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Aggregating Subjective Probabilities: Some Limitative Theorems

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1 Introduction Suppose that n individuals, labeled $1, \ldots, n$, wish to assign consensual probabilities to a sequence of events $E_1, \ldots, E_k$ which partition some set of possibilities. Let their subjective assignments be registered in an $n \times k$ matrix $P = (p_{ij})$, where p_{ij} denotes the subjective probability assigned by individual i to event E_j. The question of how to aggregate the probabilities in P into a single sequence of consensual probabilities may be abstractly modeled as the problem of choosing a mapping (hereafter called a *probability aggregation method,* or PAM)

$$F\colon \mathcal{P}(n,k) \to \mathcal{P}(k), \tag{1.1}$$

where $\mathcal{P}(n,k)$ denotes the set of all $n \times k$ matrices with nonnegative entries summing to one in each row and $\mathcal{P}(k)$ the set of k-dimensional row vectors with nonnegative entries summing to one. Given a particular choice of F, an equation $F(P) = (p_1, \ldots, p_k)$ is interpreted to assert that if individuals should assign the subjective probabilities registered in P, the resulting consensual probabilities assigned to the events $E_1, \ldots, E_k$ would be $p_1, \ldots, p_k$.

As defined above the class of PAMs includes aggregation methods which might appear contrary to the ethos of group decisionmaking. It includes, for example, *dictatorial* PAMs (for some fixed individual d, $F(P) =$ the d^{th} row of P, $\forall P \in \mathcal{P}(n,k)$), as well as PAMs which *impose* the same consensual distribution, regardless of the opinions of any of the individuals (for some fixed probability vector $(q_1, \ldots, q_k) \in \mathcal{P}(k)$, $F(P) = (q_1, \ldots, q_k)$, $\forall P \in \mathcal{P}(n,k)$).

In addition to wishing to preclude such methods of aggregation, decision theorists have often posited further restrictions on aggregation. Roughly speaking, such restrictions have arisen from the belief that the agreement of individuals on certain features of the distribution in question ought to be preserved in the corresponding consensual distribution. It has often been posited as an axiom of aggregation, for example, that if all individuals assign an event the same probability, then the consensual probability assigned to that event should respect their agreement.

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Other common features of individual distributions have also been deemed worthy of preservation in consensual distributions. Laddaga [4] and Schmitt [6] have argued, for example, that when individuals assign probabilities to atomic events $E_1, \ldots, E_k$ in such a way that some pair of events A and B in the algebra generated by $E_1, \ldots, E_k$ turns out to be independent on each of their assignments, consensual probabilities should be assigned so that A and B are independent.

Lehrer and Wagner [5] have argued against this constraint on aggregation, stressing that when individuals direct their initial acts of assessment at the probabilities of events which partition a set of possibilities, independence of events in the algebra generated by this partition is of negligible epistemic significance. This argument was supplemented by a proof that if (a) the consensual probability assigned to each atomic event depends only on the probabilities assigned by individuals to that event, (b) the consensual probability of an atomic event is zero if all individuals assign that event probability zero, and (c) consensual distributions preserve instances of independence common to all individual distributions, then aggregation must be dictatorial.[1] In the present paper we delete even the unanimity condition (b) and prove that an aggregation method satisfying just (a) and (c) above must still be either dictatorial or imposed.

2 Irrelevance of alternatives and preservation of independence Suppose that a PAM F is required to determine consensual probabilities in such a way that the consensual probability assigned to each event depends only on the probabilities assigned by individulas to that event. This restriction on aggregation may be formalized as follows:

Irrelevance of alternatives (IA) There exist functions f_j: $[0,1]^n \to [0,1]$, $j = 1, \ldots, k$, such that $\forall P = (p_{ij}) \in \mathcal{P}(n,k)$, $F(P) = (p_1, \ldots, p_k)$, where $p_j = f_j(p_{1j}, \ldots, p_{nj})$, $j = 1, \ldots, k$.

Note that the functions f_j may vary from event to event, subject only to the condition that for any set of vectors $z_1, \ldots, z_k \in [0,1]^n$,

$$z_1 + \ldots + z_k = (1, 1, \ldots, 1) \Rightarrow f_1(z_1) + \ldots + f_k(z_k) = 1 \text{ .} \qquad (2.1)$$

Condition (2.1) follows from applying F to the matrix P, the j^{th} column of which is z_j^T, $j = 1, \ldots, k$, and using the fact (implicit in the definition of a PAM) that consensual probabilities must sum to one.

As a preliminary to formalizing the requirement that consensual distributions preserve instances of independence common to all individual distributions, we recall that the algebra $\mathcal{A}$ of events generated by the partition $E_1, \ldots, E_k$ consists of the empty set, along with all possible unions of the atomic events, $E_1, \ldots, E_k$. Consensual probabilities $p_1, \ldots, p_k$ assigned to these events by a PAM F induce a consensual probability measure π on $\mathcal{A}$ in the obvious way: if $A = E_{j_1} \cup \ldots \cup E_{j_r} \in \mathcal{A}$, $\pi(A) = p_{j_1} + \ldots + p_{j_r}$. Similarly, the subjective probabilities $p_{i1}, \ldots, p_{ik}$ assigned to the atomic events by individual i induce a measure π_i on $\mathcal{A}$ by the rule $\pi_i(E_{j_1} \cup \ldots \cup E_{j_r}) = p_{ij_1} + \ldots + p_{ij_r}$.

Events A and B are usually defined to be independent with respect to a probability measure μ if $\mu(A \cap B) = \mu(A)\mu(B)$. For this symmetric definition of independence $\mu(A)$ and $\mu(B)$ may, in appropriate circumstances, take on the values zero or one. If one regards instances of independence involving the probabilities 0 or 1 as idiosyncratic one may not wish to require their preservation in consensual distributions. Thus, depending on what one regards as a "genuine" case of independence, one may formalize preservation of independence in any of the following ways:

Preservation of independence (PI) axioms

PI_0 $\pi_i(A \cap B) = \pi_i(A)\pi_i(B)$, $i = 1, \ldots, n \Rightarrow \pi(A \cap B) = \pi(A)\pi(B)$.

PI_1 $\pi_i(A)$, $\pi_i(B) > 0$ and $\pi_i(A \cap B) = \pi_i(A)\pi_i(B)$, $i = 1, \ldots, n \Rightarrow \pi(A)$, $\pi(B) > 0$ and $\pi(A \cap B) = \pi(A)\pi(B)$.

PI_2 $0 < \pi_i(A)$, $\pi_i(B) < 1$ and $\pi_i(A \cap B) = \pi_i(A)\pi_i(B)$, $i = 1, \ldots, n \Rightarrow 0 <$ $\pi(A)$, $\pi(B) < 1$ and $\pi(A \cap B) = \pi(A)\pi(B)$.

Although we shall be concerned with PAMs satisfying just IA and some PI axiom, we state, for ease of reference, a class of unanimity axioms. For each $\alpha \in [0, 1]$, let $\underline{\alpha}$ denote the n-dimensional row vector $(\alpha, \ldots, \alpha)$. For each such α, we have as a possible restriction on aggregation

$$\alpha\text{-Unanimity } (\mathrm{U}(\alpha))\colon f_j(\underline{\alpha}) = \alpha,\ j = 1, \ldots, k\ .$$

In particular, if $\alpha = 0$, we get

$$\text{Zero-Unanimity } (\mathrm{U}(0))\colon f_j(\underline{0}) = 0,\ j = 1, \ldots, k\ .$$

In terms of the aforementioned axioms, Lehrer and Wagner ([5], Theorem 1) showed that if $k \geq 3$, a PAM satisfying IA, U(0), and PI_0 or PI_1 must be dictatorial, and if $k \geq 4$, a PAM satisfying IA, U(0), and PI_2 must be dictatorial.[2] In what follows we use recent results of Aczél et al. [1] to describe, for each of the aforementioned PI conditions, the PAMs satisfying IA. We shall see that, even in this setting, dictatorial aggregation may be avoided only by imposing an external consensual distribution.

3 Three limitative theorems The following three theorems describe the PAMs satisfying IA and PI_i, for $i = 0$, 1, or 2. Proofs of these theorems appear in the Appendix.

Theorem 3.1 *A PAM F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, where $k \geq 3$, satisfies IA and PI_0 iff it is dictatorial or $\exists l \in \{1, \ldots, k\}$ such that $\forall P \in \mathcal{P}(n, k)$, $F(P) = (\delta_{1l}, \ldots, \delta_{kl})$, where $\delta_{jl} = 0$ if $j \neq l$ and $\delta_{ll} = 1$.*

Thus, if $k \geq 3$, the only nondictatorial PAMs satisfying IA and PI_0 impose a consensual distribution which assigns some fixed atomic event E_l the probability one and all other atomic events the probability zero.

Theorem 3.2 *A PAM F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, where $k \geq 3$, satisfies IA and PI_1 iff it is dictatorial.*

Theorem 3.3 *A PAM F: $\mathcal{P}(n, 4) \to \mathcal{P}(4)$ satisfies IA and PI_2 iff it is dictatorial or $F(P) = (1/4, 1/4, 1/4, 1/4)$, $\forall P \in \mathcal{P}(n, k)$. A PAM F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, where $k \geq 5$, satisfies IA and PI_2 iff it is dictatorial.*

Thus, if $k \geq 4$, the only nondictatorial PAM satisfying IA and PI_2 occurs when $k = 4$, in which case a uniform consensual distribution is imposed.

Appendix

Theorem 3.1 *A PAM F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, where $k \geq 3$, satisfies IA and PI_0 iff it is dictatorial or $\exists l \in \{1, \ldots, k\}$ such that $\forall P \in \mathcal{P}(n, k)$, $F(P) = (\delta_{1l}, \ldots, \delta_{kl})$, where $\delta_{jl} = 0$ if $j \neq l$ and $\delta_{ll} = 1$.*

Proof: Sufficiency. Dictatorial PAMs obviously satisfy IA and PI_0. The PAM $F(P) \equiv (\delta_{1l}, \ldots, \delta_{kl})$ clearly satisfies IA, and since for the measure π induced by the probabilities $\delta_{1l}, \ldots, \delta_{kl}$, $\pi(A \cap B) = \pi(A)\pi(B)$ for *every* pair of events A and $B \in \mathcal{A}$, it also satisfies PI_0.

Necessity. If F satisfies U(0) in addition to IA and PI_0, it is dictatorial by the aforementioned theorem of Lehrer and Wagner. If U(0) is violated, $\exists l \in \{1, \ldots, k\}$ such that

$$f_l(0) > 0 \ . \tag{A.1}$$

Let $j \neq l$ and choose $m \neq j, l$. For each $z = (\zeta_1, \ldots, \zeta_n) \in [0, 1]^n$, consider the matrix P_z, defined as follows: column l of P_z is $\underline{0}^T$; column j is z^T; column m is $(\underline{1} - z)^T$; all remaining columns, if any, are $\underline{0}^T$.

Let

$$A = \bigcup_{r \neq j} E_r \text{ and } B = \bigcup_{r \neq l} E_r \ . \tag{A.2}$$

For each $i = 1, \ldots, n$, $\pi_i(A \cap B) = 1 - \zeta_i = (1 - \zeta_i)(1) = \pi_i(A)\pi_i(B)$, and hence by PI_0, we must have

$$\pi(A \cap B) = \pi(A)\pi(B) \ . \tag{A.3}$$

By (2.1) and (A.2)

$$\begin{aligned} \pi(A) &= 1 - f_j(z), \\ \pi(B) &= 1 - f_l(\underline{0}), \text{ and} \\ \pi(A \cap B) &= 1 - f_j(z) - f_l(\underline{0}) \ ; \end{aligned} \tag{A.4}$$

and by (A.3) and (A.4),

$$f_j(z) f_l(\underline{0}) = 0 \ , \tag{A.5}$$

whence by (A.1), $f_j(z) = 0$, $\forall j \neq l$, $\forall z \in [0, 1]^n$. It then follows from (2.1) that $f_l(z) = 1$, $\forall z \in [0, 1]^n$ so that $F(P) = (\delta_{1l}, \ldots, \delta_{kl})$, $\forall P \in \mathcal{P}(n, k)$.

Theorem 3.2 *A PAM F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, where $k \geq 3$, satisfies IA and PI_1 iff it is dictatorial.*

Proof: Sufficiency is clear.

Necessity. We show that F must satisfy U(0) and then invoke Theorem 1 of [5]. If U(0) is assumed to be violated, we proceed as in the preceding proof to show that if $f_l(\underline{0}) > 0$ for some $l \in \{1, \ldots, k\}$, then

$$f_j(z) = 0, \ \forall j \neq l, \ \forall z \in [0, 1)^n \ , \tag{A.6}$$

the restriction of the coordinates of z to the half-open interval [0, 1) being necessary to ensure that $\pi_i(A) > 0$, as required by PI_1. But (A.6) is inconsistent with PI_1 as illustrated by the matrix P defined as follows: Let $j \neq l$ and let columns j and l of P both be $(\underline{1/2})^T$, with all remaining columns $\underline{0}^T$. Let

$$A = E_l \cup E_j \text{ and } B = E_j \ . \tag{A.7}$$

Then $\pi_i(A) = 1$ and $\pi_i(B) = 1/2$ and $\pi_i(A \cap B) = \pi_i(A)\pi_i(B)$, $i = 1, \ldots, n$, and so if PI_1 were satisfied, it would be true, among other things, that $\pi(B) > 0$. But $\pi(B) = \pi(E_j) = f_j(\underline{1/2}) = 0$ by (A.6).

The following result of Aczél et al. ([1], Theorem 2) will be used in the proof of our final limitative theorem:

Lemma *A PAM F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, where $k > 3$, satisfies IA and $U(1/k)$ iff there exists a sequence of real weights $\omega_1, \ldots, \omega_n$ such that $\forall P = (p_{ij}) \in \mathcal{P}(n, k)$, $F(P) = (p_1, \ldots, p_k)$, where*

$$p_j = \sum_{i=1}^{n} \omega_i(p_{ij} - 1/k) + 1/k, \quad j = 1, \ldots, k \ . \tag{A.8}$$

The weights may, subject to certain restrictions which need not concern us here, be negative. However, it is always the case that $\sum \omega_i \leq 1$, and if $\sum \omega_i = 1$, the weights must all be nonnegative, in which case each p_j is an ordinary weighted arithmetic mean of the entries in the j^{th} column of P, with weights invariant across j.

Theorem 3.3 *A PAM F: $\mathcal{P}(n, 4) \to \mathcal{P}(4)$ satisfies IA and PI_2 iff it is dictatorial or $F(P) = (1/4, 1/4, 1/4, 1/4)$, $\forall P \in \mathcal{P}(n, 4)$. A PAM F: $\mathcal{P}(n, k) \to \mathcal{P}(k)$, where $k \geq 5$, satisfies IA and PI_2 iff it is dictatorial.*

Proof: Sufficiency. Dictatorial PAMs clearly satisfy IA and PI_2. If $k = 4$ and A and B are events in the algebra generated by the atomic events E_1, E_2, E_3, E_4 and if $0 < \pi_i(A)$, $\pi_i(B) < 1$ and $\pi_i(A \cap B) = \pi_i(A)\pi_i(B)$, $i = 1, \ldots, n$, it is easy to check that A and B must each be the union of two atomic events and $A \cap B$ must be equal to some atomic event. The imposed PAM $F(P) \equiv (1/4, 1/4, 1/4, 1/4)$ yields $\pi(A \cap B) = 1/4 = (1/2)(1/2) = \pi(A)\pi(B)$, as required by PI_2, in all such cases. It also clearly satisfies IA.

Necessity. We show first that if $k \geq 4$, IA and PI_2 imply that $\forall j, l \in \{1, \ldots, k\}$,

$$f_j(\underline{1/k}) = f_l(\underline{1/k}) \ . \tag{A.9}$$

It then follows from (2.1) and (A.9)—by considering the matrix P with all entries identical to $1/k$—that

$$f_j(\underline{1/k}) = 1/k, j = 1, \ldots, k \ . \tag{A.10}$$

Hence IA and PI_2 imply $U(1/k)$ if $k \geq 4$.

In order to prove (A.9) consider the matrices P and P^* defined as follows: Choose indices m and r distinct from each other, and from j and l. (To simplify notation we denote $1/k$ by the Greek letter κ). Column j of P is $\underline{\kappa}^T$; columns l and r are each $(\underline{\sqrt{\kappa} - \kappa})^T$; column m is $(\underline{1 - 2\sqrt{\kappa} + \kappa})^T$; the remaining columns of P, if any, are $\underline{0}^T$. P^* is the matrix resulting from the interchange of columns j and l in P.

Denote by π_i and π_i^* the individual probability measures associated to P and P^* and let

$$\begin{aligned} A &= E_j \cup E_r \\ B &= E_m \cup E_r \\ C &= E_l \cup E_r \ . \end{aligned} \tag{A.11}$$

It is straightforward to check that $0 < \pi_i(A),\ \pi_i(B),\ \pi_i^*(C),\ \pi_i^*(B) < 1$ and that $\pi_i(A \cap B) = \pi_i(A)\pi_i(B)$ and $\pi_i^*(C \cap B) = \pi_i^*(C)\pi_i^*(B)$, $i = 1, \ldots, n$. Hence, denoting by π and π^* the consensual probability measures associated to P and P^* by F, it follows by PI_2 that

$$\pi(A \cap B) = \pi(A)\pi(B),\ 0 < \pi(A),\ \pi(B) < 1 \ ,$$

and

$$\pi^*(C \cap B) = \pi^*(C)\pi^*(B),\ 0 < \pi^*(C),\ \pi^*(B) < 1 \ . \tag{A.12}$$

Since $A \cap B = C \cap B = E_r$ and the r^{th} columns of P and P^* are identical, it follows from (A.12) by IA that

$$\pi(A)\pi(B) = \pi^*(C)\pi^*(B) \ , \tag{A.13}$$

and since the m^{th} columns of P and P^* are also identical it follows from IA and (A.11) that $\pi(B) = \pi^*(B)$. Since by (A.12), $\pi(B) > 0$, (A.13) yields

$$\pi(A) = \pi^*(C) \ ; \tag{A.14}$$

i.e., by (A.11),

$$f_j(\underline{\kappa}) + f_r(\underline{\sqrt{\kappa} - \kappa}) = f_l(\underline{\kappa}) + f_r(\underline{\sqrt{\kappa} - \kappa}) \ , \tag{A.15}$$

which implies (A.9), and hence, as argued at the outset, (A.10).

Hence F satisfies $\mathrm{U}(1/k)$ as well as IA, and so by the lemma, the aggregation functions f_j all take the identical form

$$f_j(z) = \sum_{i=1}^{n} \omega_i(\zeta_i - 1/k) + 1/k \ , \tag{A.16}$$

where $z = (\zeta_1, \ldots, \zeta_n)$ and the ω_i's are real numbers such that $\sum \omega_i \leq 1$, with all $\omega_i \geq 0$ in case $\sum \omega_i = 1$. Denoting $1 - \sum_{i=1}^{n} \omega_i$ by ω_{n+1}, we may rewrite (A.16) as

$$f_j(\zeta_1, \ldots, \zeta_n) = \sum_{i=1}^{n} \omega_i \zeta_i + \omega_{n+1}(1/k) \ . \tag{A.17}$$

We now show that if $k \geq 5$, then $\omega_{n+1} = 0$, from which it follows from (A.17) that $f_j(\underline{0}) = 0$, $j = 1, \ldots, k$. Thus for $k \geq 5$, a PAM satisfying IA and PI_2 satisfies $\mathrm{U}(0)$ and is hence, by the aforementioned theorem of Lehrer and Wagner, dictatorial.

In order to show that $\omega_{n+1} = 0$ if $k \geq 5$, we again denote $1/k$ by κ and consider the matrix P, defined as follows: columns 1 and 3 of P are $(\underline{2\kappa - 4\kappa^2})^T$; column 2 is $(\underline{4\kappa^2})^T$; column 4 is $(\underline{1 - 4\kappa + 4\kappa^2})^T$; all remaining columns are $\underline{0}^{\mathrm{T}}$. Let

$$A = E_1 \cup E_2 \text{ and } B = E_2 \cup E_3 \ . \tag{A.18}$$

Then for each of the individual probability measures π_i associated with P, we have $0 < \pi_i(A),\ \pi_i(B) < 1$ and $\pi_i(A \cap B) = \pi_i(A)\pi_i(B)$. Hence by PI_2

$$\pi(A \cap B) = \pi(A)\pi(B) \ 0 < \pi(A),\ \pi(B) < 1 \ , \tag{A.19}$$

where π is the consensual measure associated to P by F. By (A.17) and the fact that $\sum_{i=1}^{n} \omega_i = 1 - \omega_{n+1}$, we have

$$\pi(A) = \pi(B) = 2\kappa$$

and

$$\pi(A \cap B) = 4\kappa^2 + (\kappa - 4\kappa^2)\omega_{n+1} \ . \tag{A.20}$$

Hence, by (A.19) and (A.20),

$$\kappa(1 - 4\kappa)\omega_{n+1} = 0 \ . \tag{A.21}$$

Since $\kappa = 1/k \leq 1/5$, (A.21) yields $\omega_{n+1} = 0$, as desired.

Suppose finally that $k = 4$. In this case we show that either $1 - \sum_{i=1}^{4} \omega_i = \omega_5 = 0$ (whence F satisfies U(0) and is thus dictatorial, as argued above when $k \geq 5$) or $\omega_5 = 1$ and $\omega_i = 0$, $i = 1, \ldots, 4$, in which case (A.17) implies that $F(P) = (1/4, 1/4, 1/4, 1/4)$, $\forall P \in \mathcal{P}(n, 4)$.

First consider the matrix P, defined as follows: columns 1 and 3 of P are $(\underline{6/25})^T$; column 2 is $(\underline{4/25})^T$; column 4 is $(\underline{9/25})^T$. Let $A = E_1 \cup E_2$ and $B = E_2 \cup E_3$. By a now familiar argument, it follows that

$$\pi(A \cap B) = \pi(A)\pi(B) \ , \tag{A.22}$$

and, using (A.17), we find that

$$\pi(A) = \pi(B) = (1/10)\omega_5 + (2/5) \ ,$$

and

$$\pi(A \cap B) = (4/25) + (9/100)\omega_5 \ . \tag{A.23}$$

Combining (A.22) and (A.23) yields

$$\omega_5^2 - \omega_5 = 0 \ , \tag{A.24}$$

so that $\omega_5 = 0$ or 1.

If $\omega_5 = 0$, then F is dictatorial, as argued above. Suppose that $\omega_5 = 1$. Then $\omega_1 + \ldots + \omega_4 = 0$. We show in fact that $\omega_i = 0$, $i = 1, \ldots, 4$. To simplify notation we show that $\omega_1 = 0$, from which the general proof will be clear. Consider the following matrix P: column 1 of P is $(4/9, 1/9, \ldots, 1/9)^T$; columns 2 and 4 are $(\underline{2/9})^T$; column 3 is $(1/9, 4/9, \ldots, 4/9)^T$. By the usual argument we conclude that for $A = E_1 \cup E_2$ and $B = E_2 \cup E_3$

$$\pi(A \cap B) = \pi(A)\pi(B) \ . \tag{A.25}$$

Using (A.17) and the fact that $\omega_1 + \ldots + \omega_4 = 0$ (whence $\omega_2 + \omega_3 + \omega_4 = -\omega_1$) we have

$$\begin{aligned}\pi(A) &= (1/3)\omega_1 + (1/2)\\ \pi(B) &= -(1/3)\omega_1 + (1/2)\\ \pi(A \cap B) &= 1/4\ .\end{aligned} \tag{A.26}$$

Combining (A.25) and (A.26) yields

$$\omega_1^2 = 0\ , \tag{A.27}$$

and hence, $\omega_1 = 0$, as desired.

NOTES

1. This result sharpened an earlier theorem of Dalkey [3] (which posited aggregation of probabilities of all atomic events by a single continuous function f: $[0,1]^n \to [0,1]$), partly meeting the objection of Bordley and Wolff [2] that under certain assumptions less restrictive than those of Dalkey, preservation of independence might be accommodated in the framework of nondictatorial aggregation.
2. It is easy to see that if $k \leq 3$, PI_2 is (vacuously) satisfied by every PAM.

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