff **P4** iff **P5** iff **P6**; and
A4 is impossible.
(8.6) If **A4**: **E2**, **L5**, **L6**, and **L8**.

To these results we may add, for ease of reference, three results reported previously in connection with Table 5.

(8.7) If not **T** and not **F**, then **E1**.
(8.8) **P6** iff **P7**.
(8.9) **P3** iff **P4**.

One result reported above (in 8.2) is less trivial than the rest to prove, namely that if **E4**(Q) (and nothing stronger) then not **C5**(Q). Suppose **E4**(Q) and **C5**(Q), and suppose $(QS)[P]$, for some S and P. By Rule A, $(QS)[S \& P]$, so because **E4**(Q) $(Q \sim S)[\sim(S \ \& \ P)]$. By Rule A again, $(Q \sim S)[\sim S \ \& \sim(S \& P)]$. By Rule E, $(Q \sim S)[\sim S]$. Since **E4**(Q), it follows that $(QS)[S]$. Since **C5**(Q) we get $\sim(QS)[S]$, which is a contradiction. So for any S and P, $\sim(QS)[P]$. Thus **F**(Q), contradicting the stipulation that Q have no property stronger than **E4**.

One other collection of elementary results is available to simplify further the process of sorting out the various subtypes of square of opposition. These results are all of the following form: if Q has one form of subcontrariety and another form of contrariety, then it has one or more forms of equivalence properties. These results are all recorded in Table 6, in which the antecedents appear as coordinates, and the consequents as the entries, of the array.

Table 6. Implications of Some Pairs of Properties

	C2	**C4**	**C5**	**C6**	**C8**
S2	...	**E4**	**E6**	**E5 & E8**	**E6**
S4	**E4**	...	**E6 & E8**	**E5 & E6 & E8**	**E5 & E6**
S5	**E6**	**E6 & E8**	...	**E2 & E4**	**E4**
S6	**E5 & E8**	**E5 & E6 & E8**	**E2 & E4**	...	**E2 & E4**
S8	**E6**	**E5 & E6**	**E4**	**E2 & E4**	...

Let us use expressions like '**SLLSE**' to designate combinations of properties of a single quantifier—in this case, the combination **S2**, **L4**, **L5**, **S6**, and **E8**. In the light of 8.7–8.9 we need not give separate mention to the associated properties from columns 1, 3, and 7. Thousands of combinations of properties are lexicographically possible, but fewer than 100 such combinations are consistent with all the results in 8.1–8.9 and Table 6. Of these, some can be associated with one another, using the results in Table 5, as generating the same subtype of square of opposition. For example, if Q has the combination **SLSSL**, then Q' will have the combination **SLLSS**, Q^* will have the combination **CLLCC**, and ^{-}Q will have the combination **CLCCL**. We could therefore choose any one of these four combinations to represent the subtype of square of opposition which Q, Q', Q^*, and ^{-}Q fit. It is convenient to stipulate the "alphabetic" order **L**, **C**, **S**, **E**, **A**, **F**, **T** for the letters involved, and let this generate a lexicographic order among the combinations. Then when a set of combinations are associated with one another by the results in Table 5, as belonging to the same subtype of square, the lexicographically earliest such combination can be taken as representative of the rest of the set. This procedure puts the subtypes in order by type, and chooses as the representative quantifier, in a set of cognate quantifiers, one which can fit in the upper left corner of the square (the A-position, by traditional labeling) in the standard arrangement of the square given in Figure 3.

When we associate with one another the combinations of properties which correspond to cognate quantifiers, using Table 5, and apply all the constraints given in 8.1–8.9 and Table 6, we find that there are only 34 possible subtypes of square of opposition. Examples of quantifiers of each of these subtypes can be constructed, so there are *exactly* 34 subtypes. They are given in Table 7.

Table 7. The Subtypes of Square of Opposition

Type I	Type II	Type III	Type IV
LLLLL	**CLLLL**	**ELLLL**	**FFFFF**
LLLLC	**CLLLC**	**ELCCC**	
LLLLE	**CLLCL**	**ECLLL**	
LLLCL	**CLLCC**	**ECCCC**	
LLLCC	**CLCLC**	**EELLL**	
LLCLC	**CLCCC**	**EEEEE**	
LLCCC	**CCLLL**	**EALLL**	
LCLLL	**CCLLC**		
LCLLC	**CCLCL**		
LCLCL	**CCLCC**		
LCLCC	**CCCLC**		
LCCLC	**CCCCC**		
LCCCC	**CCCCE**		

9 Rules of immediate inference By a classical rule of immediate inference I mean any rule of the form $A/\therefore B$ or the form $A :: B$, where A and B are cognate categorical schemata. Every possible classical rule of immediate inference corresponds to one or another of the properties of quantifiers introduced in Section 8 and, with the exception of the trivial properties **E1** and **L2–L8**, each of the properties of quantifiers corresponds to a rule of immediate inference. Some of these correspondences are evident from our choice of names for the properties. Thus, for example, any quantifier with the property **E8** will permit a rule of contraposition. Other correspondences are less obvious. For example, any quantifier Q with the property **C6** permits a rule of "conversion by limitation" (although the term "limitation" may not be as apt here as it was in the traditional analysis of categorical propositions), i.e., a rule of the form $QXY/\therefore Q^*YX$. Some of the rules of immediate inference that correspond to properties of quantifiers are entirely unfamiliar. Thus, for example, if Q has the property **A3** then it permits the rule $QXY :: {}^-Q \sim XY$.

Once we have determined the collection of properties a quantifier has, and therefore have determined what properties its cognates have, we will not only have determined what subtype of square of opposition they fall into, but will also have determined the classical rules of immediate inference they support. Indeed, in determining which subtypes of square of opposition are possible, we have in effect determined what combinations of classical rules of immediate inference are possible. For each distinct subtype, there is a distinct set of rules of immediate inference that characterize it. The infinitely many quantifiers of the forms 'all but n', 'with fewer than n exceptions', 'with at most n exceptions', 'with at least n exceptions', and 'with more than n exceptions' each has the same collection of properties as 'all'. We therefore know that each generates a square of opposition of the same subtype (namely **LLLLE**) as 'all' and that each is subject to exactly the same classical rules of immediate inference. Similarly, their duals 'exactly n', 'at least n', 'more than n', 'fewer than n', and 'at most n', respectively, all have the same properties, and therefore support the same classical rules of immediate inference, as 'some'. Thus in a few strokes we obtain complete information about the classical rules of immediate inference associated with an infinite collection of quantifiers available in natural discourse.

NOTES

1. The formal semantics I will offer is equivalent to the semantics given by Barwise and Cooper [2], if we interpret them as intending it to be a defining feature of quantifiers (in my sense of 'quantifiers') that they denote quantificational functions (as I define them in 3.3 below) rather than arbitrary functions from subsets of the domain to families of subsets of the domain. The Barwise and Cooper semantics is a descendent of that given by Montague [8].

2. I am indebted to the referee for the observation that if we are willing to introduce distinguished quantifier constants for universal and existential quantification, and to add any standard set of rules and axioms for first-order logic, we get a system for which the completeness proof has essentially been given in Barwise [1]. The referee evidently refers to Barwise's sketch of a proof of completeness for indexical quantifiers. By dropping from the proof all references to indices, and by modifying the construction of quantificational functions in the canonical model so as to conform to the fact that all our quantifiers are relativized, it is indeed possible to adapt Barwise's proof to show the completeness of *GQ* extended by (or treated as an extension of) first-order logic. A technical problem arises from the fact that, in the canonical model, we must define the value of a quantificational function even for arguments which do not happen to be the extensions of any formulas with one free variable. It suffices, however, to let each quantificational function take the empty set as value for each such argument. I second the referee's conjecture that, even without distinguished quantifiers for universal and existential quantification, *GQ* (understood to incorporate any appropriate formulation of the propositional calculus) is complete.

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