Notre Dame Journal of Formal Logic
Volume 25, Number 2, April 1984

Conjunctive Normal Forms and Weak Modal Logics Without the Axiom of Necessity

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In *S5* it is known that any formula can be reduced to conjunctive normal form (CNF) of degree 1; completeness easily follows from this fact (see, e.g., [5]). The purpose of this paper is to extend this method to prove completeness for very weak modal logics, and to give some applications. Twenty modal logics are dealt with here. We call them Lpq, $DLpq$ ($p = 0, 1$ and $q = 0, 2$), $L3$, $DL3$, and $\mathbf{L}_N$, where $\mathbf{L}$ is one of $L00$–$DL3$. We first define $\mathbf{L}$-tautologies corresponding to each logic $\mathbf{L}$, where $\mathbf{L}$ is either Lpq or $DLpq$, and characterize them by the set $\mathbf{L}^*$ of value-assignments having certain properties. Then, we show that $\Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$ is provable in a modal logic $\mathbf{L}$ iff (1) C is a tautology, or (2) $A \vee B_i$ is an $\mathbf{L}$-tautology for some i, where C, but not necessarily A or B_i, contains no modal operator. Completeness for Lpq (or $DLpq$) follows from this equivalence. For completeness of $\mathbf{L}_N$, where $\mathbf{L}$ is either Lpq or $DLpq$, we shall also make use of the above equivalence for $\mathbf{L}$. For the remaining logics, a more direct method will be used.

The modal logics dealt with in this paper are defined in Section 1, $\mathbf{L}$-tautology definitions and characterizations are given in Section 2, completeness proofs in Section 3, and applications in Section 4.

1 Weak modal logics We are given a countable set, Π, of propositional variables, logical connectives, $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\Box$, and parentheses, (,). The other connectives and formulas are defined as usual. We shall consider modal systems obtained by adding the following axiom schemata and rules of inference to the classical logical base.

A1 $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
A2 $\Box A \rightarrow \Diamond A$

Received December 22, 1982; revised June 17, 1983

R00 $PC \vdash A \to B \Rightarrow \vdash \Box A \to \Box B$

R10 $PC \vdash \bigwedge_{i=1}^{n} A_i \to (A \to B) \Rightarrow \vdash \Box A \to \Box B$, where A_i is an axiom

R02 $PC \vdash \bigwedge_{i=1}^{n} (\Box A_i \to \Box B_i) \to (A \to B), \vdash \bigwedge_{i=1}^{n} (\Box A_i \to \Box B_i) \Rightarrow \vdash \Box A \to \Box B$

R12 $PC \vdash \bigwedge_{i=1}^{n_1} A_i \to \left(\bigwedge_{j=1}^{n_2} (\Box B_j \to \Box C_j) \to (A \to B)\right), \vdash \bigwedge_{j=1}^{n_2} (\Box B_j \to \Box C_j) \Rightarrow$
$\vdash \Box A \to \Box B$, where A_i is an axiom

R3 $\vdash A \to B \Rightarrow \vdash \Box A \to \Box B$

$R0_N$ $PC \vdash A \Rightarrow \vdash \Box A$

$R1_N$ $\vdash \Box A$, where A is an axiom or a tautology

$R2_N$ $\vdash \Box (A \to B) \Rightarrow \vdash \Box(\Box A \to \Box B)$

$R3_N$ $\vdash A \Rightarrow \vdash \Box A$.

In the above, $\bigwedge_{i=1}^{0} A_i \to A$ means A.

Now, we define the following modal logics:

$L00 = A1, R00 : L10 = A1, R10 : L02 = A1, R02$
$L12 = A1, R12 : L3 = A1, R3 : L00_N = A1, R0_N$
$L10_N = A1, R1_N : L02_N = A1, R0_N, R2_N$
$L12_N = A1, R1_N, R2_N : L3_N = A1, R3_N$
$D\mathbf{L} = A2, \mathbf{L}$, where $\mathbf{L}$ is one of the above logics $L00$–$L3_N$.

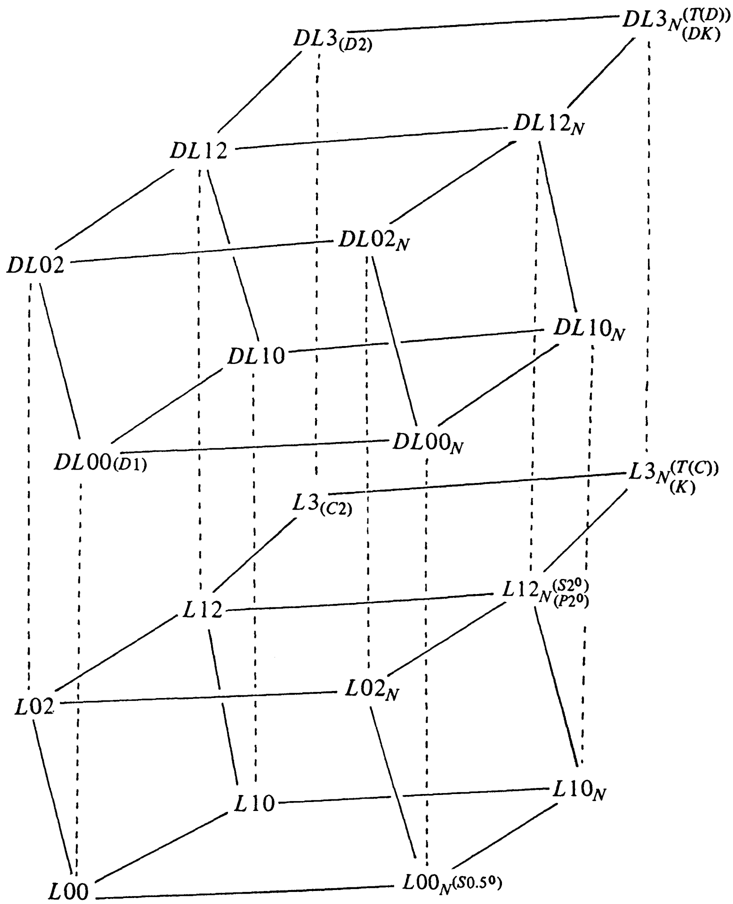
$DL00$ is Lemmon's $D1$, $L00_N$ is $S0.5^0$, and $L12_N$ is $P2^0$. $L3$, $L3_N$, $DL3$, and $DL3_N$ are Lemmon's $C2$, $T(C)$, $D2$, and $T(D)$, respectively. $T(C)$ and $T(D)$ are often called K and DK, respectively.

Lemma 1

1. $L00$ (*or* $L00_N$) $\vdash \Box(A \wedge B) \longleftrightarrow (\Box A \wedge \Box B)$
2. $L00$ (*or* $L00_N$) $\vdash \Diamond(A \vee B) \longleftrightarrow (\Diamond A \vee \Diamond B)$
3. $L00$ (*or* $L00_N$) $\vdash \sim\Diamond\sim A \longleftrightarrow \Box A$
4. $L00$ (*or* $L00_N$) $\vdash \sim\Box A \longleftrightarrow \Diamond\sim A$
5. $L00$ (*or* $L00_N$) $\vdash \sim\Diamond A \longleftrightarrow \Box\sim A$
6. $DL00$ (*or* $DL00_N$) $\vdash \Diamond A$, *where A is a tautology*
7. *For any formula A, $L00$ (or $L00_N$) $\vdash A \longleftrightarrow (\Diamond A_1 \vee \Box B_{11} \vee \ldots \vee \Box B_{1n_1} \vee C_1) \wedge \ldots \wedge (\Diamond A_m \vee \Box B_{m1} \vee \ldots \vee \Box B_{mn_m} \vee C_m)$, where C contains no modal operator.*

Proof: Straightforward.

Remark: We only need axioms of the form $\Box(X \to Y) \to (\Box X \to \Box Y)$ and $\Box X \to \Diamond X$, where both X and Y are Boolean combinations of subformulas of A and B in the proof. We do not claim in 7 that A_i and B_{ij} are also in normal form or $\deg(A_i) = \deg(B_{ij}) = 0$, since we observe here only outermost modal operators.



2 Characterization of L-tautologies Henceforth, unless otherwise specified, we use **L** as a variable ranging over the modal logics Lpq and $DLpq$ ($p = 0, 1$ and $q = 0, 2$).

Definition 1 A formula A is an **L**-tautology iff there exist axioms A_i of **L** and **L**-tautologies $B_j \to C_j$ such that $\bigwedge_{i=1}^{n_1} A_i \to \left(\bigwedge_{j=1}^{n_2} (\Box B_j \to \Box C_j) \to A\right)$ is a tautology, where $n_1 = 0$ if **L** $= L0q$ (or $DL0q$) and $n_2 = 0$ if **L** $= Lp0$ (or $DLp0$).

Definition 1 is a definition by induction. So we can also define **L**-tautologies as follows: (0) each axiom of **L** is an **L**-tautology ($p = 0$), (1) if A_i is an axiom of **L**, $B_j \to C_j$ is an **L**-tautology, and $\bigwedge_{i=1}^{n_1} A_i \to \left(\bigwedge_{j=1}^{n_2} (\Box B_j \to \Box C_j) \to A\right)$ is a tautology, then A is an **L**-tautology.

Lemma 2 *$A \to B$ is an $Lp2$ (or $DLp2$)-tautology iff $\Box A \to \Box B$ is an $Lp2$ (or $DLp2$)-tautology.*

Proof: Since $(\Box A \to \Box B) \to (\Box A \to \Box B)$ is a tautology, if $A \to B$ is an $Lp2$ (or $DLp2$)-tautology, so is $\Box A \to \Box B$ by Definition 1. Suppose now $\Box A \to \Box B$ is an $Lp2$ (or $DLp2$)-tautology. The proof proceeds by induction on the construction of $Lp2$ (or $DLp2$)-tautologies: (1) if $\Box A \to \Box B$ is a tautology, so is $A \to B$, hence $A \to B$ is an $Lp2$ (or $DLp2$)-tautology; (2) let

$$\text{(a)} \qquad \bigwedge_{i=1}^{n_1} A_i \to \left(\bigwedge_{j=1}^{n_2} (\Box B_j \to \Box C_j) \to (\Box A \to \Box B)\right)$$

be a tautology, where A_i is a formula of the form $\Box(B \to C) \to (\Box B \to \Box C)$ or $\Box B \to \sim\Box\sim B$ and $B_j \to C_j$ is an $Lp2$ (or $DLp2$)-tautology. Substitute X for each subformula of the form $\Box X$ in (a) which does not occur in the scope of any other $\Box$. Then we have tautologies $(B \to C) \to (B \to C)$ and $B \to \sim\sim B$ for $\Box(B \to C) \to (\Box B \to \Box C)$ and $\Box B \to \sim\Box\sim B$, respectively, so $\bigwedge_{j=1}^{n_2} (B_j \to C_j) \to (A \to B)$ is a tautology. Thus, $A \to B$ is an $Lp2$ (or $DLp2$)-tautology, since $B_j \to C_j$ is an $Lp2$ (or $DLp2$)-tautology.

Definition 2 Let Φ be the set $\Pi \cup \{\Box A | A$ is a formula$\}$, where Π is the set of propositional variables $p_1, p_2, \ldots$. $\overline{V}$ is the usual extension of $\{\mu | \mu : \Phi \to 2\}$ for $\sim, \wedge, \vee, \to$. $\mathbf{L}^*$ is the set $\{\mu \in \overline{V} |$ for any $\mathbf{L}$-tautology A $\mu(A) = 1\}$.

Lemma 3 (Characterization lemma) *A is an $\mathbf{L}$-tautology iff for any μ in $\mathbf{L}^*$, $\mu(A) = 1$.*

Proof: Let A be an $\mathbf{L}$-tautology. Then for any μ in $\mathbf{L}^*$, $\mu(A) = 1$, by the definition. Now, assume A is not an $\mathbf{L}$-tautology. We construct μ in $\mathbf{L}^*$ such that $\mu(A) = 0$. It is sufficient to consider the case of $DL12$ since the others are special cases of this one. We adopt the convention of association to the right for omitting parentheses. Thus, $A_1 \to A_2 \to \ldots \to A_n$ means $A_1 \to (A_2 \to \ldots \to (A_{n-1} \to A_n) \ldots)$. We say B is a K-formula iff B is of the form $\Box(B_1 \to B_2) \to \Box B_1 \to \Box B_2$. If B is a K-formula $\Box(B_1 \to B_2) \to \Box B_1 \to \Box B_2$, then let $SF(B) = \Box(B_1 \to B_2)$. Assume the given formula A is a Boolean combination of $p_1, \ldots, p_t$, $\Box B_1, \ldots, \Box B_m$, and let $\Box A_1, \ldots, \Box A_n$ be an enumeration of $\Box T$, $\Box\sim T$, $\Box B_1, \ldots, \Box B_m$, where T is a fixed tautology, and let Z be the set of $\Box(B_{i_1} \to \ldots \to B_{i_k})$ such that $k \geq 2$, B_{i_j} is one of $A_1, \ldots, A_n$, and $B_{i_j} \neq B_{i_h}$ if $j \neq h$. Now let $AS = \{B | B$ is a K-formula such that $SF(B)$ in $Z\}$.

It is clear AS is a finite set. Let $\{\Box A_i' | i = 1, 2, \ldots, r\}$ be the set of formulas such that both $\Box A_i'$ and $\Box\sim A_i'$ occur among $\Box A_1, \ldots, \Box A_n$, and let $\{B_1^{(j)} \to \ldots \to B_{q_j}^{(j)} | j = 1, 2, \ldots, s\}$ be the set of $DL12$-tautologies such that $\Box B_k^{(j)}$ occurs among $\Box A_1, \ldots, \Box A_n$ and $B_h^{(j)} \neq B_i^{(j)}$ if $h \neq i (k = 1, 2, \ldots, q_j)$. Since A is not a $DL12$-tautology,

$$(\#) \qquad \bigwedge_{i=1}^{r} (\Box A_i' \to \Diamond A_i') \to \bigwedge_{B \in AS} B \to \bigwedge_{j=1}^{s} (\Box B_1^{(j)} \to \Box(B_2^{(j)} \to \ldots \to B_{q_j}^{(j)})) \to A$$

is not a tautology. So, there exists some μ' in $\overline{V}$ such that

$$\mu'\left(\bigwedge_{i=1}^{r}(\Box A_i' \to \Diamond A_i')\right) = \mu'\left(\bigwedge_{B \in AS} B\right) =$$

$$\mu'\left(\bigwedge_{j=1}^{s}(\Box B_1^{(j)} \to \Box(B_2^{(j)} \to \ldots \to B_{q_j}^{(j)}))\right) = 1 \text{ and } \mu'(A) = 0.$$

Let $\Box A_{n+1}, \ldots$ be an enumeration of all the formulas of the form $\Box B$ except $\Box A_1, \ldots, \Box A_n$. Now, let

$$\mu(B) = \begin{cases} \mu'(B) & \text{if } B \text{ is } p_i \text{ or } \Box A_j (1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant n) \\ e_i & \text{if } B \text{ is } p_i (i > t)(e_i = 0 \text{ or } 1)\ . \end{cases}$$

The next step is an extension of μ to formulas of the form $\Box A_j$, for $j \geqslant n+1$. There are two cases to consider. In both we assume that μ is defined for all $i < j$.

Case 1. $\mu'(\Box T) = 0$. Put $\mu(\Box A_j) = 0$ $(j \geqslant n+1)$. $\mu'(\Box A_j) = 0$ for $j \leqslant n$, since $A_j \to T$ is a $DL12$-tautology and $\mu'(\Box A_j \to \Box T) = 1$. Hence, for all j, $\mu(\Box A_j) = 0$. This fact and the definition of $DL12$-tautologies imply that μ belongs to $DL12^*$ and $\mu(A) = 0$.

Case 2. $\mu'(\Box T) = 1$. For $j \geqslant n+1$,

(1°) if there exists some $i < j$ such that $\sim A_i \longleftrightarrow A_j$ is a tautology and $\mu(\Box A_i) = 1$, then let $\mu(\Box A_j) = 0$,

(2°) if there exist $i_1, \ldots, i_k < j$ such that $\bigwedge_{q=1}^{k} A_{i_q} \to A_j$ is a $DL12$-tautology and $\mu\left(\bigwedge_{q=1}^{k} \Box A_{i_q}\right) = 1$, then let $\mu(\Box A_j) = 1$,

(3°) if there exist $i_1, \ldots, i_k < j$ such that $\bigwedge_{q=1}^{k-1} A_{i_q} \to A_j \to A_{i_k}$ is a $DL12$-tautology, $\mu\left(\bigwedge_{q=1}^{k-1} \Box A_{i_q}\right) = 1$, and $\mu(\Box A_{i_k}) = 0$, then let $\mu(\Box A_j) = 0$,

(4°) otherwise, let $\mu(\Box A_j) = e_j$.

If both (1°) and (2°) hold for some j, we cannot define $\mu(\Box A_j)$, since on the one hand $\mu(\Box A_j)$ must be 0, by (1°), and on the other hand, it must be 1, by (2°). The same situation occurs if both (2°) and (3°) hold. So we must show that at most one of (1°) and (2°) holds, and similarly that at most one of (2°) and (3°) holds.

If $A_{i_1} \to \ldots \to A_{i_k}$ being a $DL12$-tautology implies $\mu(\Box A_{i_1} \to \ldots \to \Box A_{i_k}) = 1$ for all $i_1, \ldots, i_k < j$, then at most one of (1°) and (2°) holds. For assume that $\sim A_i \longleftrightarrow A_j$ is a tautology, $\bigwedge_{q=1}^{k} A_{i_q} \to A_j$ is a $DL12$-tautology, and $\mu(\Box A_i) = \mu\left(\bigwedge_{q=1}^{k} \Box A_{i_q}\right) = 1$, for some $i, i_1, \ldots, i_k < j$. Then, $A_i \wedge \bigwedge_{q=1}^{k} A_{i_q} \to \sim T$ is a $DL12$-tautology, since $\sim A_j \wedge A_j \to \sim T$ and $(A_i \to \sim A_j) \wedge \left(\bigwedge_{q=1}^{k} A_{i_q} \to A_j\right) \to \left(A_i \to \bigwedge_{q=1}^{k} A_{i_q} \to \sim A_j \wedge A_j\right)$ are tautologies.

$\mu\left(\Box A_i \to \bigwedge_{q=1}^{k} \Box A_{i_q} \to \Box{\sim}T\right) = 1$ by the hypothesis and $\mu(\Box A_i) = \mu\left(\bigwedge_{q=1}^{k} \Box A_{i_q}\right) = 1$ by the assumption, so we have $\mu(\Box{\sim}T) = 1$. This contradicts $\mu(\Box T \to {\sim}\Box{\sim}T) = 1$ and $\mu(\Box T) = 1$. Furthermore, if $A_{i_1} \to \ldots \to A_{i_k}$ being a $DL12$-tautology implies $\mu(\Box A_{i_1} \to \ldots \to \Box A_{i_k}) = 1$ for all $i_1, \ldots, i_k < j$, then at most one of (2°) and (3°) holds. For assume that $\bigwedge_{q=1}^{k_1} A_{i_q} \to A_j$ and $\bigwedge_{r=1}^{k_2-1} A_{i'_r} \to A_j \to A_{i'_{k_2}}$ are $DL12$-tautologies, $\mu\left(\bigwedge_{q=1}^{k_1} \Box A_{i_q}\right) = \mu\left(\bigwedge_{r=1}^{k_2-1} \Box A_{i'_r}\right) = 1$, and $\mu(\Box A_{i'_{k_2}}) = 0$, for some $i_1, \ldots, i_{k_1}, i'_1, \ldots, i'_{k_2} < j$. Then, $\bigwedge_{q=1}^{k_1} A_{i_q} \to \bigwedge_{r=1}^{k_2-1} A_{i'_r} \to A_{i'_{k_2}}$ is a $DL12$-tautology, since $\left(\bigwedge_{q=1}^{k_1} A_{i_q} \to A_j\right) \wedge \left(A_j \to \bigwedge_{r=1}^{k_2-1} A_{i'_r} \to A_{i'_{k_2}}\right) \to \left(\bigwedge_{q=1}^{k_1} A_{i_q} \to \bigwedge_{r=1}^{k_2-1} A_{i'_r} \to A_{i'_{k_2}}\right)$ is a tautology. $\mu\left(\bigwedge_{q=1}^{k_1} \Box A_{i_q}\right) = \mu\left(\bigwedge_{r=1}^{k_2-1} \Box A_{i'_r}\right) = 1$ by the assumption, so we have $\mu(\Box A_{i'_{k_2}}) = 1$, by the hypothesis. But this contradicts $\mu(\Box A_{i'_{k_2}}) = 0$ in (3°). If both (2°) and (3°) hold, then we can put $\mu(\Box A_j) = 0$ without any difficulty. If none of (1°)–(3°) hold, we may put $\mu(\Box A_j) = e_j$ ($e_j = 0$ or 1) without any restriction. Thus, μ is well-defined.

We will now show by induction on the construction of $DL12$-tautologies that μ is in $DL12^*$. For any axiom of the form $\Box B \to {\sim}\Box{\sim}B$, let $\Box B$ be $\Box A_h$ and $\Box{\sim}B$ be $\Box A_k$ in the enumeration of the formulas of the form $\Box C$. If $h, k \leqslant n$, then $\mu(\Box B \to {\sim}\Box{\sim}B) = \mu'(\Box B \to {\sim}\Box{\sim}B) = 1$. Now let $n < h$ or $n < k$. If $h < k$ and $\mu(\Box A_h) = 1$, then $\mu(\Box A_k) = 0$, by (1°), since ${\sim}A_h = {\sim}B \longleftrightarrow {\sim}B = A_k$. For the case $\mu(\Box A_h) = 0$, it is trivial that $\mu(\Box A_h \to {\sim}\Box A_k) = 1$. If $k < h$ and $\mu(\Box A_k) = 1$, then $\mu(\Box A_h) = 0$ by (1°), since ${\sim}A_k = {\sim}{\sim}B \longleftrightarrow B = A_h$. For an axiom of the form $\Box(B_1 \to B_2) \to (\Box B_1 \to \Box B_2)$, we have $\mu(\Box(B_1 \to B_2) \to (\Box B_1 \to \Box B_2)) = 1$ by an argument similar to the one for $\Box B \to {\sim}\Box{\sim}B$ except that we use (2°) or (3°) instead of (1°). Let B be a $DL12$-tautology. Then there exist B_i, B_{1j}, B_{2j}, C_k, and D_k such that $\bigwedge_{i=1}^{n_1} (\Box B_i \to {\sim}\Box{\sim}B_i) \to \bigwedge_{j=1}^{n_2} (\Box(B_{1j} \to B_{2j}) \to (\Box B_{1j} \to \Box B_{2j})) \to \bigwedge_{k=1}^{n_3} (\Box C_k \to \Box D_k) \to B$ is a tautology, where each $C_k \to D_k$ is a $DL12$-tautology and $\mu(\Box B_i \to {\sim}\Box{\sim}B_i) = \mu(\Box(B_{1j} \to B_{2j}) \to (\Box B_{1j} \to \Box B_{2j})) = 1$. And $\mu(\Box C_k \to \Box D_k) = 1$, by (2°) or (3°), since $C_k \to D_k$ is a $DL12$-tautology.

Corollary *$\Diamond T$ is a DL12-tautology but not a DL10-tautology.*

Proof: $(\Box T \to {\sim}\Box{\sim}T) \to (\Box{\sim}T \to \Box T) \to {\sim}\Box{\sim}T$ is a tautology, so $\Diamond T$ is a $DL12$-tautology. Let $\mu(\Box B) = 1$ for every formula B whose negation is a tautology, and otherwise let $\mu(\Box B) = 0$. Then μ belongs to $DL10^*$ and $\mu(\Diamond T) = 0$. So $\Diamond T$ is not a $DL10$-tautology.

3 *Completeness*

Definition 3 A structure $\mathfrak{M} = \langle W, R, Q, P\rangle$ consists of

(a) a nonempty set $W \subseteq \overline{V}$,
(b) a relation $R \subseteq W \times W$,
(c) sets $Q, P \subseteq W$ such that $Q \cap P = \phi$.

If a structure $\mathfrak{M} = \langle W, R, P\rangle$, then $\mathfrak{M}$ is such that $Q = \phi$. A structure $\mathfrak{M}$ is called *serial* iff for any $\mu \in W - Q \cup P$ there exists a $\nu \in W$ such that $\mu R\nu$. A structure $\mathfrak{M}$ is an **L**-structure iff both $P \subseteq \mathbf{L}^*$ and $\mathfrak{M}$ is serial if $\mathbf{L} = DLpq$.

Definition 4 Let $\mathfrak{M} = \langle W, R, Q, P\rangle$ be a structure. Then for any $\mu \in W$ and formulas A and B,

(1) if A is a propositional variable, then $\langle \mathfrak{M}, \mu\rangle \models A$ iff $\mu(A) = 1$,
(2) for $X = {\sim}A$, $A \vee B$, $A \wedge B$, or $A \to B$ $\langle \mathfrak{M}, \mu\rangle \models X$ is defined as usual,
(3) $\langle \mathfrak{M}, \mu\rangle \models \Box A$ iff (i) $\mu \notin Q \cup P$ and for any ν such that $\mu R\nu$, $\langle \mathfrak{M}, \nu\rangle \models A$, or (ii) $\mu \in P$ and $\mu(\Box A) = 1$.

We will write:

$\mathfrak{M} \models A$ iff for any $\mu \in W - P$ $\langle \mathfrak{M}, \mu\rangle \models A$
$\mathbf{L} \models A$ iff for any **L**-structure $\mathfrak{M} = \langle W, R, Q, P\rangle$ $\mathfrak{M} \models A$
$\mathbf{L}_N \models A$ iff for any **L**-structure $\mathfrak{M} = \langle W, R, P\rangle$ $\mathfrak{M} \models A$.

In the following lemmas, C is a formula which contains no modal operator.

Lemma 4 *If* **L** *(or* $\mathbf{L}_N$*)* $\models \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$, *then either (a) C is a tautology, or (b) $A \vee B_i$ is an* **L**-*tautology, for some i ($1 \leqslant i \leqslant n$).*

Proof: Assume neither (a) nor (b) holds. Then there exist $\mu_0, \mu_1, \ldots, \mu_n$ by Lemma 3 such that $\mu_0 \neq \mu_i$, $\mu_i \in \mathbf{L}^*$ and which satisfy $\mu_0(C) = 0$ and $\mu_i(A \vee B_i) = 0$ ($1 \leqslant i \leqslant n$).

Let $W = \{\mu_0, \mu_1, \ldots, \mu_n\}$, $P = \{\mu_1, \ldots, \mu_n\}$, $Q = \phi$, $\mu_0 R\mu_i$ ($1 \leqslant i \leqslant n$), and $\mathfrak{M} = \langle W, R, Q, P\rangle$. Then $\langle \mathfrak{M}, \mu_0\rangle \not\models \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$. This is a contradiction. So either (a) or (b) holds. In the case where C is missing, we can choose μ_0 such that $\mu_0 \neq \mu_i$, for $i = 1, 2, \ldots, n$.

Remark: In Lemma 4, $\Diamond A$ or $\Box B_i$ is possibly missing even if C is missing or is not tautology. If $\Diamond A$ is missing then $\mathbf{L}_N \models \Box B_1 \vee \ldots \vee \Box B_n \vee C$, and if $n = 0$, then $\mathbf{L} = DLpq$.

Lemma 5 *If $A \to B$ is an* **L**-*tautology, then* $\mathbf{L} \vdash A \to B$ *and* $\mathbf{L} \vdash \Box A \to \Box B$.

Proof: We proceed by induction on the construction of **L**-tautologies. If $\bigwedge_{i=1}^{n_1} A_i \to \bigwedge_{j=1}^{n_2} (\Box B_j \to \Box C_j) \to (A \to B)$ is a tautology, A_i is an axiom, and $B_j \to C_j$ is an **L**-tautology, then $\mathbf{L} \vdash \Box B_j \to \Box C_j$, by induction hypothesis, so $\mathbf{L} \vdash A \to B$, and $\mathbf{L} \vdash \Box A \to \Box B$, by $R00$–$R12$, respectively.

Corollary *Let* $\mathbf{L} = DL00$–$DL12$. *If A is an* **L**-*tautology, then* $\mathbf{L} \vdash \Diamond A$.

Proof: Assume A is an $\mathbf{L}$-tautology. Then so is $\sim A \to \sim T$, where T is a given fixed tautology. By Lemma 5, $\mathbf{L} \vdash \Box\sim A \to \Box\sim T$. So, $\mathbf{L} \vdash \Diamond T \to \Diamond A$. $\mathbf{L} \vdash \Diamond A$, since $DL00 \vdash \Diamond T$, by 6 in Lemma 1.

In the following lemmas, $\mathbf{L}_N$ means one of $L00_N$–$DL12_N$.

Lemma 6 *If A is an $\mathbf{L}$-tautology, then $\mathbf{L}_N \vdash A$ and $\mathbf{L}_N \vdash \Box A$.*

Proof: Assume $\bigwedge_{i=1}^{n_1} A_i \to \bigwedge_{j=1}^{n_2} (\Box B_j \to \Box C_j) \to A$ is a tautology, A_i is an axiom, and $B_j \to C_j$ is an $\mathbf{L}$-tautology. By induction hypothesis, $\mathbf{L}_N \vdash \Box(B_j \to C_j)$, so $\mathbf{L}_N \vdash \Box B_j \to \Box C_j$, therefore, $\mathbf{L}_N \vdash A$.

$\mathbf{L}_N \vdash \Box\left(\bigwedge_{i=1}^{n_1} A_i \to \bigwedge_{j=1}^{n_2}(\Box B_j \to \Box C_j) \to A\right)$ by $R0_N$ (or $R1_N$), so $\mathbf{L}_N \vdash$ $\bigwedge_{i=1}^{n_1} \Box A_i \to \bigwedge_{j=1}^{n_2} \Box(\Box B_j \to \Box C_j) \to \Box A$, therefore $\mathbf{L}_N \vdash \Box A$, by $R0_N$ (or $R1_N$) and $R2_N$ with $\Box(B_j \to C_j)$ being provable.

Corollary *Let $\mathbf{L} = DL00$–$DL12$. If A is an $\mathbf{L}$-tautology, then $\mathbf{L}_N \vdash \Diamond A$.*

Proof: $\mathbf{L}_N \vdash \Box A$. So $\mathbf{L}_N \vdash \Diamond A$.

Remark: In the proof of Lemma 6, we may assume that each A_i, B_j, and C_j, is a Boolean combination of subformulas of A, by Lemma 3 and (#). Furthermore, there exists a proof of $\Box \bigwedge_{i=1}^{n} X_i \longleftrightarrow \bigwedge_{i=1}^{n} \Box X_i$ where we need only modal axioms of the form $\Box(X \to Y) \to (\Box X \to \Box Y)$ such that both X and Y are Boolean combinations of subformulas of $\bigwedge_{i=1}^{n} X_i$. Thus, by induction on the construction of $\mathbf{L}$-tautologies, we have a proof of $\Box A$ in Lemma 6 where both X and Y occurring in each modal axiom of the form $\Box(X \to Y) \to (\Box X \to \Box Y)$ or $\Box X \to \Diamond X$ are Boolean combinations of subformulas of $\Box A$. We say in such a case that $\Box A$ has the *M*-subformula property. We also have the same result for Lemma 5.

Lemma 7 *If $\mathbf{L}$ (or $\mathbf{L}_N$) $\vDash \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$, then $\mathbf{L}$ (or $\mathbf{L}_N$) $\vdash \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$.*

Proof: By Lemma 4, we have either (1) C is a tautology, or (2) $A \vee B_i$ is an $\mathbf{L}$-tautology for some i ($1 \leqslant i \leqslant n$).

If (1) holds, then trivially $\mathbf{L}$ (or $\mathbf{L}_N$) $\vdash \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$. If (2) holds, we have three cases: (1°) If A is missing, then $\mathbf{L}_N \vDash \Box B_1 \vee \ldots \vee \Box B_n \vee C$, by the Remark following Lemma 4, and for some i, B_i is an $\mathbf{L}$-tautology by Lemma 4. Thus, we have $\mathbf{L}_N \vdash \Box B_i$ by Lemma 6. Thus, $\mathbf{L}_N \vdash \Box B_1 \vee \ldots \vee \Box B_n \vee C$.

(2°) If $n = 0$, then $\mathbf{L} = DLpq$, by the Remark following Lemma 4, and A is an $\mathbf{L}$-tautology. So, $\mathbf{L}$ (or $\mathbf{L}_N$) $\vdash \Diamond A$, by the corollaries to Lemma 5 and Lemma 6.

(3°) In the remaining case, $\mathbf{L}$ (or $\mathbf{L}_N$) $\vdash \Box{\sim}A \to \Box B_i$, by Lemma 4 and Lemma 5 (or Lemma 6). Thus, $\mathbf{L}$ (or $\mathbf{L}_N$) $\vdash \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$.

Remark: It is easily seen that there exists a proof of $\Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$ with the same property mentioned in the Remark following Lemma 6.

If a structure $\mathcal{M} = \langle W, R, Q\rangle$ or $\langle W, R\rangle$, then $\mathcal{M}$ is such that $P = \phi$ or $P = Q = \phi$, respectively. $\mathcal{M}$ is an $L3$-structure iff $\mathcal{M} = \langle W, R, Q\rangle$, and $\mathcal{M}$ is a $DL3$-structure iff $\mathcal{M} = \langle W, R, Q\rangle$ and $\mathcal{M}$ is serial. Let $\mathbf{L}$ be $L3$ or $DL3$. $\mathbf{L} \vDash A$ iff for any $\mathbf{L}$-structure $\mathcal{M} = \langle W, R, Q\rangle$, $\mathcal{M} \vDash A$, and $\mathbf{L}_N \vDash A$ iff for any structure $\mathcal{M} = \langle W, R\rangle$, $\mathcal{M} \vDash A$.

Lemma 8 *Let $\mathcal{M}_i = \langle W_i, R_i, Q_i\rangle$ $(i \in I)$, $W_i \cap W_j = \phi$ if $i \neq j$ $(i, j \in I)$, and $\mathcal{M} = \langle W, R, Q\rangle$, where $W = \bigcup\limits_i W_i$, $R = \bigcup\limits_i R_i$, and $Q = \bigcup\limits_i Q_i$. Then, for any formula A and any $i \in I$, $\mu \in W_i$, $\langle \mathcal{M}, \mu\rangle \vDash A$ iff $\langle \mathcal{M}_i, \mu\rangle \vDash A$.*

Proof: Straightforward.

Lemma 9 *Let $\mathcal{M}_i = \langle W_i, R_i, Q_i\rangle$ and $\mu_i \in W$ $(1 \leqslant i \leqslant m)$. For each $\mathcal{M}_i$, μ_i, and propositional variables $p_1, \ldots, p_n$, there exists $\mathcal{M}_i' = \langle W_i', R_i', Q_i'\rangle$ such that $\mu_i' \in W_i'$, $W_i' \cap W_j' = \phi$ if $i \neq j$, and $\langle \mathcal{M}_i, \mu_i\rangle \vDash A$ iff $\langle \mathcal{M}_i', \mu_i'\rangle \vDash A$, for any formula A having only propositional variables $p_1, \ldots, p_n$.*

Proof: Straightforward.

Lemma 4′ *Let $\mathbf{L}$ be $L3$ or $DL3$. If $\mathbf{L}$ (or $\mathbf{L}_N$) $\vDash \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$, then either (1) C is a tautology, or (2) $\mathbf{L}$ (or $\mathbf{L}_N$) $\vDash A \vee B_i$ for some i.*

The proof is similar to the proof of Lemma 4 except that we make use of Lemma 9.

Thus, Lemma 7 with the Remark following it hold for $L3$ (or $L3_N$) and $DL3$ (or $DL3_N$), by induction on modal degrees of formulas.

The completeness theorem follows by 7 in Lemma 1, Lemma 7, and by checking that each axiom is valid and that each rule of inference preserves validity.

Theorem *Let $\mathbf{L}$ be $L00$–$DL3$. Then $\mathbf{L}$ (or $\mathbf{L}_N$) $\vDash A$ iff $\mathbf{L}$ (or $\mathbf{L}_N$) $\vdash A$.*

4 Applications

Derivability of the rule $\vdash \Box A \Rightarrow \vdash A$ $S2^0$ based on the PC is formulated as $L12_N + \{\vdash \Box A \Rightarrow \vdash A\}$ in Zeman [10] and Bowen [1].

We show here the rule $\vdash \Box A \Rightarrow \vdash A$ is derivable in $L00_N$–$DL3_N$. Hence, we can see this rule is redundant in the $S2^0$ formulation.

Corollary *Let $\mathbf{L}$ be $L00$–$DL3$. If $\mathbf{L}_N \vdash \Box A$, then $\mathbf{L}_N \vdash A$.*

Proof: Suppose $\mathbf{L}_N \vdash \Box A$. Then $\mathbf{L}_N \vDash \Box A$. So $\mathbf{L}_N \vDash A$. Therefore, $\mathbf{L}_N \vdash A$.

Decidability First we show that it is decidable whether A is an $\mathbf{L}$-tautology or not, where $\mathbf{L}$ is one of $L00$–$DL12$. We consider only the case of $DL12$ as in the proof of Lemma 3. Our proof proceeds by induction on the construction of $DL12$-tautologies and modal degrees of formulas. It is decidable whether A is an axiom of $DL12$ or not. Assume that for any formula A such that $\deg(A) < n$, we can decide whether A is a $DL12$-tautology or not. Let A be a Boolean combination of $p_1, \ldots, p_t, \Box B_1, \ldots, \Box B_m$ and $\deg(A) = n$. Then we can construct (#), as in the proof of Lemma 3, in a finite number of steps since $\deg(B_1^{(j)} \to \ldots \to B_{q_j}^{(j)}) < n$. It is easy to decide whether (#) is a tautology or not. Thus, we can decide whether or not A is a $DL12$-tautology. We can also decide whether or not $\mathbf{L}$ (or $\mathbf{L}_N$) $\vdash \Diamond A \vee \Box B_1 \vee \ldots \vee \Box B_n \vee C$ by Lemmas 4–6. For $L3$–$DL3_N$, we have the same result by Lemma 4′ and induction on modal degrees of formulas. So each formula in conjunctive normal form is decidable. Any formula is equivalent to its conjunctive normal form, by 7 in Lemma 1, and for any given formula we have a method for deriving its conjunctive normal form. Therefore, each logic here is decidable.

M-subformula property Let $\mathbf{L}$ be $L00$–$DL3_N$. We say a formula A provable in $\mathbf{L}$ has the M-subformula property iff there exists a proof of A in $\mathbf{L}$ such that each formula X, Y occurring in any modal axiom of the form $\Box(X \to Y) \to (\Box X \to \Box Y)$ or $\Box X \to \Diamond X$ in the proof consists of a Boolean combination of subformulas of A. A logic $\mathbf{L}$ has the M-subformula property iff each formula provable in $\mathbf{L}$ has the M-subformula property.

Theorem *Let $\mathbf{L}$ be $L00$–$DL3_N$. Then $\mathbf{L}$ has the M-subformula property.*

Proof: Assume $\mathbf{L} \vdash A$. By 7 in Lemma 1, $\mathbf{L} \vdash A \longleftrightarrow \bigwedge_{i=1}^{m} (\Diamond A_i \vee \Box B_{i1} \vee \ldots \vee \Box B_{in_i} \vee C_i)$. Since $\mathbf{L} \vdash A$, $\mathbf{L} \vdash \Diamond A_i \vee \Box B_{i1} \vee \ldots \vee \Box B_{in_i} \vee C_i$, for each i. Thus, this formula has the M-subformula property by the Remark following Lemma 7. Hence, A has the M-subformula property by the Remark following the proof of Lemma 1.

REFERENCES

[1] Bowen, K. A., *Model Theory for Modal Logic*, Reidel Publishing Company, Amsterdam, 1979.

[2] Cresswell, M. J., "A conjunctive normal form for S3.5," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 253–255.

[3] Fine, K., "Normal forms in modal logic," *Notre Dame Journal of Formal Logic*, vol. 16 (1975), pp. 229–237.

[4] Fitting, M., "Subformula results in some propositional modal logics," *Studia Logica*, vol. 37 (1978), pp. 387–391.

[5] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen Co., Ltd., London, 1968.

[6] Kripke, S. A., "Semantical analysis of modal logic I," *Zeitschrift für Mathematische Logik und Grundlagen der mathematik*, vol. 9 (1963), pp. 67–96.

[7] Lemmon, E. J., "New foundations for Lewis modal systems," *The Journal of Symbolic Logic*, vol.22 (1957), pp. 176–186.

[8] Lemmon, E. J., "Algebraic semantics for modal logic I," *The Journal of Symbolic Logic*, vol. 31 (1966), pp.46–65.

[9] Ōhama, S., "$S2^0 = P2^0 - \{\vdash \Box A \Rightarrow \vdash A\}$," *Journal of Toyota National Technical College*, vol. 15 (1982), pp. 91–96.

[10] Zeman, J. J., "Lemmon-style bases for the system $S1^0 - S4^0$," *The Journal of Symbolic Logic*, vol. 33 (1968), pp. 458-461.

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